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Numerical Solution of Differential Equations by Using Chebyshev Wavelet Collocation Method

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Abstract: In this article, a new method known as the Chebyshev wavelet collocation method is presented for the solution of second-order linear ordinary differential equations (ODEs). The method is based on the approximation of the truncated Chebyshev wavelet series. By using the Chebyshev collocation points, an algebraic equation system has been obtained and solved. Hence the implicit forms of the approximate solution of second-order linear ordinary differential equations have been obtained. This present method has been applied to the Bessel differential equation of order zero and the Lane–Emden equation. These calculations demonstrate that the accuracy of the Chebyshev wavelet collocation method is quite high even in the case of a small number of grid points. The present method is a very reliable, simple, fast, computationally efficient, flexible, and convenient alternative method.
Keywords: Chebyshev wavelet; Collocation; Legendre wavelet; approximate solution.

1. Introduction

In recent years, several simple and accurate methods based on orthogonal functions and polynomial series, including wavelets, have been used to approximate the solution of various problems [1–6]. The main characteristic of using an orthogonal basis is that it reduces these problems into solving systems of algebraic equations. The approach is based on converting the underlying differential equations into integral equations through integration, approximating various signals involved in the equation by the truncated orthogonal series

$$y(t) \cong y_N(t) = \sum_{i=0}^{N-1} c_i \phi_i$$

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and eliminating the integral operations by the operational matrix P of integration. The matrix P is given by

$$\int_0^t \Phi(s)ds \cong P\Phi(t),$$

where $\Phi(t) = [\phi_0(t), \phi_1(t), \dots, \phi_{N-1}(t)]^T$ and matrix P can be uniquely determined on the basis of particular orthogonal functions defined for a certain interval $[a,b]$. Special attention has been given to the applications of Legendre polynomials [2], Chebyshev polynomials [3], the Walsh function [7], the block-pulse function [8] and Legendre series [9]. All the orthogonal functions given above are supported on the whole interval $a \le t \le b$; however, this kind of global support is evidently a drawback for certain analyses, notably systems involving abrupt variations or local functions vanishing outside a short interval of time or space.

Wavelets, as very well-localized functions, are considerably useful for solving differential and integral equations and provide accurate solutions. The wavelet technique allows the creation of very fast algorithms when compared with the algorithms ordinarily used. Since the cancellation of many terms is required in order to obtain a reasonable degree of accuracy, the wavelets analysis could be a possible tool for solving this difficulty in physics, communication and image processing [5]. Gu and Jiang [10] derived the Haar wavelets operational matrix of integration. In the literature, special attention has been given to the applications of Legendre wavelets [6, 11]. The Legendre and Chebyshev wavelets operational matrixes of integration and product operation matrix have been introduced in [12-15]. These matrices can be used to solve problems such as identification, analysis and optimal control.

Our analyses show some disadvantages to applying the Legendre wavelet and Chebyshev wavelet methods. Although second integrations of the $\Psi(t)$ differs from P^2, it is taken as P^2. Moreover, some expressions, obtained from integrating $\Psi(t)$ and that are different from elements of $\Psi(t)$, have not been used in the construction of the matrix P.

In this study, the Chebyshev wavelet collocation method is presented as the solution of second-order linear ordinary differential equations. In the present method, there is no disadvantage, given above, of the Legendre wavelet and Chebyshev wavelet methods.

The proposed method is based on the approximation of the truncated Chebyshev wavelets series. By using the Chebyshev collocation points, an algebraic equation system has been obtained and by solving this algebraic equation, the coefficients of the Chebyshev wavelet series can be found. Hence, we have the implicit form of the approximate solution of second-order linear ordinary differential equations. This present method has been applied to three problems, and calculations demonstrate that the accuracy of the Chebyshev wavelet collocation method is quite high even in the case of a small number of grid points. The present method is a very reliable, simple, fast, computationally efficient, flexible, and convenient alternative method.

2. Chebyshev wavelet and Collocation Method

In recent years, wavelets have been used in many different fields of science and engineering. They constitute a family of functions constructed from the dilation and translation of a single function known as the mother wavelet. When the dilation parameter *a* and the translation parameter *b* vary continuously, we have the following family of continuous wavelets [10]:

$$\psi_{a,b}(t) = |a|^{1/2}\psi\left(\frac{t-b}{a}\right), \quad a,b \in R, \quad a \neq 0. \tag{2.1}$$

Chebyshev wavelets $\psi_{nm}(t) = \psi(k,n,m,t)$ have four arguments,
$k = 0, 1, 2, \ldots,$ $n = 1, 2, \ldots, 2^k$, *m* is the degree of the Chebyshev polynomials of the first kind and *t* denotes normalized time. They are defined on the interval $[0,1)$ by:

$$\psi_{nm}(t) = \begin{cases} \dfrac{\alpha_m 2^{k/2}}{\sqrt{\pi}} T_m(2^{k+1}t - 2n + 1), & \dfrac{n-1}{2^k} \le t \le \dfrac{n}{2^k}, \\ 0 & otherwise \end{cases} \tag{2.2}$$

where

$$\alpha_m = \begin{cases} \sqrt{2} & m = 0 \\ 2 & m = 1, 2, \ldots \end{cases}$$

and $T_m(t)$ are Chebyshev polynomials of the first kind of degree m which are orthogonal with respect to the weight function $w = 1/\sqrt{1-t^2}$ on $[-1, 1]$ and can be generated from the following recursive formula [16]:

$$T_0(t) = 1, \quad T_1(t) = t, \quad T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t) \quad m = 1, 2, 3, \ldots$$

Similarly, the set of Chebyshev wavelets are orthogonal with respect to the weight function $w_n(t) = w(2^{k+1}t - 2n + 1)$

A function $f(t) \in L_w^2[0,1]$ may be expanded as:

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} f_{nm} \psi_{nm}(t) \tag{2.3}$$

where

$$f_{nm} = \langle f(t), \psi_{nm}(t) \rangle \tag{2.4}$$

In Equation (2.4), $\langle .,. \rangle$ denotes the inner product with weight function $w_n(t)$

If the infinite series in Equation (2.3) is truncated, then Equation (2.3) can be written as:

$$f(t) \cong \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} f_{nm} \psi_{nm}(t) = C^T \Psi(t) \tag{2.5}$$

where C and $\Psi(t)$ are $2^k M \times 1$ matrices given by:

$$C^T = [f_{10}, f_{11}, \ldots, f_{1M-1}, f_{20}, \ldots, f_{2M-1}, \ldots, f_{2^k 0}, \ldots, f_{2^k M-1}] \tag{2.6}$$

$$\Psi(t) = [\psi_{10}, \psi_{11}, \ldots, \psi_{1M-1}, \psi_{20}, \ldots, \psi_{2M-1}, \ldots, \psi_{2^k 0}, \ldots, \psi_{2^k M-1}]^T \tag{2.7}$$

3. Chebyshev Wavelet Collocation Method

The integration of $\psi_{nm}(t)$ given in (2.2) can be shown as

$$p_{nm}(t) = \int_0^t \psi_{nm}(s)ds \tag{3.1}$$

For $m = 0, m = 1$ and $m > 1$, $p_{nm}(t)$ can be obtained as

$$p_{n0}(t) = \begin{cases} 0 & 0 \le t < \dfrac{n-1}{2^k} \\ \dfrac{\alpha_0 2^{-k/2-1}}{\sqrt{\pi}}\Big[T_1(2^{k+1}t - 2n + 1) + T_0(2^{k+1}t - 2n + 1)\Big] & \dfrac{n-1}{2^k} \le t < \dfrac{n}{2^k} \\ \dfrac{\alpha_0 2^{-k/2}}{\sqrt{\pi}} T_0(2^{k+1}t - 2n + 1) & \dfrac{n}{2^k} \le t < 1 \end{cases}$$

$$p_{n1}(t) = \begin{cases} 0 & 0 \le t < \dfrac{n-1}{2^k} \\ \dfrac{\alpha_1 2^{-k/2-3}}{\sqrt{\pi}}\Big[T_2(2^{k+1}t - 2n + 1) - T_0(2^{k+1}t - 2n + 1)\Big] & \dfrac{n-1}{2^k} \le t < \dfrac{n}{2^k} \\ 0 & \dfrac{n}{2^k} \le t < 1 \end{cases}$$

$$p_{nm}(t) = \begin{cases} 0 & 0 \le t < \dfrac{n-1}{2^k} \\ \dfrac{\alpha_m 2^{-k/2-2}}{\sqrt{\pi}}\left[\dfrac{T_{m+1}(u) - (-1)^{m+1}}{m+1} - \dfrac{T_{m-1}(u) - (-1)^{m-1}}{m-1}\right] & \dfrac{n-1}{2^k} \le t < \dfrac{n}{2^k} \\ \dfrac{\alpha_m 2^{-k/2-2}}{\sqrt{\pi}}\left[\dfrac{1 - (-1)^{m+1}}{m+1} - \dfrac{1 - (-1)^{m-1}}{m-1}\right] & \dfrac{n}{2^k} \le t < 1 \end{cases}$$

respectively, where $u = 2^{k+1}t - 2n + 1$. The integration of $\Psi(t)$ can be represented as

$$P(t)=\int_0^t \Psi(s)ds=[p_{10},\, p_{11},\, ...,\, p_{1M-1},\, p_{20},\, ...,\, p_{2M-1},\, ...,\, p_{2^k0},\, ...,\, p_{2^kM-1}\,]^T \qquad (3.2)$$

The second integration of $\psi_{nm}(t)$ can be shown as

$$q_{nm}(t)=\int_0^t p_{nm}(s)ds$$

For $m=0$, $m=1$, $m=2$ and $m>2$, $q_{nm}(t)$ can be obtained as

$$q_{n0}(t)=\begin{cases} 0 & 0\le t<\dfrac{n-1}{2^k} \\ \dfrac{\alpha_0 2^{-3k/2-4}}{\sqrt{\pi}}\left[T_2(u)+4T_1(u)+3T_0(u)\right] & \dfrac{n-1}{2^k}\le t<\dfrac{n}{2^k} \\ \dfrac{\alpha_0 2^{-k/2}}{\sqrt{\pi}}\left[\dfrac{1}{2^{k+1}}+t-\dfrac{n}{2^k}\right] & \dfrac{n}{2^k}\le t<1 \end{cases}$$

$$q_{n1}(t)=\begin{cases} 0 & 0\le t<\dfrac{n-1}{2^k} \\ \dfrac{\alpha_1 2^{-3k/2-4}}{\sqrt{\pi}}\left[\dfrac{T_3(u)}{6}-\dfrac{3T_1(u)}{2}-\dfrac{4T_0(u)}{3}\right] & \dfrac{n-1}{2^k}\le t<\dfrac{n}{2^k} \\ \dfrac{\alpha_1 2^{-3k/2-4}}{\sqrt{\pi}}\left[\dfrac{-8}{3}\right] & \dfrac{n}{2^k}\le t<1 \end{cases}$$

$$q_{n2}(t)=\begin{cases} 0 & 0\le t<\dfrac{n-1}{2^k} \\ \dfrac{\alpha_2 2^{-3k/2-3}}{\sqrt{\pi}}\left[\dfrac{T_4(u)-1}{24}-\dfrac{T_2(u)-1}{3}-\dfrac{2}{3}T_1(u)-\dfrac{2}{3}T_0(u)\right] & \dfrac{n-1}{2^k}\le t<\dfrac{n}{2^k} \\ -\dfrac{\alpha_2 2^{-k/2}}{3\sqrt{\pi}}\left[\dfrac{1}{2^{k+1}}+t-\dfrac{n}{2^k}\right] & \dfrac{n}{2^k}\le t<1 \end{cases}$$

$$q_{nm}(t)=\begin{cases}0 & 0\le t<\dfrac{n-1}{2^k}\\[2ex] \dfrac{\alpha_m 2^{-3k/2-3}}{\sqrt{\pi}}\left[\begin{array}{l}\dfrac{T_{m+2}(u)-(-1)^{m+2}}{2(m+1)(m+2)}-\dfrac{T_m(u)-(-1)^m}{2(m+1)m}-\dfrac{T_m(u)-(-1)^m}{2m(m-1)}\\ +\dfrac{T_{m-2}(u)-(-1)^{m-2}}{2(m-1)(m-2)}+\left(1+T_1(u)\right)\left[\dfrac{(-1)^{m-1}}{m-1}-\dfrac{(-1)^{m+1}}{m+1}\right]\end{array}\right] & \dfrac{n-1}{2^k}\le t<\dfrac{n}{2^k}\\[2ex] \dfrac{\alpha_m 2^{-3k/2-3}}{\sqrt{\pi}}\left[\begin{array}{l}\dfrac{1-(-1)^{m+2}}{2(m+1)(m+2)}-\dfrac{1-(-1)^m}{2(m+1)m}-\dfrac{1-(-1)^m}{2m(m-1)}\\ +\dfrac{1-(-1)^{m-2}}{2(m-1)(m-2)}+2\left(\dfrac{(-1)^{m-1}}{m-1}-\dfrac{(-1)^{m+1}}{m+1}\right)\\ +2^{k+1}\left(t-\dfrac{n}{2^k}\right)\left[\dfrac{1-(-1)^{m+1}}{m+1}-\dfrac{1-(-1)^{m-1}}{m-1}\right]\end{array}\right] & \dfrac{n}{2^k}\le t<1\end{cases}$$

respectively, where $u=2^{k+1}t-2n+1$.

The second integration of $\Psi(t)$ can be represented as

$$Q(t)=\int_0^t\int_0^s \Psi(w)dwds=\int_0^t P(s)ds = [q_{10}, q_{11}, ..., q_{1M-1}, q_{20}, ..., q_{2M-1}, ..., q_{2^k 0}, ..., q_{2^k M-1}]^T \tag{3.3}$$

The collocation points can be taken as $2^{k+1}t_{ni}-2n+1=\cos\dfrac{((M+1)-i)\pi}{(M+1)}$ or

$$t_{ni}=\frac{1}{2^{k+1}}\left(2n-1+\cos\frac{((M+1)-i)\pi}{(M+1)}\right),\ i=1,2,...,M,\ n=1,2,...,2^k \tag{3.4}$$

which are also called the turning points of $T_{M+1}(2^{k+1}t-2n+1)$. Substituting the Chebyshev collocation points into the (2.7), (3.2) and (3.3), a discretized form of the vectors $\Psi(t_{ni}), P(t_{ni})$ and $Q(t_{ni})$ can be obtained. Hence, by the discretized form of the vectors $\Psi(t_{ni}), P(t_{ni})$ and $Q(t_{ni})$, the matrices Ψ, P and Q, which have the dimension $2^k M \times 2^k M$, are achieved respectively.

In the [13] and [14] the operational matrix of integration P has been derived as

$$\int_0^t \Psi(s)ds \cong P\Psi(t)$$

$$\int_0^t\int_0^s \Psi(w)dwds \cong P^2\Psi(t)$$

where $\Psi(t)$ is given in (2.7) and P is a $2^k M \times 2^k M$ matrix. There are some mistakes in the application of the Legendre wavelet and Chebyshev wavelet methods. First of all, direct integral calculations show that P^2 does not hold. For instance, from the Chebyshev wavelet method, we have

$$\int_0^t \Psi(s)ds \cong P\Psi(t) = \begin{bmatrix} 1/4 & \sqrt{2}/8 & 1/2 & 0 \\ -\sqrt{2}/16 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & \sqrt{2}/8 \\ 0 & 0 & -\sqrt{2}/16 & 0 \end{bmatrix}\Psi(t)$$

$$\int_0^t\int_0^s \Psi(w)dwds \cong \begin{bmatrix} 3/64 & \sqrt{2}/32 & 0 & \sqrt{2}/16 \\ -\sqrt{2}/48 & -3/128 & -\sqrt{2}/96 & 0 \\ 0 & 0 & 3/64 & \sqrt{2}/32 \\ 0 & 0 & -\sqrt{2}/48 & -3/128 \end{bmatrix}\Psi(t) \neq P^2\Psi(t)$$

for $M = 2, k = 1$. Secondly, some expressions obtained from the integration of (2.7) have not been used in the construction of the matrix P. For example, in the Chebyshev wavelet method, T_M and T_{M+1} obtained from the first and second integrations of the $\Psi(t)$ have not been used in the construction of P and P^2. These are the disadvantages of the Legendre wavelet and Chebyshev wavelet methods.

In Chebyshev wavelet collocation method, Ψ, P and Q matrices are constructed from $\Psi(t)$ and the first and second integrations of $\Psi(t)$ by using collocation points (3.4) respectively. Values of T_M and T_{M+1}, obtained from the first and second integrations of $\Psi(t)$, in the collocation points are also used in the computation of P and Q. Hence the present method is proves to be superior to the Legendre wavelet and Chebyshev wavelet methods.

4. Numerical examples

In this section, we implement our proposed method to solve three examples. In the first example, the computed results have been compared with other methods such as the Chebyshev wavelet method, Legendre wavelet method and Bernstein polynomial basis. In the second example, the effect of parameter M is observed. In the third example, the application of the Chebyshev wavelet collocation method has been shown for a boundary value problem, and the effect of the parameter k is observed.

Example 1. Consider the following Bessel differential equation of order zero

$$\begin{aligned} &ty''(t) + y'(t) + ty(t) = 0 \\ &y(0) = 1, \quad y'(0) = 0 \end{aligned} \tag{4.1}$$

The solution for the given equation is known as the Bessel function of the first kind of order zero and denoted as

$$J_0(t) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(i!)^2} \left(\frac{t}{2}\right)^{2i}$$

First, it is assumed that $y''(t)$ can be expanded in terms of Chebyshev wavelets as

$$y''(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} f_{nm}\psi_{nm}(t) = C^T \Psi(t) \quad (4.2)$$

By integrating Equation (4.2) twice with respect to t from 0 to t, following equations are obtained

$$y'(t) = \int_0^t y''(s)ds + y'(0) = \int_0^t C^T \Psi(s)ds = C^T P(t) \quad (4.3)$$

$$y(t) = \int_0^t y'(s)ds + y(0) = \int_0^t C^T P(s)ds + y(0) = C^T Q(t) + 1 \quad (4.4)$$

Substituting equations (4.2)-(4.4) into the equation (4.1), we have

$$C^T \left(t\Psi(t) + P(t) + tQ(t)\right) = -t \quad (4.5)$$

Substituting the Chebyshev collocation points into (4.5), we can obtain an algebraic equation system the matrix notation of which is

$$C^T U = B \quad (4.6)$$

where U is a $2^k M \times 2^k M$ matrix. C and B are $2^k M \times 1$ vectors. Hence, by solving algebraic equation system (4.6) obtained from Equation (4.1), we can find the coefficients of the Chebyshev wavelet series that satisfies the equation and its boundary conditions. By substituting the Chebyshev wavelet coefficients into equation (4.4), we have the implicit form of the approximate solution of (4.1). Table 1 shows the absolute error in the equally divided interval $[0,1]$ with the Legendre wavelet method, Chebyshev wavelet method, Chebyshev wavelet collocation method for $M = 3$, $k = 1$ and the Bernstein polynomial basis [17] and Chebyshev wavelet collocation method for $M = 5$ and $k = 0$. As seen from Table 1, it is clear that the result obtained by the present method is superior to that obtained by the Legendre wavelets method, Chebyshev wavelet method and the Bernstein polynomial basis method.

TABLE 1. Absolute error of exact and approximated solution of Example 1 with the Legendre, Chebyshev wavelets methods, Bernstein polynomial basis method and the Chebyshev wavelet collocation method

t	Legendre wavelet $M=3, k=1$	Chebyshev wavelet $M=3, k=1$	Chebyshev wavelet collocation $M=3, k=1$	Bernstein [17] $M=5$	Chebyshev wavelet collocation $M=5, k=0$
0.0	0.0963 e-3	0.0601e-3	0	0.41506e-6	0
0.1	0.0287 e-3	0.0615e-3	0.773992e-7	0.16138e-6	0.431792e-8
0.2	0.0360 e-3	0.0599e-3	0.306218e-8	0.75736e-7	0.334664e-9
0.3	0.0183e-3	0.0090e-3	0.168318e-6	0.12007e-6	0.908061e-8
0.4	0.0412e-3	0.0524e-3	0.110450e-6	0.38093e-7	0.932330e-8
0.5	0.2695e-3	0.1695e-3	0.182181e-8	0.13032e-6	0.672725e-9
0.6	0.0922e-3	0.1602e-3	0.307575e-6	0.28912e-7	0.613608e-8
0.7	0.0826e-3	0.1140e-3	0.359019e-6	0.12445e-6	0.196136e-8
0.8	0.0688e-3	0.0784e-3	0.233400e-7	0.68492e-7	0.829334e-8
0.9	0.1026e-3	0.1577e-3	0.158813e-6	0.16395e-6	0.782187e-8
1.0	0.2689e-3	0.1636e-3	0.462420e-6	0.41524e-6	0.127796e-8

Example 2. Consider the Lane–Emden equation given in [7]

$$\begin{gathered} y''(t) + \frac{2}{t} y'(t) - 2(2t^2 + 3) y(t) = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{gathered} \tag{4.7}$$

If we assume the unknown function $y''(t)$ is given by

$$y''(t) = C^T \Psi(t) \tag{4.8}$$

and then use the boundary conditions in (4.7), we find that

$$y'(t) = C^T P(t), \qquad y(t) = C^T Q(t) + 1 \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.7) we obtain

$$C^T\left(\Psi(t)+\frac{2}{t}P(t)-2(2t^2+3)Q(t)\right)=2(2t^2+3) \tag{4.10}$$

Substituting the Chebyshev collocation points into (4.10), we obtain the algebraic equation system given in (4.6). By solving this algebraic equation, we are able to find the coefficients of the Chebyshev wavelet series. Hence, we have the implicit form of the approximate solution of (4.7). In Table 2, absolute errors of results of the Chebyshev wavelet collocation method are shown for $M = 7, 9, 11$ and $k = 0$. From Table 2, it can be easily concluded that as M increases, the computed values and actual solutions overlap.

TABLE 2. Absolute error of exact and approximated solution of Example 2 with the Chebyshev wavelet collocation method

t	Chebyshev wavelet collocation M=7	Chebyshev wavelet collocation M=9	Chebyshev wavelet collocation M=11
0.0	0	0	0
0.1	0.165428e-6	0.154780e-7	0.181277e-9
0.2	0.179282e-5	0.465503e-8	0.178702e-9
0.3	0.356063e-6	0.109366e-7	0.314875e-9
0.4	0.167239e-5	0.311962e-7	0.239934e-9
0.5	0.103464e-5	0.427988e-8	0.101895e-9
0.6	0.261666e-5	0.216496e-7	0.225457e-9
0.7	0.398640e-6	0.323641e-7	0.290981e-9
0.8	0.146700e-5	0.177243e-8	0.368848e-9
0.9	0.221316e-5	0.163085e-8	0.202763e-9
1.0	0.212668e-6	0.265502e-8	0.273084e-10

Example 3. Consider the following Bessel differential equation of order zero

$$\begin{aligned} &ty''(t) + y'(t) + ty(t) = 0 \\ &y'(0) = 0, \quad y(1) = 1 \end{aligned} \tag{4.11}$$

The solution to the given equation is known as the Bessel function of the first kind of order zero. The exact solution of this example is

$$y(t) = \frac{J_0(t)}{J_0(1)}.$$

If we assume the unknown function $y''(t)$ is given by

$$y''(t) = C^T \Psi(t) \tag{4.12}$$

and use the boundary conditions in (4.7), we find that

$$y'(t) = C^T P(t), \tag{4.13}$$

$$y(t) = C^T Q(t) + y(0) \tag{4.14}$$

By using the condition $y(1) = 1$,

$$y(0) = 1 - C^T Q(1) \ (4.15)$$

is obtained. Hence, equation (4.14) can be written as

$$y(t) = C^T (Q(t) - Q(1)) + 1 \tag{4.16}$$

Substituting (4.12), (4.13) and (4.16) into (4.11) we obtain

$$C^T \big(t\Psi(t) + P(t) + t(Q(t) - Q(1)\big) = -t \tag{4.17}$$

Substituting the Chebyshev collocation points into (4.17), the algebraic equation system given in (4.6) is obtained. From the solution of this algebraic equation, the coefficients of the Chebyshev wavelet series can be found. Hence, we have the implicit form of the approximate solution of (4.11). In Table 3, the absolute errors of results of the Chebyshev wavelet collocation method are shown for $k = 0,1,2,3$ and $M = 3$. From Table 3, it can be easily concluded that as k increases, the computed values and actual solutions overlap.

TABLE 3. Absolute error of exact and approximated solution of Example 3 with the Chebyshev wavelet collocation method

t	Chebyshev. wavelet collocation $M = 3, k = 0$	Chebyshev. wavelet collocation $M = 3, k = 1$	Chebyshev. wavelet collocation $M = 3, k = 2$	Chebyshev. wavelet collocation $M = 3, k = 3$
0.0	0.772803e-6	0.789748e-6	0.652787e-7	0.439940e-8
0.1	0.212098e-5	0.686626e-6	0.650477e-7	0.442417e-8
0.2	0.550307e-5	0.777869e-6	0.669096e-7	0.433848e-8
0.3	0.480595e-5	0.992046e-6	0.570156e-7	0.389152e-8
0.4	0.570957e-6	0.902814e-6	0.617307e-7	0.348496e-8
0.5	0.827113e-5	0.738774e-6	0.502688e-7	0.320908e-8
0.6	0.143824e-4	0.318299e-6	0.348398e-7	0.287675e-8
0.7	0.152176e-4	0.226742e-6	0.395705e-7	0.233237e-8
0.8	0.945930e-5	0.637852e-6	0.125396e-7	0.113874e-8
0.9	0.648928e-6	0.430195e-6	0.216590e-7	0.405126e-9
1.0	0	0	0	0

5. Conclusion

In this paper, the Chebyshev wavelet collocation method is proposed for the solution of second-order linear ordinary differential equations. Approximate solutions of the differential equations, obtained by computer simulation, are compared with the exact solutions. These calculations demonstrate that the accuracy of the Chebyshev wavelet collocation method is quite high even in the case of a small number of grid points. In this method, there is no complex integral or methodology. Application of this proposed method is very simple and gives the implicit form of the approximate solutions to the problems. These are the main advantages of the method. This method is also very convenient for solving the boundary value problems, since the boundary conditions in the solution are handled automatically. Hence, this proposed method is a very reliable, simple, fast, computationally efficient, flexible, and convenient alternative method.

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