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The Proposed Modified Schnute Model

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Abstract: Statistical modeling with growth data is the most efficient way to get more objective results and also understand growth. Schnute growth model is a large-scale model that includes many nonlinear growth models, it offers optimum parameter estimates compared to other growth models, especially regardless of whether the growth curve has an asymptotic feature. In recent years, in the literature studies, there are approaches in the theoretical and applicational fields in which the equations are modified to make sense of growth equations. The purpose of modifying the equations is to convert growth parameters into meaningful parameters such as maximum value, A; rapid specific rate of growth μ_m and lag time λ. In this study, it is demonstrated that by which mathematical operations the parameters of the Schnute growth model are converted to significant parameters and a new modified Schnute growth model is presented to the literature.

Schnute Büyüme Modelinde Yeni Bir Modifikasyon

Anahtar Kelimeler
Büyüme Modelleri,
Schnute Modeli,
Modifiye Büyüme Modelleri

Özet: Büyüme verileri ile istatistiksel modelleme, daha objektif sonuçlar elde etmenin ve dahası büyümeyi anlamanın en etkili yoludur. Schnute büyüme modeli, birçok doğrusal olmayan büyüme modelini içeren büyük ölçekli bir model olup, özellikle büyüme eğrisinin asimptotik özellik taşıyıp taşımadığına bakılmaksızın, diğer büyüme modellerine göre optimum parametre tahminleri sunmaktadır. Son yıllarda, literatür çalışmalarında, teorik ve uygulama alanlarında, büyüme denklemlerini anlamlandırmak için denklemlerin modifiye edildiği yaklaşımlar bulunmaktadır. Denklem modifikasyonunun amacı, büyüme parametrelerini, örneğin, maksimum değer, A; büyüme oranı μ_m ve gecikme süresi λ gibi anlamlı parametrelere dönüştürmektir. Bu çalışmada, Schnute büyüme modeline ait parametrelerin hangi matematiksel işlemlerle anlamlı parametrelere dönüştürüldüğü gösterilmiş ve modifiye edilmiş yeni bir Schnute büyüme modeli literatüre sunulmuştur.

1. Introduction

Growth doesn't just have to be about living things. Sometimes an increase or decrease (or negative increase) in the number of individuals in a population or an increase (or negative increase) in prices can also be described as growth. All examples such as the growth and reproduction of a bacterium in the laboratory, the increase of a certain population, the differentiation of the size of any mass in the body, the change in the sales amount of a product, an increase and a decrease in inflation, represent a growth. All these can be modeled mathematically with growth functions. Such mathematical modeling is used to determine the rate of growth, the level it will reach at a given moment, or its lifetime. Lifetime determination can be used not only in organism or in the population, but also even in estimating the failure time of an electronic device. That is, growth functions are used in many different areas from organisms to any increase in cancer cells and from inflation to the lifetime of a household air conditioner [1], [2], [3]. Growth functions examine the time-dependent changes of all these conditions using certain parameters, interpret

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them by expressing them with growth curves and make predictions. Thanks to these parameters, it is possible to understand the growth process and to determine the factors that affect growth during this process [4], [5].

The ability to comment on these areas of use and the quantitative status of the organism or perhaps the population studied (at certain times) has led to numerous studies on growth models. [6], [7]. While defining growth for any organism, some quantitative changes in the organism per unit time are considered [8].

For example, data such as length and weight used to determine the age of fish and growth models are obtained. Thanks to these models, decisions can be made about what kind of feed the fish should be fed or when to fish [8]. In the same way, thanks to modeling growth in trees, it is possible to compare the growth characteristics of trees with different factors [9]. Thanks to the growth curves created with growth models, biological development can be understood, and this information can be used in the treatment of diseases [10], [11]. Wide usage areas of growth models include computer science. For example, growth models are used in computer science to model the change in the number of software errors [12].

Also, theses that comprehensively address the growth models used in economics [13] or studies in which asymptotic properties are examined by focusing on the problem of determining models in econometric models are available in the literature [14]. For example, Ridley and Llaugel [15] examined the limits of growth rate and growth by considering the CDR economic growth model. In his book, Kaldor [16] emphasized that the purpose of economic growth theory is to show the nature of growth and mentioned the importance of determining the trend of growth. For this reason, supporting the extensive literature on growth models in economics mainly theoretically and proposing new models to practitioners will allow for both stronger and more detailed applications.

Since the growth models may vary there are many growth models in the literature. Some of these models are linear and some are nonlinear growth models. Linear, Quadratic, Cubic, Negative Exponential, Brody, Gompertz, Logistics, Bertalanffy, Richard, Schnute are the most known growth models. Although the growth rate of a living thing can be constant in certain periods of its lifetime, this feature does not continue throughout its life [17], [1], [5]. That is, the growth model should be able to show growth at different times and situations [18]. Therefore, linear models may be insufficient in modeling the growth of organisms over the lifetime and nonlinear models are preferred instead of these models [17], [19]. In case of growing at decreasing rates, models such as Negative Exponentials and Brody with late-term asymptotes are used. Models such as Logistics, Gompertz, Bertalanffy and Richard are preferred when there is varying growth rates [20].

Working with nonlinear functions is more difficult in terms of modelling or prediction, and the results are approximated using iteration instead of analytical methods. Complex estimation methods such as Draper and Smith [21] or Marquardt [22] methods may be required [23], [24]. The results obtained with such non-analytical methods can be obtained approximately.

Therefore, by trying to determine the structure of the growth curve with partial derivatives, more effective and more precise parameter estimates may be done. For example, Bilgin and Esenbuğa [20] used partial derivatives of nonlinear growth models such as "Negative Exponential, Brody, Gompertz, Logistics, Bertalanffy and Richard" in the estimation of model parameters.

The origin of all these growth functions is based on Actuarial science. Survival functions are defined by examining the distribution of the random variable $T(\mathrm{X})$ which indicates the lifetime of an individual until the age of X or death at the age of X. The starting point of survival functions is based on basic probability information. The probability of an individual to live more than age x can be described as survival function and can be given as

$$\mathrm{s}(\mathrm{x}) = 1 - \mathrm{F_X}(\mathrm{x}), = \mathrm{P}(\mathrm{X} > \mathrm{x}),\ \ \mathrm{x} \geq 0,$$

where $\mathrm{F_X}(\mathrm{x}) = \mathrm{P}(\mathrm{X} \leq \mathrm{x}),\ \mathrm{x} \geq 0$ is the distribution function of an individual to live at age $\mathrm{X} = \mathrm{x}$ [25]. With a similar perspective, $\mathrm{P}(\mathrm{T}(\mathrm{x}) > \mathrm{t})$ is the probability of a person of age x to be alive at age $(\mathrm{x} + \mathrm{t})$.

In general terms, the death of the newborn between the ages of x and z can be expressed with the conditional probability as follows,
$\mathrm{P}(\mathrm{x} < \mathrm{X} \leq \mathrm{z}/\mathrm{X} > \mathrm{x}) = \frac{\mathrm{F_X}(\mathrm{z}) - \mathrm{F_X}(\mathrm{x})}{1 - \mathrm{F_X}(\mathrm{x})}$.

If $\mathrm{x} + \varepsilon$, $\varepsilon > 0$ is taken instead of z then the statement will be $\mathrm{P}(\mathrm{x} < \mathrm{X} \leq \mathrm{x} + \varepsilon/\mathrm{X} > \mathrm{x}) = \frac{\mathrm{F_X}(\mathrm{x}+\varepsilon) - \mathrm{F_X}(\mathrm{x})}{1 - \mathrm{F_X}(\mathrm{x})}$.

Since $\mathrm{F'_X}(\mathrm{x}) = \mathrm{f_X}(\mathrm{x})$ exists for continuous random variable X, $\mathrm{P}(\mathrm{x} < \mathrm{X} \leq \mathrm{x} + \varepsilon/\mathrm{X} > \mathrm{x}) \cong \frac{\mathrm{f_X}(\mathrm{x})\varepsilon}{1 - \mathrm{F_X}(\mathrm{x})}$ can be obtained which defines the survival probability of a person at the age x for a short period of ε. In this equation, if the coefficient of ε is expressed as $\frac{\mathrm{f_X}(\mathrm{x})}{1 - \mathrm{F_X}(\mathrm{x})} = \mu(\mathrm{x}) = \frac{-\mathrm{s}'(\mathrm{x})}{\mathrm{s}(\mathrm{x})}$ and called as force of mortality [25]. The force of mortality can be seen in different forms for different organisms and by using this function, different growth functions can be defined [11], [26], [27], [28], [29], [30].

In this study, in Section 2, Schnute growth model which is a nonlinear growth model is discussed. In section 3, after reviewing the modified Schnute equation defined by Schnute [30], a new Modified Schnute Model is given.

2. Material and Method

Schnute Model [30] can represent the models such as "Chapman-Richards, Gompertz, Richards, Von Bertalanfy" which are frequently used in the literature, with special selection of parameters. This model is obtained with the calculation of $\frac{dy}{dt}$ growth rate according to the size of a living at the age of t being Y(t). The relative value of this growth rate with respect to size can be given as

$$Z = \frac{1}{Y}\frac{dy}{dt} = \frac{d}{dt}\log(Y) \tag{1}$$

and is called as relative growth rate. Since the derivative measures the change of Y(t) with respect to t, so Z defines the rate of growth clearly.

Let us assume that $\frac{1}{Z}\frac{dz}{dt}$ is linear with respect to Z, then it can be expressed as

$$\frac{1}{Z}\frac{dz}{dt} = -(a + bZ) \tag{2}$$

Here a and b are constants and can be positive, negative or null. The relative growth rate indicates the logarithmic growth rate, since the growth curve has an increasing and then decreasing structure. Examining the differential equations giving the relative value of growth rate with time Schnute growth model is obtained in the following form [30],

$$Y(t) = \left[{Y_1}^b + \left({Y_2}^b - {Y_1}^b\right)\frac{1 - e^{-a(t-\tau_1)}}{1 - e^{-a(\tau_2-\tau_1)}}\right]^{\frac{1}{b}}$$

where $b \neq 0$, $a \neq 0$. For different a and b cases Schnute Growth Model is obtained as,

$$Y(t) = Y_1\exp\left\{\log\left(\frac{Y_2}{Y_1}\right)\frac{1\text{-}e^{-a(t-\tau_1)}}{1\text{-}e^{-a(\tau_2-\tau_1)}}\right\} \text{ when } b = 0,\ a \neq 0$$

$$Y(t) = \left[{Y_1}^b + \left({Y_2}^b\text{-}{Y_1}^b\right)\frac{(t\text{-}\tau_1)}{(\tau_2\text{-}\tau_1)}\right]^{\frac{1}{b}} \text{ when } b \neq 0,\ a = 0$$

$$Y(t) = \left[Y_1\exp\left\{\left(\log\frac{Y_2}{Y_1}\right)\frac{t\text{-}\tau_1}{\tau_2\text{-}\tau_1}\right\}\right] \text{ when } b = 0,\ a = 0.$$

These equations depend on growth parameters a and b and the size of the organisms $Y_1 = Y(\tau_1)$, $Y_2 = Y(\tau_2)$ at the age of τ_1, τ_2 respectively. Where τ_1 and τ_2 being the first and second ages which the researcher determined. The special cases of the Schnute growth model given below express the growth equations that are frequently used in the literature [30].

For example;
Gompertz Growth Equation is obtained for $a > 0$ and $b = 0$,

Richards Growth Equation is obtained for $a > 0$ and $b < 0$,

Von Bertalanffy Equation is obtained for $a > 0$ and $b > 0$.

As can be seen from the equations of the growth models, there are regression parameters in the models. Instead of these regression parameters, it may be necessary to use parameters that are more meaningful to obtain the parameter estimates of the model. For this reason, by examining the relation between regression parameters and meaningful parameters theoretically, the modified equations can be obtained. In this study, a New Modified Schnute Equation is proposed (PMS-The Proposed Modified Schnute Model).

3. Results: A New Modified Growth Equation for the Schnute Growth Model

It was mentioned that Schnute Model depends on growth parameters such as, a, b, $Y_1 = Y(\tau_1)$ and $Y_2 = Y(\tau_2)$. These growth parameters should be transformed to meaningful parameters such as μ_m, A and λ, to derive the modified form of Schnute Model. Because the applicability of a growth equation gains convenience by writing it in terms of meaningful parameters. Here, μ_m is maximum specific growth rate; A is maximum growth value, and λ is lag time.

For the derivation of modified form theoretically, the second derivative of the Schnute equation should be examined in order to determine the inflection point.

That is, $\frac{d^2y}{dt^2} = 0$ should be analyzed. Using the following growth rate,

$$Y(t) = \left[Y_1 + \left({Y_2}^b - {Y_1}^b\right)\frac{1-e^{-a(t-\tau_1)}}{1-e^{-a(\tau_2-\tau_1)}}\right]^{\frac{1}{b}} \tag{3}$$

with $b \neq 0, a \neq 0$ and taking the first derivative of this growth rate, the following equation is reached,

$$Y'(t) = \frac{1}{b}\left\{{Y_1}^b + \frac{\left({Y_2}^b - {Y_1}^b\right)1 - e^{-a(t-\tau_1)}}{1 - e^{-a(\tau_2-\tau_1)}}\right\}^{\frac{1}{b}-1} . \left\{a\frac{({Y_2}^b - {Y_1}^b)e^{-a(t-\tau_1)}}{1 - e^{-a(\tau_2-\tau_1)}}\right\}$$

By simplifying above equation, the following form can be reached,

$$Y'(t) = \frac{1}{b}[Y(t)]^{1-b}\frac{a\left({Y_2}^b - {Y_1}^b\right)}{1 - e^{-a(\tau_2-\tau_1)}}\ e^{-a(t-\tau_1)}.$$

Taking the second derivative of a growth function and finding the zero values of this form lead us to reach inflection points of the curve. So, let's find the second derivative as follows,

$$Y''(t) = \frac{1}{b}\left(\frac{1}{b} - 1\right)$$
$$.\left\{y_1{}^b + \frac{\left(Y_2{}^b - Y_1{}^b\right)1 - e^{-a(t-\tau_1)}}{1 - e^{-a(\tau_2-\tau_1)}}\right\}^{\frac{1}{b}-2}$$
$$.\left\{a\frac{\left(Y_2{}^b - Y_1{}^b\right)e^{-a(t-\tau_1)}}{1 - e^{-a(\tau_2-\tau_1)}}\right\}^2$$
$$+\frac{1}{b}\left\{Y_1{}^b + \frac{\left(Y_2{}^b - Y_1{}^b\right)1 - e^{-a(t-\tau_1)}}{1 - e^{-a(\tau_2-\tau_1)}}\right\}^{\frac{1}{b}-1}$$

$.\left\{(-a^2)\frac{(Y_2{}^b - Y_1{}^b)e^{-a(t-\tau_1)}}{1 - e^{-a(\tau_2-\tau_1)}}\right\}$ can be found. Similar as Schnute [30], to analyze $\frac{d^2y}{dt^2} = Y\,Z\,\{(1-b)\,Z - a\}$ for every τ^* when $b \neq 0, a \neq 0$, the value $Z = \frac{a}{1-b}$ can be found (since Y and Z can't be null) by using Equation 1 and 2.

Now, if k which is a function of Y and Z is defined as $k = Y^b(a + bZ)$, then $\frac{dk}{dt} = bY^b\left(\frac{1}{Y}\frac{dy}{dt}(a + bZ) + \frac{dz}{dt}\right)$ is obtained.

As similar approach to Schnute [30], when $\frac{dz}{dt} = -Z(a + bZ)$ and $\frac{dy}{dt} = YZ$ statements are used in equation; it is observed that the first derivative is zero. That is, k is a constant function.

Therefore, $Y = Y_1$ and $Z = Z_1$ can be written. That is; $k = \frac{a\left(Y_2{}^b e^{a\tau_2} - Y_1{}^b e^{a\tau_1}\right)}{e^{a\tau_2} - e^{a\tau_1}}$ is obtained by using $Z_1 = \frac{a\left(Y_2{}^b - Y_1{}^b\right)}{bY_1{}^b\left[1 - e^{-a(\tau_2-\tau_1)}\right]}$ in the equation of $k = Y^b(a + bZ)$ (Appendix A1-A9).

$Y = \left(\frac{k}{(a+bZ)}\right)^{1/b}$ can be found from the equation $k = Y^b(a + bZ)$. It is observed here that Y is a function of Z, in this case, there is a Y^* for every Z^*. So, $Y^*(t) = \left(\frac{k}{a+bZ^*}\right)^{\frac{1}{b}}$ can be written. Using $Z^* = \frac{a}{1-b}$ and $k = \frac{a\left(Y_2{}^b e^{a\tau_2} - Y_1{}^b e^{a\tau_1}\right)}{e^{a\tau_2} - e^{a\tau_1}}$,

$$Y^*(t) = Y(\tau^*)\left[(1-b)\frac{\left(Y_2{}^b e^{a\tau_2} - Y_1{}^b e^{a\tau_1}\right)}{e^{a\tau_2} - e^{a\tau_1}}\right]^{\frac{1}{b}}$$

$$Y^*(t) = [(1-b)M]^{\frac{1}{b}} \tag{4}$$

is obtained, where $M = \frac{\left(Y_2{}^b e^{a\tau_2} - Y_1{}^b e^{a\tau_1}\right)}{e^{a\tau_2} - e^{a\tau_1}}$.

If $Y^*(t) = Y(\tau^*)$ is expressed in Equation 3, then

$$Y^*(t) = Y(\tau^*) = \left\{(1-b)\frac{\left(Y_2{}^b e^{a\tau_2} - Y_1{}^b e^{a\tau_1}\right)}{e^{a\tau_2} - e^{a\tau_1}}\right\}$$
$$= \left\{Y_1{}^b + \left(Y_2{}^b - Y_1{}^b\right)\frac{1 - e^{-a(t-\tau_1)}}{e^{a\tau_2} - e^{a\tau_1}}e^{a\tau_2}\right\} \tag{5}$$

is obtained. Thus, from the Equation 4 and 5;

$$e^{-a(t-\tau_1)} = be^{-a\tau_2}\frac{\left(Y_2{}^b e^{a\tau_2} - Y_1{}^b e^{a\tau_1}\right)}{\left(Y_2{}^b - Y_1{}^b\right)} \tag{6}$$

is obtained. The inflection point, $t = \tau^*$ is obtained by taking t from the Equation 6:

$$\tau^* = \tau_1 + \tau_2 - \frac{1}{a}\log\left[b\frac{\left(Y_2{}^b e^{a\tau_2} - Y_1{}^b e^{a\tau_1}\right)}{\left(Y_2{}^b - Y_1{}^b\right)}\right]$$

Using the Equation 4 and 6; maximum specific growth rate (the slope of the inflection point) can be found as follows:

$$Y'(\tau^*) = \mu_m = \frac{a}{1-b}(1-b)^{\frac{1}{b}}\left(\frac{Y_2{}^b e^{a\tau_2} - Y_1{}^b e^{a\tau_1}}{e^{a\tau_2} - e^{a\tau_1}}\right)^{\frac{1}{b}}$$

$$\mu_m = a(1-b)^{\frac{1}{b}-1}M^{\frac{1}{b}}$$

Finally, the tangent equation with slope μ_m at $(\lambda, 0)$ is used to find the lag time parameter λ.

If $Y = 0, Y_1 = Y(\tau^*), m = \mu_m = Y'(\tau^*), x = \lambda, X_1 = \tau^*$ are taken in tangent equation $Y - Y_1 = m(X - X_1)$, then

$$-Y(\tau^*) = Y'(\tau^*)(\lambda - \tau^*) \tag{8}$$

is obtained. By using the Equation 4 and 7;

$$-[(1-b)M]^{\frac{1}{b}} = a(1-b)^{\frac{1}{b}-1}M^{\frac{1}{b}}(\lambda - \tau^*) \tag{9}$$

and

$$\lambda = \tau^* - \frac{(1-b)}{a} \tag{10}$$

can be reached.

Taking $\tau^* = \tau_1 + \tau_2 - \frac{1}{a}\log\left[b\frac{\left(Y_2{}^b e^{a\tau_2} - Y_1{}^b e^{a\tau_1}\right)}{\left(Y_2{}^b - Y_1{}^b\right)}\right]$ in Equation 10,

$$a\tau_1 + a\tau_2 - \log\left[b\frac{\left(Y_2{}^b e^{a\tau_2} - Y_1{}^b e^{a\tau_1}\right)}{\left(Y_2{}^b - Y_1{}^b\right)}\right] - a\lambda = (1-b)$$

$$e^{a\tau_1}e^{a\tau_2}e^{a\lambda} = b\frac{\left(Y_2{}^b e^{a\tau_2} - Y_1{}^b e^{a\tau_1}\right)}{\left(Y_2{}^b - Y_1{}^b\right)} \tag{11}$$

and

$$\left(Y_2{}^b - Y_1{}^b\right) = bM(e^{a\tau_2} - e^{a\tau_1})e^{-a\tau_1 - a\tau_2 + a\lambda + (1-b)} \tag{12}$$

are obtained.

Discussing the Schnute formula given in Equation 3 theoretically,

$$Y(t) = \left[Y_1 + \left({Y_2}^b - {Y_1}^b\right)\frac{1 - e^{-a(t-\tau_1)}}{1 - e^{-a(\tau_2-\tau_1)}}\right]^{\frac{1}{b}}$$

$$= \left[{Y_1}^b + \left({Y_2}^b - {Y_1}^b\right)\frac{(e^{at}-e^{a\tau_1})e^{a\tau_2}}{e^{at}(e^{a\tau_2}-e^{a\tau_1})}\right]^{\frac{1}{b}}$$

$$Y(t) = \left[\frac{e^{a\tau_2}e^{a\tau_1}\left({Y_1}^b - {Y_2}^b\right)}{e^{at}(e^{a\tau_2} - e^{a\tau_1})} + M\right]^{\frac{1}{b}}$$

$$Y(t) = \left[\frac{\left({Y_2}^b - {Y_1}^b\right)}{e^{at}(e^{-a\tau_2}-e^{-a\tau_1})} + \left(\frac{Y'(\tau^*)}{a(1-b)^{\frac{1}{b}-1}}\right)^{\frac{1}{b}}\right]^{\frac{1}{b}} \qquad (13)$$

can be found. Using $M = \left[\frac{Y'(\tau^*)}{a(1-b)^{\frac{1}{b}-1}}\right]^{\frac{1}{b}}$ in Equation 13, the following result is found,

$$Y(t) = \left[\frac{\left({Y_2}^b - {Y_1}^b\right)}{e^{at}(e^{-a\tau_2}-e^{-a\tau_1})} + \left(\frac{Y'(\tau^*)}{a(1-b)^{\frac{1}{b}-1}}\right)^{\frac{1}{b}}\right]^{\frac{1}{b}}.$$

Similarly, using $Y'(\tau^*) = \mu_m$ and M in Equation 12,

$$Y(t) = \left[\frac{bM(e^{a\tau_2} - e^{a\tau_1})e^{-a\tau_1 - a\tau_2 + a\lambda + (1-b)}}{e^{at}(e^{-a\tau_2} - e^{-a\tau_1})} + \left(\frac{Y'(\tau^*)}{a(1-b)^{\frac{1}{b}-1}}\right)^{\frac{1}{b}}\right]^{\frac{1}{b}}$$

$$= \left[\left(\frac{Y'(\tau^*)}{a(1-b)^{\frac{1}{b}-1}}\right)^{\frac{1}{b}} - \frac{bMe^{a\lambda+(1-b)}}{e^{at}}\right]^{\frac{1}{b}}$$

$$= \left[\left(\frac{Y'(\tau^*)}{a(1-b)^{\frac{1}{b}-1}}\right)^{\frac{1}{b}} - b\left(\frac{Y'(\tau^*)}{a(1-b)^{\frac{1}{b}-1}}\right)^{\frac{1}{b}} e^{a\lambda+(1-b)-at}\right]^{\frac{1}{b}}$$

$$Y(t)_{proposed} = \left[\frac{{\mu_m}^{\frac{1}{b}}(1\text{-}be^{a\lambda+1\text{-}b\text{-}at})}{a^{\frac{1}{b}}(1\text{-}b)^{(1\text{-}b)}}\right]^{\frac{1}{b}}$$

can be reached which can be called as Proposed Modified Schnute (PMS) Growth Equation for $a \neq 0$ and $b \neq 0$.

4. Conclusion

When dealing with datasets with a sigmoidal growth curve, working with mathematically significant parameters may not work as well as meaningful parameters in modeling the data set. Therefore, it would be more accurate to make statistical predictions with growth models containing significant parameters.

In this study, meaningful parameters are obtained with some theoretical calculations in order to define the modified Schnute growth model. Thus, a new form for the modified Schnute function has been proposed to the literature. The remarkable point here is due to the elimination of the asymptotic approach restriction in the classical Schnute growth model. Since this restriction is removed, A, which is the common parameter of modified growth equations, is not included in the newly proposed modified Schnute (PMS) growth model.

It is possible to perform comparative analysis of meaningful parameters (A, μ_m and λ) with modified equations. This study has been prepared theoretically and does not include analyses made by statistical methods. In future studies, it is aimed to compare the parameter estimates of the PMS growth model and other modified growth models. Thus, it will be possible topredict the determination of the growth model that best describes the data set.

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Appendices

Appendix A

Let's assume that $\frac{dz}{-Z(a+bZ)} = dt$.

$\frac{-1}{Z(a+bZ)} = \frac{A(a+bZ)+BZ}{Z(a+bZ)} = \frac{-\frac{1}{a}}{Z} + \frac{\frac{b}{a}}{(a+bZ)} = dt$ is obtained.

The operations are continued through relative growth rate $Z(t)$.

$$-\frac{dz}{Z} + \frac{bdz}{(a+bZ)} = a\,dt$$

The statement is regularized, its integral is taken, and the Equation 3 is found.

$$-\log Z + \log(a+bZ) = at$$

$$\log\left(\frac{a+bZ}{Z}\right) = at + c \tag{A1}$$

$$c = \left(\log\frac{a+bZ_1}{Z_1}\right) - a\tau_1$$

Using c in (A1) and solving for initial condition of

$Z(\tau_1) = Z_1$,

$$\log\left(\frac{a+bZ}{Z}\right) - \log\left(\frac{a+bZ_1}{Z_1}\right) = at - a\tau_1$$

$$\frac{1}{a}\log\left\{\left(\frac{a+bZ}{Z}\right)\left(\frac{Z_1}{a+bZ_1}\right)\right\} = (t-\tau_1) \tag{A2}$$

may observed. Then,

$$\left(\frac{a+bZ}{Z}\right)\left(\frac{Z_1}{a+bZ_1}\right) = e^{a(t-\tau_1)}$$

$$Z = \frac{aZ_1e^{-a(t-\tau_1)}}{a+bZ_1\left[1-e^{-a(t-\tau_1)}\right]} \tag{A3}$$

When the Equation 2 is used instead of Z in (A3),

$$\frac{d}{dt}(\log Y) = \frac{aZ_1e^{-a(t-\tau_1)}}{a+bZ_1\left[1-e^{-a(t-\tau_1)}\right]} \tag{A4}$$

$\frac{d}{dt}(\log Y) = \frac{baZ_1e^{-a(t-\tau_1)}}{b\{a+bZ_1[1-e^{-a(t-\tau_1)}]\}}$ can be obtained.

Taking the integration of the following equation and using the initial condition of $Y(\tau_1) = Y_1$

$$\frac{d}{dt}(\log Y) = \frac{1}{b}\frac{d}{dt}\left[\log\left\{a + bZ_1\left(1 - e^{-a(t-\tau_1)}\right)\right\}\right]$$

$$\frac{d}{dt}\left[\log Y - \frac{1}{b}\left[\log\left\{a + bZ_1\left(1 - e^{-a(t-\tau_1)}\right)\right\}\right]\right] = 0$$

$$\log Y - \frac{1}{b}\left[\log\left\{a + bZ_1\left(1 - e^{-a(t-\tau_1)}\right)\right\}\right] = c_1 \tag{A5}$$

$$c_1 = \log Y_1 - \frac{1}{b}[\log(a)]$$

can be found. Then the following is obtained:

$$\log Y - \frac{1}{b}\left[\log\left\{a + bZ_1\left[1 - e^{-a(t-\tau_1)}\right]\right\}\right] = \log Y_1 - \frac{1}{b}\log(a)$$

$$\log\frac{Y}{Y_1} = \frac{1}{b}\log\left[\frac{a + bZ_1\left[1 - e^{-a(t-\tau_1)}\right]}{a}\right]$$

$$Y(t) = Y_1\left[\frac{a+bZ_1\left[1-e^{-a(t-\tau_1)}\right]}{a}\right]^{\frac{1}{b}} \tag{A6}$$

Since $Y(\tau_2) = Y_2$ and $\lim_{b\to 0}(1+\beta b)^{\frac{1}{b}} = e^{\beta}$ then

$$Y_2 = Y_1\left[\frac{a + bZ_1\left[1 - e^{-a(\tau_2-\tau_1)}\right]}{a}\right]^{1/b}$$

$$Z_1 = \frac{a\left({Y_2}^b - {Y_1}^b\right)}{b{Y_1}^b\left[1-e^{-a(\tau_2-\tau_1)}\right]} \tag{A7}$$

for $b \neq 0$, $a \neq 0$. Using (A7) in Y_2 and writing this new form in (A6).

Schnute Growth model is expressed as following for $b \neq 0$, $a \neq 0$

$$Y(t) = \left[{Y_1}^b + \left({Y_2}^b - {Y_1}^b\right)\frac{1-e^{-a(t-\tau_1)}}{1-e^{-a(\tau_2-\tau_1)}}\right]^{\frac{1}{b}}. \tag{A9}$$