SAKARYA ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ DERGİSİ

Sakarya University Journal of Science
SAUJS

ISSN 1301-4048 e-ISSN 2147-835X Period Bimonthly Founded 1997 Publisher Sakarya University
http://www.saujs.sakarya.edu.tr/

Title: Encoder Hurwitz Integers: Hurwitz Integers that have the “Division with Small Remainder” Property

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Recieved: 2022-10-24 00:00:00

Accepted: 2023-03-04 00:00:00

Article Type: Research Article

Volume: 27
Issue: 4
Month: August
Year: 2023
Pages: 792-812

How to cite
Ramazan DURAN; (2023), Encoder Hurwitz Integers: Hurwitz Integers that have the “Division with Small Remainder” Property. Sakarya University Journal of Science, 27(4), 792-812, DOI: 10.16984/saufenbilder.1248060
Access link
https://dergipark.org.tr/en/pub/saufenbilder/issue/79486/1248060

Sakarya University Journal of Science 27(4), 792-812, 2023

# Encoder Hurwitz Integers: Hurwitz Integers that have the “Division with Small Remainder” Property

Ramazan DURAN*[1]


## Abstract

Considering error-correcting codes over Hurwitz integers, prime Hurwitz integers are considered. On the other hand, considering transmission over Gaussian channel, Hurwitz integers, whose the norm is either a prime integer or not a prime integer, are considered. In this study, we consider Hurwitz integers, the greatest common divisor of components of which is one, i.e., primitive Hurwitz integers. We show, with the help of a proposition, that some primitive Hurwitz integers accompanied by a related modulo function are not suitable for constructing Hurwitz signal constellations. To solve this problem, we show, with the help of a proposition, the existence of primitive Hurwitz integers that have the "division with small remainder" property used to construct the Hurwitz constellations. We also call the set of these integers named as "Encoder Hurwitz Integers" set. Moreover, we examine some properties of the mentioned set. In addition, we investigate the performances of Hurwitz signal constellations, which are constructed accompanied by a related modulo function using Hurwitz integers, each component of which is in half-integers, for transmission over the additive white Gaussian noise (AWGN) channel by means of the constellation figure of merit (CFM), average energy, and signal-to-noise ratio (SNR).


**Keywords**: Quaternion integers, Hurwitz integers, residual class, signal constellations, code constructions

## 1. NTRODUCTION

A Gaussian integer is a complex number, each component of which is in integers. The set of Gaussian integers that is denoted by $\mathbb{Z}[i]$ is shown by $\mathbb{Z}[i]=\{\alpha=\alpha_1+\alpha_2 i:\alpha_1,\alpha_2\in\mathbb{Z}, i^2=-1\}$. Let $\alpha=\alpha_1+\alpha_2 i$ be a Gaussian integer. The conjugate of a Gaussian integer $\alpha$ is equal to $\bar{\alpha}=\alpha_1-\alpha_2 i$. The norm of a Gaussian integer $\alpha$ is equal to $N(\alpha)=\alpha_1^2+\alpha_2^2$. The inverse of a Gaussian integer $\alpha$ is equal to $\alpha^{-1}=\dfrac{\bar{\alpha}}{N(\alpha)}$, where $N(\alpha)\neq 0$. A Gaussian integer $\alpha$ is called a prime Gaussian integer if its norm is a prime integer. A Gaussian integer $\alpha$ is called a primitive Gaussian integer if the greatest common divisor (gcd) of its components is one, i.e. $\gcd(\alpha_1,\alpha_2)=1$. In [1], codes over Gaussian integers were first presented by Huber. His original idea is to


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regard a finite field as a residue class of the Gaussian integer ring modulo a prime Gaussian integer. Moreover, the Euclidean division is used to get a unique element of the minimal norm in each residue class, which represents each element of a finite field. Therefore, each element of a finite field can be represented by a Gaussian integer with the minimal Galois norm in the residue class. The visualization of the residue classes of Gaussian integers, Eisenstein-Jacobi integers, quaternion integers, Lipschitz integers, or Hurwitz integers, respectively, is called a signal constellation, which is a communication term. In coding theory, each element of the signal constellation refers to a complex-value codeword. Huber is used prime Gaussian integers such that $1 \equiv p \mod 4$, where $p = \alpha\bar{\alpha}$ and $\alpha_1 > \alpha_2 > 0$. In this study, we consider primitive Gaussian integers, the norm of which is either a prime integer or not a prime integer, where $\alpha_1 > \alpha_2 > 0$. Codes over Gaussian integer rings were studied in papers [2-5].

Quaternions are a number system that extends complex numbers. Let $\pi = \pi_1 + \pi_2 i + \pi_3 j + \pi_4 k$ be a quaternion. Here $\pi_1$ is the real part, and $\pi_2 i + \pi_3 j + \pi_4 k$ is the imaginary part. Multiplication of two quaternions has no commutative property, in general. Multiplication of two quaternions has commutative property if their imaginary parts are parallel or conjugate to each other. A quaternion $\pi$ is called a quaternion integer just if $\pi_1$, $\pi_2$, $\pi_3$ and $\pi_4$ are in integers. In [6], Özen and Güzeltepe studied codes over quaternion integers, which have the commutative property. Codes over quaternion integers were studied in papers [6-10]. A quaternion integer $\pi$ is called a Lipschitz integer just if its components are in integers. A Lipschitz integer $\pi$ is called a primitive Lipschitz integer if the greatest common divisor of its components is one. Codes over Lipschitz integers were studied in papers [11-14, 19, 27].

A quaternion integer $\pi$ is called a Hurwitz integer just if its components are either in $\mathbb{Z}$ or in $\mathbb{Z} + \frac{1}{2}$. A Hurwitz integer $\pi$ is called a primitive Hurwitz integer if the greatest common divisor of its components is one. In [15], Güzeltepe studied the classes of linear codes over Hurwitz integers equipped with a new metric that refer as the Hurwitz metric. In [16], Rohweder et al. presented a new algebraic construction technique to construct finite sets of Hurwitz integers by a respective modulo function. Moreover, they investigated the performances of Hurwitz signal constellations constructed by Lipschitz integers for transmission over the additive white Gaussian noise (AWGN) channel. Codes over Hurwitz integers were studied in papers [15-21].

This work is organized as follows: In the next section, we give some basic information used throughout this paper. In Section III, we define a new set named "encoder Hurwitz integers". This set comprises to the Hurwitz integers that have the "division with small remainder" property. In Section IV, we investigate the performances of Hurwitz signal constellations constructed by primitive Hurwitz integers, whose components are in $\mathbb{Z} + \frac{1}{2}$, for transmission over the AWGN channel by means of constellation figure of merit (CFM), average energy, and signal-noise-to ratio (SNR). Finally, we conclude the paper in Section V.

## 2. PRELIMINARIES

We begin with some basic definitions.

**Definition 2.1** Let $\pi = \pi_1 + \pi_2 i + \pi_3 j + \pi_4 k$ be a quaternion. A quaternion integer $\pi$ is called a Hurwitz integer just if either $\pi_1, \pi_2, \pi_3, \pi_4 \in \mathbb{Z}$ or $\pi_1, \pi_2, \pi_3, \pi_4 \in \mathbb{Z} + \frac{1}{2}$. The set of all Hurwitz integers that is denoted by $\mathcal{H}$ is shown by

$$\mathcal{H}=\left\{\begin{matrix}\pi_1+\pi_2 i+\pi_3 j+\pi_4 k:\pi_1,\pi_2,\pi_3,\pi_4\in\mathbb{Z}\\ or\ \pi_1,\pi_2,\pi_3,\pi_4\in\mathbb{Z}+\frac{1}{2}\end{matrix}\right\}$$

$$=\mathcal{H}(\mathbb{Z})\bigcup\mathcal{H}\left(\mathbb{Z}+\frac{1}{2}\right).$$

For instance, $\pm1\pm\frac{1}{2}i\pm\frac{1}{2}j\pm\frac{1}{2}k$ and $\pm\frac{1}{2}\pm\frac{1}{2}j$ are not Hurwitz integers, but $\pm\frac{3}{2}\pm\frac{5}{2}i\pm\frac{1}{2}j\pm\frac{7}{2}k$ is a Hurwitz integer and so on. The ring of Hurwitz integers forms a subring of the ring of all quaternions since it is closed under multiplication and addition. The conjugate of a Hurwitz integer $\pi$ is $\bar{\pi}=\pi_1-\pi_2 i-\pi_3 j-\pi_4 k.$ The norm of a Hurwitz integer $\pi$ is $N(\pi)=\pi\cdot\bar{\pi}=\pi_1^2+\pi_2^2+\pi_3^2+\pi_4^2$. The inverse of a Hurwitz integer $\pi$ is $\pi^{-1}=\frac{\bar{\pi}}{N(\pi)}$, where $N(\pi)\neq0$.

**Definition 2.2** Let $\pi$ be a Hurwitz integer. The Hurwitz integer $\pi$ is called a prime Hurwitz integer if its norm is a prime integer.

For instance, $\pi=2-3i+j+3k$ and $\beta=\frac{3}{2}+\frac{5}{2}i-\frac{3}{2}j+\frac{7}{2}k$ are the prime Hurwitz integers since $N(\pi)=2^2+(-3)^2+1^2+3^2=23$ and $N(\beta)=\left(\frac{3}{2}\right)^2+\left(\frac{5}{2}\right)^2+\left(-\frac{3}{2}\right)^2+\left(\frac{7}{2}\right)^2=23$.

**Definition 2.3** Let $\pi=\pi_1+\pi_2 i+\pi_3 j+\pi_4 k$ be a Hurwitz integer. If $\pi$ is a Hurwitz integer, whose each component is in integers, then it is called a primitive Hurwitz integer just if the greatest common divisor of its components is one, i.e. $\gcd(\pi_1,\pi_2,\pi_3,\pi_4)=1$. If $\pi$ is a Hurwitz integer, whose each component is in half-integers, then it is called a primitive Hurwitz integer just if the greatest common divisor of its numerators is one.

Note that, in this study, unless otherwise stated, we consider primitive Hurwitz integers, the norm of which is either a prime integer or not a prime integer, where $\pi_1\geq\pi_2\geq\pi_3\geq\pi_4>0$.

**Definition 2.4** [15] Let $\alpha$ and $\pi$ be Hurwitz integers. If there exists $\lambda\in\mathcal{H}$ such that $q_1-q_2=\lambda\alpha$, then $q_1,q_2\in\mathcal{H}$ are said to be right congruent modulo $\pi$. This relation is denoted by $q_1\equiv_r q_2$. Here, $\equiv_r$ is represented as the right congruent. This relation $q_1\equiv_r q_2$ is an equivalence relation. The elements in the right ideal $\langle\pi\rangle=\{\lambda\pi:\lambda\in\mathcal{H}\}$ define a normal subgroup of the additive group of the ring $\mathcal{H}$. The set of cosets to $\langle\pi\rangle$ in $\mathcal{H}$ defines the Abelian group denoted by $\mathcal{H}_\pi=\mathcal{H}/\langle\pi\rangle$. Analogous results are valid for left congruent modulo $\pi$.

Note that we consider the left congruent modulo an element $\pi$ in the Hurwitz integers rings. Therefore, we consider the elements in the left ideal $\langle\pi\rangle=\{\pi\lambda:\lambda\in\mathcal{H}\}$.

**Definition 2.5** A notation for the nearest integer rounding is denoted by $[\![\bullet]\!]$. It is rounding a rational number to the integer closest to it. Each component is rounded to the integer closest to it for a quaternion, respectively.

Considering half-integers, the rounding is done by the following. We take an example $\frac{\pi_1}{2}$, where $\pi_1$ is an odd integer. If $\pi_1$ is an odd negative integer, then we round it as

- $$\left[\!\!\left[\frac{\pi_1}{2}\right]\!\!\right]=\frac{\pi_1}{2}+\frac{1}{2}, \quad (1)$$

- $$\left[\!\!\left[-\frac{\pi_1}{2}\right]\!\!\right]=-\left[\!\!\left[\frac{\pi_1}{2}\right]\!\!\right]=-\left(\frac{\pi_1}{2}+\frac{1}{2}\right). \quad (2)$$

If $\pi_1$ is a positive integer, then we round it as

- $\left[\!\left[\frac{\pi_1}{2}\right]\!\right]=\frac{\pi_1}{2}-\frac{1}{2},$ (3)
- $\left[\!\left[-\frac{\pi_1}{2}\right]\!\right]=-\left[\!\left[\frac{\pi_1}{2}\right]\!\right]=-\left(\frac{\pi_1}{2}-\frac{1}{2}\right).$ (4)

For instance, let $\pi=\frac{5}{4}+\frac{1}{2}i-\frac{1}{2}j-\frac{5}{2}k$ be a Hurwitz quaternion. By eq. (1), eq. (2), eq. (3) and eq. (4), we get

$$[\![\pi]\!]=\left[\!\left[\frac{5}{4}+\frac{1}{2}i-\frac{1}{2}j-\frac{5}{2}k\right]\!\right]$$

$$=\left[\!\left[\frac{5}{4}\right]\!\right]+\left[\!\left[\frac{1}{2}\right]\!\right]i-\left[\!\left[\frac{1}{2}\right]\!\right]j-\left[\!\left[\frac{5}{2}\right]\!\right]k$$

$$=1+0\cdot i-0\cdot j-2\cdot k=1-2k\,.$$

In the rest of this study, we consider eq. (3) and eq. (4) since $\pi$ is a primitive Hurwitz integer.

**Definition 2.6** Let $\pi$ be a primitive Hurwitz integer, and let $z\in\mathbb{Z}_{N(\pi)}$. The modulo function $\mu:\mathbb{Z}_{N(\pi)}\rightarrow\mathcal{H}_\pi$ is defined by

$$\mu_\pi(z)=z\bmod\pi=z-\pi\cdot\left[\!\left[\pi^{-1}z\right]\!\right]=z-\pi\left[\!\left[\frac{\bar{\pi}z}{N(\pi)}\right]\!\right]. \quad (5)$$

Here, $\mathbb{Z}_{N(\pi)}$ is the well-known residual class set of ordinary integers ring with $N(\pi)$ elements, $\mathcal{H}_\pi$ is the left residual class set of $z$ with respect to the modulo function in eq. (5), and $\mu_\pi(z)$ is given the remainder of $z$ with respect to the modulo function in eq. (5). The quotient ring of the Hurwitz integers modulo this equivalence relation, which we denote as $\mathcal{H}_\pi=\left\{z\bmod\pi \mid z\in\mathbb{Z}_{N(\pi)}\right\}$. This set contains $N(\pi)$ elements. If $\pi$ is a prime Hurwitz integer, then the modulo function $\mu$ defines a bijective mapping from $\mathbb{Z}_{N(\pi)}$ to $\mathcal{H}_\pi$. Therefore, the modulo function $\mu$ is a ring isomorphism. Because there exists an inverse map [22], and we have $\mu(z_1+z_2)=\mu(z_1)+\mu(z_2)$ and $\mu(z_1z_2)=\mu(z_1)\mu(z_2)$, for any $z_1,z_2\in\mathbb{Z}_{N(\pi)}$. If $\pi$ is a primitive Hurwitz integer, the modulo function $\mu$ is a group isomorphism with respect to addition between $\mathbb{Z}_{N(\pi)}$ and $\mathcal{H}_\pi$.

A signal constellation is a physical diagram describing all the possible symbols a signaling system uses to transmit data. It is an aid in designing better communications systems. [23]. These symbols represent the codewords. In other words, they represent the elements, defined as the complex-value codewords, in the set of the residual class of Hurwitz integers ring. Thus, in the rest of this study, we use the "signal constellation" term instead of "the set of residual class" term. We can take an example, "Hurwitz signal constellation" instead of the "residue classes of modulo an element $\pi$ in the Hurwitz integers rings". You can find more details about signal constellation in [23].

You can find more details about the arithmetic properties of quaternions and Hurwitz integers in [24-25].

## 3. ENCODER HURWITZ INTEGERS

The Euclid division algorithm states that given positive integers $a$ and $b$, there exist unique integers $q$ and $r$ such that $a=bq+r$ and $0\le r<b$. Here, $a$ is the dividend, $b$ is the divisor, $q$ is the quotient, and $r$ is the remainder. Considering Hurwitz integers, the Euclid division algorithm states that given Hurwitz integers $\theta$ and $\pi$, there exist unique Hurwitz integers $\beta$ and $\gamma$ such that $\theta=\pi\beta+\gamma$ and $0\le N(\gamma)\le N(\pi)$. In other words, Hurwitz integers, each component of which is in $\mathbb{Z}+\frac{1}{2}$, satisfy the Euclid division

algorithm but Hurwitz integers, each component of which is in integers, do not. We call that Hurwitz integers that satisfy the Euclid division have the “division with small remainder” property. The key point is that the Euclid division algorithm is not worked with some Hurwitz integers. Note that, in this study, we consider the primitive Hurwitz integers. Therefore, in this study, we investigate which primitive Hurwitz integers satisfy the Euclid division or not.

The following proposition shows that the remainder and dividend are equal to each other for the primitive Hurwitz integers, each component of which is an odd integer.

**Proposition 3.1** Let $\pi$ be a primitive Hurwitz integer, each component of which is an odd integer. Then,

$$N\left(\mu_{\pi}\left(\frac{N(\pi)}{2}\right)\right)=N(\pi) \tag{6}$$

with respect to the modulo function in eq. (5).

**Proof** Let $\pi=\pi_1+\pi_2 i+\pi_3 j+\pi_4 k$ be a primitive Hurwitz integer, each component of which is an odd integer. By eq. (5),

$$\mu_{\pi}\left(\frac{N(\pi)}{2}\right)=\frac{N(\pi)}{2}-\pi\left[\!\left[\frac{\frac{\overline{\pi}N(\pi)}{2}}{N(\pi)}\right]\!\right]$$

$$=\frac{N(\pi)}{2}-\pi\left[\!\left[\frac{\overline{\pi}}{2}\right]\!\right]$$

$$=\frac{N(\pi)}{2}-\left(\pi_1+\pi_2 i+\pi_3 j+\pi_4 k\right)\left[\!\left[\frac{\pi_1-\pi_2 i-\pi_3 j-\pi_4 k}{2}\right]\!\right].$$

By eq. (3) and eq. (4), $\left[\!\left[\frac{\pi_1}{2}\right]\!\right]=\frac{\pi_1}{2}-\frac{1}{2}$, $\left[\!\left[-\frac{\pi_2}{2}\right]\!\right]=-\left(\frac{\pi_2}{2}-\frac{1}{2}\right)$, $\left[\!\left[-\frac{\pi_3}{2}\right]\!\right]=-\left(\frac{\pi_3}{2}-\frac{1}{2}\right)$, and $\left[\!\left[-\frac{\pi_4}{2}\right]\!\right]=-\left(\frac{\pi_4}{2}-\frac{1}{2}\right)$. Then, we get

$$\mu_{\pi}\left(\frac{N(\pi)}{2}\right)=\frac{N(\pi)}{2}-\left(\pi_1+\pi_2 i+\pi_3 j+\pi_4 k\right)\left[\left(\frac{\pi_1}{2}-\frac{1}{2}\right)\right.$$

$$\left.-\left(\frac{\pi_2}{2}-\frac{1}{2}\right)i-\left(\frac{\pi_3}{2}-\frac{1}{2}\right)j-\left(\frac{\pi_4}{2}-\frac{1}{2}\right)k\right]$$

$$=\frac{N(\pi)}{2}-\left[\frac{\pi_1^2-\pi_1}{2}-\left(\frac{\pi_1\pi_2-\pi_1}{2}\right)i-\left(\frac{\pi_1\pi_3-\pi_1}{2}\right)j\right.$$

$$-\left(\frac{\pi_1\pi_4-\pi_1}{2}\right)k+\left(\frac{\pi_2\pi_1-\pi_2}{2}\right)i+\left(\frac{\pi_2^2-\pi_2}{2}\right)$$

$$-\left(\frac{\pi_2\pi_3-\pi_2}{2}\right)k+\left(\frac{\pi_2\pi_4-\pi_2}{2}\right)j+\left(\frac{\pi_3\pi_1-\pi_3}{2}\right)j$$

$$+\left(\frac{\pi_3\pi_2-\pi_3}{2}\right)k+\left(\frac{\pi_3^2-\pi_3}{2}\right)-\left(\frac{\pi_3\pi_4-\pi_3}{2}\right)i$$

$$+\left(\frac{\pi_4\pi_1-\pi_4}{2}\right)k-\left(\frac{\pi_4\pi_2-\pi_4}{2}\right)j$$

$$\left.+\left(\frac{\pi_4\pi_3-\pi_4}{2}\right)i+\left(\frac{\pi_4^2-\pi_4}{2}\right)\right]$$

$$=\frac{N(\pi)}{2}-\left[\frac{\pi_1^2+\pi_2^2+\pi_3^2+\pi_4^2-\pi_1-\pi_2-\pi_3-\pi_4}{2}\right.$$

$$+\left(\frac{-\pi_1\pi_2+\pi_1+\pi_2\pi_1-\pi_2-\pi_3\pi_4+\pi_3+\pi_4\pi_3-\pi_4}{2}\right)i$$

$$+\left(\frac{-\pi_1\pi_3+\pi_1+\pi_2\pi_4-\pi_2+\pi_3\pi_1-\pi_3-\pi_4\pi_2+\pi_4}{2}\right)j$$

$$\left.+\left(\frac{-\pi_1\pi_4+\pi_1-\pi_2\pi_3+\pi_2+\pi_3\pi_2-\pi_3+\pi_4\pi_1-\pi_4}{2}\right)k\right]$$

$$=\frac{N(\pi)}{2}-\frac{N(\pi)}{2}+\frac{\pi_1+\pi_2+\pi_3+\pi_4}{2}$$

$$+\left(\frac{-\pi_1+\pi_2-\pi_3+\pi_4}{2}\right)i+\left(\frac{-\pi_1+\pi_2+\pi_3-\pi_4}{2}\right)j$$

$$+\left(\frac{-\pi_1-\pi_2+\pi_3+\pi_4}{2}\right)k$$

$$=\frac{\pi_1+\pi_2+\pi_3+\pi_4}{2}+\left(\frac{-\pi_1+\pi_2-\pi_3+\pi_4}{2}\right)i$$

$$+\left(\frac{-\pi_1+\pi_2+\pi_3-\pi_4}{2}\right)j+\left(\frac{-\pi_1-\pi_2+\pi_3+\pi_4}{2}\right)k.$$

Therefore, we get

$$N\left(\mu_\pi\left(\frac{N(\pi)}{2}\right)\right)=N\left(\frac{\pi_1+\pi_2+\pi_3+\pi_4}{2}\right.$$

$$+\left(\frac{-\pi_1+\pi_2-\pi_3+\pi_4}{2}\right)i+\left(\frac{-\pi_1+\pi_2+\pi_3-\pi_4}{2}\right)j$$

$$\left.+\left(\frac{-\pi_1-\pi_2+\pi_3+\pi_4}{2}\right)k\right)$$

$$=\frac{4\left(\pi_1^2+\pi_2^2+\pi_3^2+\pi_4^2\right)}{4}=\pi_1^2+\pi_2^2+\pi_3^2+\pi_4^2.$$

Consequently, $N(\mu_\pi(\frac{N(\pi)}{2}))=N(\pi)$. This completes the proof.

The following proposition implies that the primitive Hurwitz integers, each component of which is in integers, do not have the "division with small remainder" property.

**Proposition 3.2** Let $\pi$ be a primitive Hurwitz integer, each component of which is in integers, and let $\beta$ be a Hurwitz integer, where $N(\beta)\neq 0$. Then, $N\left(\mu_\pi\left(\beta\right)\right)\leq N\left(\pi\right)$.

**Proof** Let $\pi$ be a primitive Hurwitz integer, each component of which is in integers, and let $\beta=\beta_1+\beta_2 i+\beta_3 j+\beta_4 k$ be a Hurwitz integer, where $N\left(\beta\right)\neq 0$. By eq. (5), we get

$$\mu_\pi\left(\beta\right)=\beta-\pi\left[\!\left[\frac{\overline{\pi}\beta}{N\left(\beta\right)}\right]\!\right].$$

Then, we get

$$\pi^{-1}\mu_\pi(\beta)=\pi^{-1}\beta-\pi^{-1}\pi\left[\!\left[\frac{\overline{\pi}\beta}{\overline{\pi}\pi}\right]\!\right]=\pi^{-1}\beta-\left[\!\left[\frac{\beta}{\pi}\right]\!\right]$$

$$=\pi^{-1}\left(\beta_1+\beta_2 i+\beta_3 j+\beta_4 k\right)$$

$$-\left[\!\left[\frac{\beta_1+\beta_2 i+\beta_3 j+\beta_4 k}{\pi}\right]\!\right]$$

$$=\pi^{-1}\beta_1+\pi^{-1}\beta_2 i+\pi^{-1}\beta_3 j+\pi^{-1}\beta_4 k$$

$$-\left[\!\left[\pi^{-1}\beta_1+\pi^{-1}\beta_2 i+\pi^{-1}\beta_3 j+\pi^{-1}\beta_4 k\right]\!\right]$$

$$=\pi^{-1}\beta_1+\pi^{-1}\beta_2 i+\pi^{-1}\beta_3 j+\pi^{-1}\beta_4 k$$

$$-\left[\!\left[\pi^{-1}\beta_1\right]\!\right]-\left[\!\left[\pi^{-1}\beta_2\right]\!\right]i-\left[\!\left[\pi^{-1}\beta_3\right]\!\right]j-\left[\!\left[\pi^{-1}\beta_4\right]\!\right]k$$

$$=\pi^{-1}\beta_1-\left[\!\left[\pi^{-1}\beta_1\right]\!\right]+\left(\pi^{-1}\beta_2-\left[\!\left[\pi^{-1}\beta_2\right]\!\right]\right)i$$

$$+\left(\pi^{-1}\beta_3-\left[\!\left[\pi^{-1}\beta_3\right]\!\right]\right)j+\left(\pi^{-1}\beta_4-\left[\!\left[\pi^{-1}\beta_4\right]\!\right]\right)k.$$

Hereby, we get

$$\left|\pi^{-1}\beta_1-\left[\!\left[\pi^{-1}\beta_1\right]\!\right]\right|\leq\frac{1}{2},$$

$$\left|\pi^{-1}\beta_2-\left[\!\left[\pi^{-1}\beta_2\right]\!\right]\right|\leq\frac{1}{2},$$

$$\left|\pi^{-1}\beta_3-\left[\!\left[\pi^{-1}\beta_3\right]\!\right]\right|\leq\frac{1}{2},$$

$$\left|\pi^{-1}\beta_4-\left[\!\left[\pi^{-1}\beta_4\right]\!\right]\right|\leq\frac{1}{2}.$$

Then, we get

$$\left(\pi^{-1}\beta_1-\left[\!\left[\pi^{-1}\beta_1\right]\!\right]\right)^2\leq\frac{1}{4},$$

$$\left(\pi^{-1}\beta_2-\left[\!\left[\pi^{-1}\beta_2\right]\!\right]\right)^2\leq\frac{1}{4},$$

$$\left(\pi^{-1}\beta_3-[\![\pi^{-1}\beta_3]\!]\right)^2\le\frac{1}{4},$$

$$\left(\pi^{-1}\beta_4-[\![\pi^{-1}\beta_4]\!]\right)^2\le\frac{1}{4}.$$

Therefore, we get

$$N\left(\pi^{-1}\mu_\pi(\beta)\right)=\left(\pi^{-1}\beta_1-[\![\pi^{-1}\beta_1]\!]\right)^2$$

$$+\left(\pi^{-1}\beta_2-[\![\pi^{-1}\beta_2]\!]\right)^2+\left(\pi^{-1}\beta_3-[\![\pi^{-1}\beta_3]\!]\right)^2$$

$$+\left(\pi^{-1}\beta_4-[\![\pi^{-1}\beta_4]\!]\right)^2$$

$$\le\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2=1.$$

Consequently, since $N\left(\mu_\pi(\beta)\right)\le N(\pi)$ $N\left(\pi^{-1}\mu_\pi(\beta)\right)=\frac{1}{N(\pi)}N\left(\mu_\pi(\beta)\right)\le 1$. This completes the proof.

***Example 1*** Let $\pi=3+i+j+3k$ be a primitive Hurwitz integer. By eq. (5), the Hurwitz signal constellation is

$$\mathcal{H}_\pi=\left\{\begin{array}{l}\mu_\pi(0)=0,\ \mu_\pi(1)=1,\ \mu_\pi(2)=2,\\ \mu_\pi(3)=3,\mu_\pi(4)=-2-2j,\\ \mu_\pi(5)=-1-2j,\mu_\pi(6)=-2j,\\ \mu_\pi(7)=1-2j,\mu_\pi(8)=2-2j,\\ \mu_\pi(9)=3-2j,\mu_\pi(10)=4-2j,\\ \mu_\pi(11)=-3+2j,\mu_\pi(12)=-2+2j,\\ \mu_\pi(13)=-1+2j,\mu_\pi(14)=2j,\\ \mu_\pi(15)=1+2j,\mu_\pi(16)=2+2j,\\ \mu_\pi(17)=-3,\ \mu_\pi(18)=-2,\mu_\pi(19)=-1\end{array}\right\}. \quad (7)$$

This set must contain twenty elements, but it contains nineteen elements since $\mu_\pi(10)\equiv\pi\equiv 0 \bmod \pi$, where $N\left(\mu_\pi(10)\right)=N(\pi)=20$. Consequently, the Hurwitz integer $\pi=3+i+j+3k$ does not have the "division with small remainder" property (see Proposition 3.2). In other words, the Euclidean division algorithm does not work for the primitive Hurwitz integer $\pi=3+i+j+3k$. In addition, to be a Euclidean metric, the inequality $d(\alpha,\beta)+d(\beta,\theta)\ge d(\alpha,\theta)$ should be verified, where $\alpha,\beta,\theta\in\mathcal{H}_\pi$. Because the conditions i) $d(\alpha,\beta)=0$ if and only if $\alpha=\beta$, and ii) $d(\alpha,\beta)=d(\beta,\alpha)$ are supplied. We consider $\alpha=\mu_\pi(10)=4-2j$, $\beta=\mu_\pi(19)=-1$ and $\theta=\mu_\pi(18)=-2$ in (7). $d(\alpha,\beta)=29$ since $N(\beta-\alpha)=29$, $d(\beta,\theta)=1$ since $N(\theta-\beta)=1$, and $d(\alpha,\theta)=40$ since $N(\theta-\alpha)=40$. Therefore, $29+1=30\ge 40$ since $d(\alpha,\beta)+d(\beta,\theta)\ge d(\alpha,\theta)$. This is a contradiction. The Euclidean metric does not satisfy the Hurwitz signal constellation $\mathcal{H}_\pi$ constructed by the primitive Hurwitz integer $\pi=3+i+j+3k$.

In the following definition, we define a new set named the encoder Hurwitz integer, which consists of the primitive Hurwitz integers that have the "division with small remainder" property.

**Definition 3.1** Let $\pi=\pi_1+\pi_2 i+\pi_3 j+\pi_4 k$ be a primitive Hurwitz integer. If a primitive Hurwitz integer $\pi$ does not satisfy the condition $N\left(\mu_\pi\left(\frac{N(\pi)}{2}\right)\right)=N(\pi)$ (see proposition 3.2) with respect to the related modulo function, then it is called an encoder Hurwitz integer. Note that a primitive Hurwitz integer $\pi$, each component of which is in $\mathbb{Z}+\frac{1}{2}$, is an encoder Hurwitz integer.

The above definition is flexible. So, the set of encoder Hurwitz integers is expandable or collapsible depending on the related modulo technique. According to Definition 3.1, the set of encoder Hurwitz integers in this study consists of the primitive Hurwitz integers, each component of which is not an odd integer (or each component of which is not the same parity). In other words, the set of encoder Hurwitz integers in this study consists of the primitive Hurwitz integers, each component of which is either in $\mathbb{Z}+\frac{1}{2}$ or not an odd integer (or components are not the same parity). In mathematics, parity is the property of an integer of whether it is even or odd [26].

Let us now show that the modulo function $\mu$ is a ring isomorphism when $\pi$ is an encoder Hurwitz integer.

**Theorem 3.1** Let $\pi$ be an encoder Hurwitz integer, and let $z_1, z_2 \in \mathbb{Z}_{N(\pi)}$. The map $\mu : \mathbb{Z}_{N(\pi)} \to \mathcal{H}_\pi$ is a ring homomorphism.

**Proof** Let $\pi$ be an encoder Hurwitz integer and, let $z_1, z_2 \in \mathbb{Z}_{N(\pi)}$. By eq. (5), we get

$$\mu_\pi(z_1) = z_1 \bmod \pi = z_1 - \pi\left[\!\!\left[\frac{\overline{\pi} z_1}{N(\pi)}\right]\!\!\right]$$

and

$$\mu_\pi(z_2) = z_2 \bmod \pi = z_2 - \pi\left[\!\!\left[\frac{\overline{\pi} z_2}{N(\pi)}\right]\!\!\right].$$

We suppose that $\lambda_1 = \left[\!\!\left[\frac{\overline{\pi} z_1}{N(\pi)}\right]\!\!\right]$ and $\lambda_2 = \left[\!\!\left[\frac{\overline{\pi} z_1}{N(\pi)}\right]\!\!\right]$, where $\lambda_1$ and $\lambda_2$ are Hurwitz integers, each component of which is in integers. Therefore, we get

$$z_1 = \pi\lambda_1 + \mu_\pi(z_1) \tag{8}$$

and

$$z_2 = \pi\lambda_2 + \mu_\pi(z_2), \tag{9}$$

respectively. Since $\mu_\pi(z_1+z_2) = =z_1 + z_2 - \pi\left[\!\!\left[\frac{\overline{\pi}(z_1+z_2)}{N(\pi)}\right]\!\!\right]$, eq.(8) and eq. (9), then we get

$$\mu_\pi(z_1+z_2) = \pi\lambda_1 + \mu_\pi(z_1) + \pi\lambda_2 + \mu_\pi(z_2)$$

$$-\pi\left[\!\!\left[\frac{\overline{\pi}\left(\pi\lambda_1 + \mu_\pi(z_1) + \pi\lambda_2 + \mu_\pi(z_2)\right)}{N(\pi)}\right]\!\!\right]$$

$$=\pi\lambda_1 + \mu_\pi(z_1) + \pi\lambda_2 + \mu_\pi(z_2)$$

$$-\pi\left[\!\!\left[\frac{\overline{\pi}\pi\lambda_1 + \overline{\pi}\mu_\pi(z_1) + \overline{\pi}\pi\lambda_2 + \overline{\pi}\mu_\pi(z_2)}{N(\pi)}\right]\!\!\right]$$

$$= \pi\lambda_1 + \mu_\pi(z_1) + \pi\lambda_2 + \mu_\pi(z_2)$$

$$-\pi \quad \lambda_1 + \lambda_2 \quad -\pi\left[\!\!\left[\frac{\overline{\pi}\mu_\pi(z_1) + \overline{\pi}\mu_\pi(z_2)}{N(\pi)}\right]\!\!\right].$$

Since $\lambda_1$ and $\lambda_2$ are the Hurwitz integers, each component of which is in integers, then $\lambda_1 + \lambda_2 \quad = \lambda_1 + \lambda_2$. Hereby, we get

$$\mu_\pi(z_1+z_2) = \pi\lambda_1 + \mu_\pi(z_1) + \pi\lambda_2$$

$$+\mu_\pi(z_2) - \pi\lambda_1 - \pi\lambda_2 - \pi\left[\!\!\left[\frac{\overline{\pi}\left(\mu_\pi(z_1) + \mu_\pi(z_2)\right)}{N(\pi)}\right]\!\!\right]$$

$$= \mu_\pi(z_1) + \mu_\pi(z_2) - \pi\left[\!\!\left[\frac{\overline{\pi}\left(\mu_\pi(z_1) + \mu_\pi(z_2)\right)}{N(\pi)}\right]\!\!\right].$$

By eq. (5), we get $\mu_\pi(z_1+z_2) = \left(\mu_\pi(z_1) + \mu_\pi(z_2)\right) \bmod \pi$.

On the other hand, according to the modulo function in eq. (5), we get

$$\mu_\pi(z_1 z_2) = z_1 z_2 \bmod \pi = z_1 z_2 - \pi\left[\!\!\left[\frac{\overline{\pi}(z_1 z_2)}{N(\pi)}\right]\!\!\right].$$

By eq. (8) and eq. (9), we get

$$\mu_\pi(z_1z_2) = \left(\pi\lambda_1 + \mu_\pi(z_1)\right)\left(\pi\lambda_2 + \mu_\pi(z_2)\right)$$

$$-\pi \left\llbracket \frac{\bar{\pi}\left(\pi\lambda_1 + \mu_\pi(z_1)\right)\left(\pi\lambda_2 + \mu_\pi(z_2)\right)}{N(\pi)} \right\rrbracket$$

$$= \pi\lambda_1\pi\lambda_2 + \pi\lambda_1\mu_\pi(z_2) + \mu_\pi(z_1)\pi\lambda_2$$

$$+\mu_\pi(z_1)\mu_\pi(z_2) - \pi \left\llbracket \frac{\bar{\pi}\pi\lambda_1\pi\lambda_2 + \bar{\pi}\pi\lambda_1\mu_\pi(z_2)}{N(\pi)} \right.$$

$$\left. + \frac{\bar{\pi}\mu_\pi(z_1)\pi\lambda_2 + \bar{\pi}\mu_\pi(z_1)\mu_\pi(z_2)}{N(\pi)} \right\rrbracket.$$

Since $N(\pi) = \pi\bar{\pi}$, then, we get

$$\mu_\pi(z_1z_2) = \pi\lambda_1\pi\lambda_2 + \pi\lambda_1\mu_\pi(z_2) + \mu_\pi(z_1)\pi\lambda_2$$

$$+\mu_\pi(z_1)\mu_\pi(z_2) - \pi \quad \lambda_1\pi\lambda_2 \quad - \pi\left\llbracket \lambda_1\mu_\pi(z_2) \right\rrbracket$$

$$-\pi\left\llbracket \frac{\bar{\pi}\mu_\pi(z_1)\pi\lambda_2}{N(\pi)} \right\rrbracket - \pi\left\llbracket \frac{\bar{\pi}\mu_\pi(z_1)\mu_\pi(z_2)}{N(\pi)} \right\rrbracket.$$

Since $\lambda_1\pi\lambda_2 \quad = \lambda_1\pi\lambda_2$, $\left\llbracket \lambda_1\mu_\pi(z_2) \right\rrbracket = \lambda_1\mu_\pi(z_2)$ and $\left\llbracket \frac{\bar{\pi}\mu_\pi(z_1)\pi\lambda_2}{N(\pi)} \right\rrbracket = \frac{\mu_\pi(z_1)\pi\lambda_2}{\pi}$, then we get

$$\mu_\pi(z_1z_2) = \pi\lambda_1\pi\lambda_2 + \pi\lambda_1\mu_\pi(z_2) + \mu_\pi(z_1)\pi\lambda_2$$

$$+\mu_\pi(z_1)\mu_\pi(z_2) - \pi\lambda_1\pi\lambda_2 - \pi\lambda_1\mu_\pi(z_2)$$

$$-\mu_\pi(z_1)\pi\lambda_2 - \pi\left\llbracket \frac{\bar{\pi}\mu_\pi(z_1)\mu_\pi(z_2)}{N(\pi)} \right\rrbracket$$

$$= \mu_\pi(z_1)\mu_\pi(z_2) - \pi\left\llbracket \frac{\bar{\pi}\mu_\pi(z_1)\mu_\pi(z_2)}{N(\pi)} \right\rrbracket.$$

By eq. (5), we get $\mu_\pi(z_1z_2) = \mu_\pi(z_1)\mu_\pi(z_2) \bmod \pi$.

Consequently, $\mu$ function is a ring homomorphism. This completes this proof.

**Theorem 3.2** Let $\pi$ be an encoder Hurwitz integer. Then, $\mathbb{Z}_{N(\pi)} \cong \mathcal{H}_\pi$.

**Proof** Let $\pi$ be an encoder Hurwitz integer and, let $z_1, z_2 \in \mathbb{Z}_{N(\pi)}$. According to Theorem 3.1, $\mu$ function is a ring homomorphism. The modulo function in Definition 2.6 is a surjective ring homomorphism since $\mathcal{H}_\pi = \left\{\mu_\pi(z) \mid z \in \mathbb{Z}_{N(\pi)}\right\} = \operatorname{Im}\mu$. If $z = 0$, then $\mu_\pi(0) = 0$. On the other hand, If $z \geq 1$, then $\mu_\pi(z) \geq 1$. Hereby, the modulo function $\mu$ is a bijective ring homomorphism since $Ker\,\mu = \left\{z \in \mathbb{Z}_{N(\pi)} \mid \mu_\pi(z) = 0\right\} = \{0\}$.

Consequently, the modulo function $\mu$ is a ring isomorphism since it is both a surjective ring homomorphism and a bijective ring homomorphism, i.e. $\mathbb{Z}_{N(\pi)} \cong \mathcal{H}_\pi$. This completes the proof.

The following proposition demonstrates that the encoder Hurwitz integers have the "division with small remainder" property.

**Proposition 3.3** Let $\pi$ be an encoder Hurwitz integer. Then, $N\left(\mu_\pi(z)\right) < N(\pi)$.

**Proof** Let $\pi = \pi_1 + \pi_2 i + \pi_3 j + \pi_4 k$ be an encoder Hurwitz integer. If encoder Hurwitz integer $\pi$, each component of which is in $\mathbb{Z} + \frac{1}{2}$, then $N\left(\mu_\pi(z)\right) < N(\pi)$ holds on. For encoder Hurwitz $\pi$, each component of which is in integers, let us analyze step by step.

***Case 1*** Let $\pi$ be an encoder Hurwitz integer, each component of which is in integers. Suppose that let $\pi_1$ be an even integer, and let

$\pi_2$, $\pi_3$ and $\pi_4$ be odd integers. Therefore, $N(\pi)$ is an odd integer. By eq (5), we get

$$\mu_\pi(z) = z - \pi[\![\pi^{-1}z]\!].$$

Therefore, we get

$$\pi^{-1}\mu_\pi(z) = \pi^{-1}z - \pi^{-1}\pi[\![\pi^{-1}z]\!] = \pi^{-1}z - [\![\pi^{-1}z]\!]$$

$$= \frac{\overline{\pi}z}{N(\pi)} - \left[\!\!\left[\frac{\overline{\pi}z}{N(\pi)}\right]\!\!\right].$$

Since $\overline{\pi} = \pi_1 - \pi_2 i - \pi_3 j - \pi_4 k$ , then we get

$$\pi^{-1}\mu_\pi(z) = \frac{\pi_1 z}{N(\pi)} - \left(\frac{\pi_2 z}{N(\pi)}\right)i - \left(\frac{\pi_3 z}{N(\pi)}\right)j$$

$$-\left(\frac{\pi_4 z}{N(\pi)}\right)k - \left[\!\!\left[\frac{\pi_1 z}{N(\pi)}\right]\!\!\right] + \left[\!\!\left[\frac{\pi_2 z}{N(\pi)}\right]\!\!\right]i$$

$$+\left[\!\!\left[\frac{\pi_3 z}{N(\pi)}\right]\!\!\right]j + \left[\!\!\left[\frac{\pi_4 z}{N(\pi)}\right]\!\!\right]k$$

$$= \frac{\pi_1 z}{N(\pi)} - \left[\!\!\left[\frac{\pi_1 z}{N(\pi)}\right]\!\!\right] - \left(\frac{\pi_2 z}{N(\pi)} - \left[\!\!\left[\frac{\pi_2 z}{N(\pi)}\right]\!\!\right]\right)i$$

$$-\left(\frac{\pi_3 z}{N(\pi)} - \left[\!\!\left[\frac{\pi_3 z}{N(\pi)}\right]\!\!\right]\right)j - \left(\frac{\pi_4 z}{N(\pi)} - \left[\!\!\left[\frac{\pi_4 z}{N(\pi)}\right]\!\!\right]\right)k.$$

Since $\pi_1$ is an even integer, $\pi_2$, $\pi_3$ and $\pi_4$ are odd integers, $N(\pi)$ is an odd integer, and $\frac{N(\pi)}{2}$ is not an integer, then we get

$$0 \le \left|\frac{\pi_1 z}{N(\pi)} - \left[\!\!\left[\frac{\pi_1 z}{N(\pi)}\right]\!\!\right]\right| < \frac{1}{2},$$

$$0 \le \left|\frac{\pi_2 z}{N(\pi)} - \left[\!\!\left[\frac{\pi_2 z}{N(\pi)}\right]\!\!\right]\right| < \frac{1}{2},$$

$$0 \le \left|\frac{\pi_3 z}{N(\pi)} - \left[\!\!\left[\frac{\pi_3 z}{N(\pi)}\right]\!\!\right]\right| < \frac{1}{2},$$

$$0 \le \left|\frac{\pi_4 z}{N(\pi)} - \left[\!\!\left[\frac{\pi_4 z}{N(\pi)}\right]\!\!\right]\right| < \frac{1}{2}.$$

Therefore, we get

$$N\left(\frac{\pi_1 z}{N(\pi)} - \left[\!\!\left[\frac{\pi_1 z}{N(\pi)}\right]\!\!\right]\right) + N\left(\frac{\pi_2 z}{N(\pi)} - \left[\!\!\left[\frac{\pi_2 z}{N(\pi)}\right]\!\!\right]\right)$$

$$+N\left(\frac{\pi_3 z}{N(\pi)} - \left[\!\!\left[\frac{\pi_3 z}{N(\pi)}\right]\!\!\right]\right) + N\left(\frac{\pi_4 z}{N(\pi)} - \left[\!\!\left[\frac{\pi_4 z}{N(\pi)}\right]\!\!\right]\right)$$

$$< \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1.$$

Hereby, we get

$$N\left(\pi^{-1}\mu_\pi(z)\right) = N\left(\pi^{-1}\right)N\left(\mu_\pi(z)\right)$$

$$= \frac{1}{N(\pi)}N\left(\mu_\pi(z)\right) < 1.$$

Consequently, $N(\mu_\pi(z)) < N(\pi)$.

***Case 2*** Let $\pi$ be an encoder Hurwitz integer each component of which is in integers. Suppose that let $\pi_1$ and $\pi_2$ be even integers, and let $\pi_3$ and $\pi_4$ be odd integers. We should check whether to verify or not eq. (6) in Proposition 3.1 since $N(\pi)$ is an even integer. Let $z = \frac{N(\pi)}{2}$. Then, we get

$$\mu_\pi\left(\frac{N(\pi)}{2}\right) = \frac{N(\pi)}{2} - \pi\left[\!\!\left[\pi^{-1}\frac{N(\pi)}{2}\right]\!\!\right].$$

Therefore, we get

$$\pi^{-1}\mu_\pi\left(\frac{N(\pi)}{2}\right) = \pi^{-1}\frac{N(\pi)}{2} - \pi^{-1}\pi\left[\!\!\left[\pi^{-1}\frac{N(\pi)}{2}\right]\!\!\right]$$

$$=\frac{\overline{\pi}}{2}-\left[\!\left[\frac{\overline{\pi}}{2}\right]\!\right].$$

Since $\overline{\pi}=\pi_1-\pi_2 i-\pi_3 j-\pi_4 k$, then we get

$$\pi^{-1}\mu_\pi\left(\frac{N(\pi)}{2}\right)=\frac{\pi_1}{2}-\left(\frac{\pi_2}{2}\right)i-\left(\frac{\pi_3}{2}\right)j-\left(\frac{\pi_4}{2}\right)k$$

$$-\left[\!\left[\frac{\pi_1}{2}\right]\!\right]+\left[\!\left[\frac{\pi_2}{2}\right]\!\right]i+\left[\!\left[\frac{\pi_3}{2}\right]\!\right]j+\left[\!\left[\frac{\pi_4}{2}\right]\!\right]k$$

$$=\frac{\pi_1}{2}-\left[\!\left[\frac{\pi_1}{2}\right]\!\right]-\left(\frac{\pi_2}{2}-\left[\!\left[\frac{\pi_2}{2}\right]\!\right]\right)i$$

$$-\left(\frac{\pi_3}{2}-\left[\!\left[\frac{\pi_3}{2}\right]\!\right]\right)j-\left(\frac{\pi_4}{2}-\left[\!\left[\frac{\pi_4}{2}\right]\!\right]\right)k\,.$$

$\left[\!\left[\frac{\pi_1}{2}\right]\!\right]=\frac{\pi_1}{2}$ and $\left[\!\left[\frac{\pi_2}{2}\right]\!\right]=\frac{\pi_2}{2}$ since $\pi_1$ and $\pi_2$ are even integers. By eq. (3), then $\left[\!\left[\frac{\pi_3}{2}\right]\!\right]=\frac{\pi_3-1}{2}$ and $\left[\!\left[\frac{\pi_4}{2}\right]\!\right]=\frac{\pi_4-1}{2}$ since $\pi_3$ and $\pi_4$ are odd integers. Therefore, we get

$$\pi^{-1}\mu_\pi\left(\frac{N(\pi)}{2}\right)=\frac{\pi_1}{2}-\frac{\pi_1}{2}-\left(\frac{\pi_2}{2}-\frac{\pi_2}{2}\right)i$$

$$-\left(\frac{\pi_3}{2}-\frac{\pi_3-1}{2}\right)j-\left(\frac{\pi_4}{2}-\frac{\pi_4-1}{2}\right)k=-\frac{1}{2}j-\frac{1}{2}k\,.$$

Hereby, we get

$$N\left(\pi^{-1}\mu_\pi\left(\frac{N(\pi)}{2}\right)\right)=N\left(-\frac{1}{2}j-\frac{1}{2}k\right)$$

$$N\left(\pi^{-1}\right)N\left(\mu_\pi\left(\frac{N(\pi)}{2}\right)\right)=\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2$$

$$\frac{1}{N(\pi)}N\left(\mu_\pi\left(\frac{N(\pi)}{2}\right)\right)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

$$N(\mu_\pi(\frac{N(\pi)}{2}))=\frac{N(\pi)}{2}\,.$$

Consequently, $N(\mu_\pi(\frac{N(\pi)}{2}))<N(\pi)$. On the other hand, let $z\neq\frac{N(\pi)}{2}$. Then we get

$$\mu_\pi(z)=z-\pi\left[\!\left[\pi^{-1}z\right]\!\right].$$

Hereby, we get

$$\pi^{-1}\mu_\pi(z)=\pi^{-1}z-\pi^{-1}\pi\left[\!\left[\pi^{-1}z\right]\!\right]$$

$$=\pi^{-1}z-\left[\!\left[\pi^{-1}z\right]\!\right]=\frac{\overline{\pi}z}{N(\pi)}-\left[\!\left[\frac{\overline{\pi}z}{N(\pi)}\right]\!\right].$$

Since $\overline{\pi}=\pi_1-\pi_2 i-\pi_3 j-\pi_4 k$, then we get

$$\pi^{-1}\mu_\pi(z)=\frac{\pi_1 z}{N(\pi)}-\left(\frac{\pi_2 z}{N(\pi)}\right)i-\left(\frac{\pi_3 z}{N(\pi)}\right)j$$

$$-\left(\frac{\pi_4 z}{N(\pi)}\right)k-\left[\!\left[\frac{\pi_1 z}{N(\pi)}\right]\!\right]+\left[\!\left[\frac{\pi_2 z}{N(\pi)}\right]\!\right]i$$

$$+\left[\!\left[\frac{\pi_3 z}{N(\pi)}\right]\!\right]j+\left[\!\left[\frac{\pi_4 z}{N(\pi)}\right]\!\right]k$$

$$=\frac{\pi_1 z}{N(\pi)}-\left[\!\left[\frac{\pi_1 z}{N(\pi)}\right]\!\right]-\left(\frac{\pi_2 z}{N(\pi)}-\left[\!\left[\frac{\pi_2 z}{N(\pi)}\right]\!\right]\right)i$$

$$-\left(\frac{\pi_3 z}{N(\pi)}-\left[\!\left[\frac{\pi_3 z}{N(\pi)}\right]\!\right]\right)j-\left(\frac{\pi_4 z}{N(\pi)}-\left[\!\left[\frac{\pi_4 z}{N(\pi)}\right]\!\right]\right)k\,.$$

Since $z\neq\frac{N(\pi)}{2}$, then we get

$$0\leq\left|\frac{\pi_1 z}{N(\pi)}-\left[\!\left[\frac{\pi_1 z}{N(\pi)}\right]\!\right]\right|<\frac{1}{2},$$

$$0 \le \left| \frac{\pi_2 z}{N(\pi)} - \left[\!\left[ \frac{\pi_2 z}{N(\pi)} \right]\!\right] \right| < \frac{1}{2},$$

$$0 \le \left| \frac{\pi_3 z}{N(\pi)} - \left[\!\left[ \frac{\pi_3 z}{N(\pi)} \right]\!\right] \right| < \frac{1}{2},$$

$$0 \le \left| \frac{\pi_4 z}{N(\pi)} - \left[\!\left[ \frac{\pi_4 z}{N(\pi)} \right]\!\right] \right| < \frac{1}{2}.$$

Therefore, we get

$$N\left( \frac{\pi_1 z}{N(\pi)} - \left[\!\left[ \frac{\pi_1 z}{N(\pi)} \right]\!\right] \right) + N\left( \frac{\pi_2 z}{N(\pi)} - \left[\!\left[ \frac{\pi_2 z}{N(\pi)} \right]\!\right] \right)$$

$$+N\left( \frac{\pi_3 z}{N(\pi)} - \left[\!\left[ \frac{-\pi_3 z}{N(\pi)} \right]\!\right] \right) + N\left( \frac{\pi_4 z}{N(\pi)} - \left[\!\left[ \frac{\pi_4 z}{N(\pi)} \right]\!\right] \right)$$

$$< \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1.$$

Therefore, we get

$$N\left(\pi^{-1}\mu_\pi(z)\right) = N\left(\pi^{-1}\right) N\left(\mu_\pi(z)\right)$$

$$= \frac{1}{N(\pi)} N\left(\mu_\pi(z)\right) < 1.$$

Consequently, $N\left(\mu_\pi(z)\right) < N(\pi)$.

***<u>Case 3</u>*** Let $\pi$ be an encoder Hurwitz integer, each component of which is in integers. Suppose that $\pi_1$, $\pi_2$ and $\pi_3$ are even integers, and $\pi_4$ is an odd integer. Therefore, $N(\pi)$ is an odd integer. Then we get

$$\mu_\pi(z) = z - \pi \left[\!\left[ \pi^{-1} z \right]\!\right].$$

Hereby, we get

$$\pi^{-1}\mu_\pi(z) = \pi^{-1} z - \pi^{-1}\pi \left[\!\left[ \pi^{-1} z \right]\!\right]$$

$$= \pi^{-1} z - \left[\!\left[ \pi^{-1} z \right]\!\right] = \frac{\overline{\pi} z}{N(\pi)} - \left[\!\left[ \frac{\overline{\pi} z}{N(\pi)} \right]\!\right].$$

Since $\overline{\pi} = \pi_1 - \pi_2 i - \pi_3 j - \pi_4 k$, then we get

$$\pi^{-1}\mu_\pi(z) = \frac{\pi_1 z}{N(\pi)} - \left( \frac{\pi_2 z}{N(\pi)} \right) i - \left( \frac{\pi_3 z}{N(\pi)} \right) j$$

$$-\left( \frac{\pi_4 z}{N(\pi)} \right) k - \left[\!\left[ \frac{\pi_1 z}{N(\pi)} \right]\!\right] + \left[\!\left[ \frac{\pi_2 z}{N(\pi)} \right]\!\right] i$$

$$+\left[\!\left[ \frac{\pi_3 z}{N(\pi)} \right]\!\right] j + \left[\!\left[ \frac{\pi_4 z}{N(\pi)} \right]\!\right] k$$

$$= \frac{\pi_1 z}{N(\pi)} - \left[\!\left[ \frac{\pi_1 z}{N(\pi)} \right]\!\right] - \left( \frac{\pi_2 z}{N(\pi)} - \left[\!\left[ \frac{\pi_2 z}{N(\pi)} \right]\!\right] \right) i$$

$$-\left( \frac{\pi_3 z}{N(\pi)} - \left[\!\left[ \frac{\pi_3 z}{N(\pi)} \right]\!\right] \right) j - \left( \frac{\pi_4 z}{N(\pi)} - \left[\!\left[ \frac{\pi_4 z}{N(\pi)} \right]\!\right] \right) k.$$

Since $\pi_4$ is an odd integer, $\pi_1$, $\pi_2$ and $\pi_3$ are even integers, and $N(\pi)$ is an odd integer, then we get

$$0 \le \left| \frac{\pi_1 z}{N(\pi)} - \left[\!\left[ \frac{\pi_1 z}{N(\pi)} \right]\!\right] \right| < \frac{1}{2},$$

$$0 \le \left| \frac{\pi_2 z}{N(\pi)} - \left[\!\left[ \frac{\pi_2 z}{N(\pi)} \right]\!\right] \right| < \frac{1}{2},$$

$$0 \le \left| \frac{\pi_3 z}{N(\pi)} - \left[\!\left[ \frac{\pi_3 z}{N(\pi)} \right]\!\right] \right| < \frac{1}{2},$$

$$0 \le \left| \frac{\pi_4 z}{N(\pi)} - \left[\!\left[ \frac{\pi_4 z}{N(\pi)} \right]\!\right] \right| < \frac{1}{2}.$$

Hereby, we get

$$N\left(\frac{\pi_1 z}{N(\pi)} - \left[\!\left[\frac{\pi_1 z}{N(\pi)}\right]\!\right]\right) + N\left(\frac{\pi_2 z}{N(\pi)} - \left[\!\left[\frac{\pi_2 z}{N(\pi)}\right]\!\right]\right)$$

$$+N\left(\frac{\pi_3 z}{N(\pi)} - \left[\!\left[\frac{\pi_3 z}{N(\pi)}\right]\!\right]\right) + N\left(\frac{\pi_4 z}{N(\pi)} - \left[\!\left[\frac{\pi_4 z}{N(\pi)}\right]\!\right]\right)$$

$$< \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1.$$

Therefore, we get

$$N\left(\pi^{-1}\mu_\pi(z)\right) = N\left(\pi^{-1}\right) N\left(\mu_\pi(z)\right)$$

$$= \frac{1}{N(\pi)} N\left(\mu_\pi(z)\right) < 1.$$

Then, we get $N\left(\mu_\pi(z)\right) < N(\pi)$. Consequently, $N\left(\mu_\pi(z)\right) < N(\pi)$. This completes the proof.

The following examples give an example for each case in the proof of Proposition 3.3.

***Example 2*** (*Case 1*) $\pi = 1 + 3i + 2j + k$ is an encoder Hurwitz integer. By eq. (5), the Hurwitz signal constellation $\mathcal{H}_\pi$ is

$$\mathcal{H}_\pi = \left\{ \begin{array}{l} \mu_\pi(0) = 0,\ \mu_\pi(1) = 1,\ \mu_\pi(2) = 2, \\ \mu_\pi(3) = i + j - 2k,\ \mu_\pi(4) = -1 + 2j + k, \\ \mu_\pi(5) = 2j + k,\ \mu_\pi(6) = 1 + 2j + k, \\ \mu_\pi(7) = 2 + 2j + k,\ \mu_\pi(8) = -2 - 2j - k, \\ \mu_\pi(9) = -1 - 2j - k,\ \mu_\pi(10) = -2j - k, \\ \mu_\pi(11) = 1 - 2j - k,\ \mu_\pi(12) = -i - j + 2k, \\ \mu_\pi(13) = -2,\ \mu_\pi(14) = -1, \end{array} \right\}.$$

The set contains fifteen elements since $N(\pi) = 15$. The norm of each element in the set is less than the norm of the primitive Hurwitz integer (encoder Hurwitz integer) $\pi$.

***Example 3*** (*Case 2*) $\pi = 2 + 3i + j + 2k$ is an encoder Hurwitz integer. By eq. (5), the Hurwitz signal constellation $\mathcal{H}_\pi$ is

$$\mathcal{H}_\pi = \left\{ \begin{array}{l} \mu_\pi(0) = 0,\ \mu_\pi(1) = 1,\ \mu_\pi(2) = 2, \mu_\pi(3) = 3, \\ \mu_\pi(4) = 1 + 2i + 2j - k, \mu_\pi(5) = -2 - 2j - k, \\ \mu_\pi(6) = -1 - 2j - k, \mu_\pi(7) = -2j - k, \\ \mu_\pi(8) = 1 - 2j - k, \mu_\pi(9) = 2 - 2j - k, \\ \mu_\pi(10) = -1 + 2j + k, \mu_\pi(11) = 2j + k, \\ \mu_\pi(12) = 1 + 2j + k, \mu_\pi(13) = 2 + 2j + k, \\ \mu_\pi(14) = -1 - 2i - 2j + k, \mu_\pi(15) = -2i - 2j + k, \\ \mu_\pi(16) = -2, \mu_\pi(17) = -1 \end{array} \right\}.$$

The set contains eighteen elements since $N(\pi) = 18$. The norm of each element in the set is less than the norm of a primitive Hurwitz integer (encoder Hurwitz integer) $\pi$.

***Example 4*** (*Case 3*) $\pi = 2 + 3i + 2j + 2k$ is an encoder Hurwitz integer. By eq. (5), the Hurwitz signal constellation $\mathcal{H}_\pi$ is

$$\mathcal{H}_\pi = \left\{ \begin{array}{l} \mu_\pi(0) = 0,\ \mu_\pi(1) = 1,\ \mu_\pi(2) = 2, \mu_\pi(3) = 3, \\ \mu_\pi(4) = 1 + 2i + 2j - 2k, \mu_\pi(5) = 2 + 2i + 2j - 2k, \\ \mu_\pi(6) = 3 - i - j + k, \mu_\pi(7) = -2 - i - j + k, \\ \mu_\pi(8) = -1 - i - j + k, \mu_\pi(9) = -i - j + k, \\ \mu_\pi(10) = 1 - i - j + k, \mu_\pi(11) = -1 + i + j - k, \\ \mu_\pi(12) = i + j - k, \mu_\pi(13) = 1 + i + j - k, \\ \mu_\pi(14) = 2 + i + j - k, \mu_\pi(15) = 3 + i + j - k, \\ \mu_\pi(16) = -2 - 2i - 2j + 2k, \\ \mu_\pi(17) = -1 - 2i - 2j + 2k,\ \mu_\pi(18) = -3, \\ \mu_\pi(19) = -2,\ \mu_\pi(17) = -1 \end{array} \right\}.$$

The set contains twenty-one elements since $N(\pi) = 21$. The norm of each element in the set is less than the norm of a primitive Hurwitz integer (encoder Hurwitz integer) $\pi$.

Example 2 (case 1), example 3 (case 2), and example 4 (case 3) verify all the conditions to be a Euclidean metric. Also, the Euclidean division algorithm works for these primitive Hurwitz integers (encoder Hurwitz integers).

In the following example, according to the presented algebraic construction technique in [16] by Rohweder et al., we show that a Hurwitz integer $\pi$, each component of which is in integers, does not have the "division with small remainder" property.

***Example 5*** In [16], Rohweder et al. presented the new construction method for Hurwitz integers by

$$\mu_\pi(z) = z - \left[\!\left[ \frac{z\bar{\pi}}{N(\pi)} \right]\!\right] \pi , \tag{10}$$

where $\pi$ is a primitive Hurwitz integer and $z \in \mathbb{Z}_{N(\pi)}$. They proposed four-dimensional Hurwitz signal constellations are obtained from the following mapping

$$\mathcal{H}_\pi = \mathcal{L}_\pi \bigcup \mathcal{O}_\pi , \tag{11}$$

where $\mathcal{L}_\pi$ is the subset of Lipschitz integers. It can be evaluated by

$$\mathcal{L}_\pi = \left\{ \mu_\pi(a+bj) \mid a,b \in \mathbb{Z}_{N(\pi)} \right\},$$

where $\mathbb{Z}_{N(\pi)}$ denotes the ring of integers modulo $N(\pi)$. Also, $\mathcal{O}_\pi$ in eq. (11) is the corresponding coset of half-integers. It can be calculated by

$$\mathcal{O}_\pi = \left\{ \mu_\pi(h+w) \mid h \in \mathcal{L}_\pi \right\},$$

where $w = \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k.$ With respect to the related modulo technique in [16], the size of the set in eq. (11) is $2N^2(\pi)$. We take an example $3+i$ in [16, Table I]. We have $\mu_\pi(5+5j)$ since $a = \frac{N(3+i)}{2} = 5$ and $b = \frac{N(3+i)}{2} = 5.$ By eq. (10), we get

$$\mu_\pi(5+5j) = 5+5j - \left[\!\left[ \frac{(5+5j)(3-i)}{10} \right]\!\right](3+i)$$

$$= 5+5j - \left[\!\left[ \frac{15-5i+15j+5k}{10} \right]\!\right](3+i)$$

$$= 5+5j-(1+j)(3+i) = 5+5j-3-i-3j+k$$

$$= 2-i+2j+k .$$

Since $N(\mu_\pi(5+5j)) = N(3+i)$, then we get $5+5j = 0 \bmod (3+i)$. Also, $\mu_\pi(0)=0$ since $a=0$ and $b=0$. So, we conclude that the set in eq. (11) has elements less than two hundred elements since $\mu_\pi(0) = \mu_\pi(5+5j) \equiv 0 \bmod (3+i)$. This contradicts the size of the Hurwitz signal constellation with two hundred elements. By Proposition 3.1, we say that the Hurwitz (Lipschitz) integer $3+i$ is not a suitable Hurwitz (Lipschitz) integer for constructing the Hurwitz signal constellation. Similarly, we can show that the primitive Hurwitz (Lipschitz) integer $2+2i+j+k$ is not a suitable Hurwitz (Lipschitz) integer for constructing the Hurwitz signal constellation with respect to the algebraic construction technique in [16].

## 4. PERFORMANCES OF HURWITZ SIGNAL CONSTELLATIONS FOR TRANSMISSION OVER AWGN CHANNEL

In this section, we present some distance and performance measures. In addition, we investigate the performances of Hurwitz signal constellations constructed by primitive Hurwitz integers (encoder Hurwitz integers), each component of which is in halves-integers, for transmission over the additive white Gaussian noise (AWGN) channel by the agency of the constellation figure of merit (CFM), average energy, and signal-to-noise ratio (SNR) gain.

We follow the procedures in [27] for distance, performance measures, and set partitioning property. The average energy of a signal constellation denoted by $\mathcal{E}_\pi$ is computed by

$$\mathcal{E}_\pi = \frac{1}{N(\pi)} \sum_{z=0}^{N(\pi)-1} N(\mu_\pi(z)).$$

The squared Euclidean distance of two Hurwitz integers in the Hurwitz signal constellation is defined as

$$d_E(\theta,\varphi) = N(\varphi-\theta),$$

and the minimum squared Euclidean distance of the signal constellation is

$$\delta_\pi^2 = \min_{\alpha\neq\beta} d_E(\theta,\varphi),$$

where $\theta,\varphi \in \mathcal{H}_\pi$. In [28], Forney and Wei proposed the constellation figure of merit (CFM) to compare signal constellations of different dimensions. The CFM is the ratio of the minimum squared Euclidean distance and the average energy per two-dimension. The CFM of a $M-$dimensional signal constellation is computed by

$$CFM = \frac{M\delta_\pi^2}{2\mathcal{E}_\pi}.$$

A higher CFM leads to better performance for transmission over an AWGN channel [27]. Signal-to-noise ratio (SNR or S/N) is a measure used in science and engineering that compares the level of a desired signal to the level of background noise [29]. SNR is defined by the ratio of signal power to the noise power, often expressed in decibels. A ratio higher than 1:1 (greater than 0 dB) indicates more signal than noise [29]. Asymptotic coding gain means a higher signal-to-noise ratio (SNR) [1]. The SNR of signal and noise power is computed by

$$SNR_{signal} = 10\cdot\log_{10}(CFM\ of\ signal) \tag{12}$$

and

$$SNR_{noise} = 10\cdot\log_{10}(CFM\ of\ noise), \tag{13}$$

respectively. As the noise, we consider the Gaussian signal constellation $\mathcal{G}_\alpha$, where $\alpha$ is a primitive Gaussian integer. Therefore, the SNR code gain of a Hurwitz signal constellation over the AWGN channel is

$$SNR_{dB} = SNR_{\mathcal{H}_\pi} - SNR_{\mathcal{G}_\alpha},$$

where Hurwitz signal constellation $\mathcal{H}_\pi$, and Gaussian signal constellation $\mathcal{G}_\alpha$. By eq. (12) and eq. (13),

$$SNR_{dB} = 10\cdot\log_{10}(CFM\ of\ \mathcal{H}_\pi)$$

$$-10\cdot\log_{10}(CFM\ of\ \mathcal{G}_\alpha)$$

$$=10\cdot\log_{10}\left(\frac{CFM\ of\ \mathcal{H}_\pi}{CFM\ of\ \mathcal{G}_\alpha}\right).$$

Note that the number of elements in the Hurwitz signal constellation and the Gaussian signal constellation should be the same to compare performances over the AWGN channel. According to the modulo function in Definition 2.6, the Hurwitz signal constellations that have the same size as Gaussian signal constellations almost show the same performances for transmission over the AWGN channel. Moreover, the squared Euclidean distance of the Hurwitz signal constellations and the Gaussian signal constellations, the size of which is the same, is one. The set partitioning aims to find a subset with a large squared Euclidean distance. Therefore, we obtain the Hurwitz signal constellations, which have the larger CFM, showing better performance for transmission over the AWGN channel.

A residue class ring of Hurwitz integers $\mathcal{H}_\lambda$ arises from the residue class ring of integers $\mathbb{Z}_{N(\lambda)} = \{0,1,\ldots,N(\lambda)-1\}$ for an integer $N(\lambda)$, where $\lambda$ is a proposed primitive Hurwitz integer. If $N(\lambda)$ is not a prime integer, then we can partition the set $\mathcal{H}_\lambda$ into

subsets of equal size. Let $N(\lambda)=c\cdot d$. In other words, we can partition the set $\mathcal{H}_\lambda$ into $c$ subsets $\mathcal{H}_\lambda^{(0)},\dots,\mathcal{H}_\lambda^{(c-1)}$ each with $d$ elements. The subsets correspond to the Hurwitz signal constellations $\mathcal{H}_\lambda^{(0)},\dots,\mathcal{H}_\lambda^{(c-1)}$, where
$\mathcal{H}_\lambda^{(0)}=\left\{\mu_\lambda(0),\mu_\lambda(c),\mu_\lambda(2c),\dots,\mu_\lambda((d-1)c)\right\}$,
and $\mathcal{H}_\lambda^{(1)},\dots,\mathcal{H}_\lambda^{(c-1)}$ are the cosets of $\mathcal{H}_\lambda^{(0)}$, i.e. $\mathcal{H}_\lambda^{(l)}=\left\{\mu_\lambda(z):\mu_\lambda(z-l)\in\mathcal{H}_\lambda^{(0)}\right\}$, where

$$\mathcal{H}_\lambda^{(l)}=\left\{\begin{matrix}\mu_\lambda(l),\mu_\lambda(l+2c),\mu_\lambda(l+3c)\\ ,\dots,\mu_\lambda(l+(d-1)c)\end{matrix}\right\}.$$

The subset $\mathcal{H}_\lambda^{(0)}$ is an additive subgroup of $\mathcal{H}_\lambda$ since the modulo function $\mu$ is an isomorphism with respect to addition.

The SNR gain of the subset of a proposed Hurwitz signal constellation over the AWGN channel is computed by

$$SNR_{dB}=10\cdot\log_{10}\left(\frac{CFM_{\mathcal{H}_\pi^{(0)}}}{CFM_{\mathcal{G}_\alpha}}\right),$$

where the subset of a proposed Hurwitz signal constellation $\mathcal{H}_\pi^{(0)}$, and Gaussian signal constellation $\mathcal{G}_\alpha$.

In the rest of this paper, we consider primitive Hurwitz integers (encoder Hurwitz integers) such that $\pi_1\geq\pi_2\geq\pi_3\geq\pi_4>0$. We investigate the performance of Hurwitz signal constellations constructed by primitive Hurwitz integers (encoder Hurwitz integers), each component of which is in $\mathbb{Z}+\frac{1}{2}$, over the AWGN channel.

In Table 1, we present the performance of Hurwitz signal constellations constructed by primitive Hurwitz integers (encoder Hurwitz integers), each component of which is in $\mathbb{Z}+\frac{1}{2}$, over the AWGN channel by means of average energy, CFM, and SNR coding gain. In Table 1, the Hurwitz signal constellations obtained by the modulo function in Definition 2.6 have almost similar properties as Lipschitz signal constellations in the paper of Freudenberger et al. [27]. The performance of Hurwitz signal constellations in Table 1 is not so good but better than nothing according to the performance of the Lipschitz signal constellations in [27, Table I] over the AWGN channel. Moreover, the performances of proposed Hurwitz signal constellations constructed by primitive Hurwitz integers (encoder Hurwitz integers), each component of which is in integers, are the same as the performances of proposed Lipschitz signal constellations in [27, Table I].

In Table 2, we present the performance of the proposed Hurwitz signal constellation constructed by proposed primitive Hurwitz integers (encoder Hurwitz integers), each component of which is in $\mathbb{Z}+\frac{1}{2}$, over the AWGN channel by means of average energy, CFM, and SNR coding gain. The proposed Hurwitz signal constellations in Table 2 have advantage performances for transmission over the AWGN channel by set partitioning property. There also exist different proposed primitive Hurwitz integers (encoder Hurwitz integers) used to construct proposed Hurwitz signal constellations that have higher CFM and lower average energy in equal size. You can see the following examples. Moreover, the below examples are given clues about the construction of tables.

***Example 6*** We consider the proposed Hurwitz signal constellation with $N=3\cdot 13=39$ elements. There exist four different proposed primitive Hurwitz integers (encoder Hurwitz integers) used to construct the proposed Hurwitz signal constellation with $N=39$. These proposed primitive Hurwitz integers (encoder Hurwitz integer) are $\frac{7}{2}+\frac{7}{2}i+\frac{7}{2}j+\frac{3}{2}k$, $\frac{9}{2}+\frac{5}{2}i+\frac{5}{2}j+\frac{5}{2}k$, $\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k$, and $\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k$.

There is no Gaussian signal constellation with $N=39$ elements. Note that proposed primitive Hurwitz integers (encoder Hurwitz integers) are not to be the same size as primitive Gaussian integers. Therefore, we could use set partitioning property on proposed primitive Hurwitz integers (encoder Hurwitz integers). Firstly, we consider proposed primitive Hurwitz integers (encoder Hurwitz integer) $\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k$ and $\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k$. For the proposed Hurwitz signal constellation $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}$, the minimum squared Euclidean distance, average energy and CFM are $1$, $12.5128$ and $0.1598$, respectively. The proposed Hurwitz signal constellation $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}$ is partition the $c=3$ different subsets with each set $d=13$ elements. The minimum squared Euclidean distance, average energy, and CFM of Hurwitz signal constellation $\mathcal{H}^{(0)}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}$ are $9$, $11.5385$, and $1.5600$, respectively. The minimum squared Euclidean distance, average energy, and CFM of the Gaussian signal constellation $\mathcal{G}_{3+2i}$ with 13 elements are $1$, $2.1539$, and $0.4643$, respectively. Therefore, the SNR coding gain of the proposed Hurwitz signal constellation $\mathcal{H}^{(0)}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}$ is

$$SNR_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}=10\log\left(\frac{CFM\ of\ \mathcal{H}^{(0)}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}}{CFM\ of\ \mathcal{G}_{3+2i}}\right)$$

$$=10\log\left(\frac{1.5600}{0.4643}\right)=5.26\ dB.$$

The minimum squared Euclidean distance, average energy, and CFM of the proposed Hurwitz signal constellation $\mathcal{H}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ are $1$, $12.5128$, and $0.1598$, respectively. The proposed Hurwitz signal constellation $\mathcal{H}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ is partition the $c=3$ different subsets with each set of $d=13$ elements. The minimum squared Euclidean distance, average energy, and CFM of Hurwitz signal constellation $\mathcal{H}^{(0)}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ are $9$, $11.5385$, and $1.5600$, respectively. The average energy, minimum squared Euclidean distance and CFM of Hurwitz signal constellations $\mathcal{H}^{(0)}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}$ and $\mathcal{H}^{(0)}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ are the same. So, Hurwitz signal constellations $\mathcal{H}^{(0)}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}$ and $\mathcal{H}^{(0)}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ have the same performances for transmission over the AWGN channel. Lastly, we consider proposed primitive Hurwitz integers (encoder Hurwitz integers) $\frac{7}{2}+\frac{7}{2}i+\frac{7}{2}j+\frac{3}{2}k$, and $\frac{9}{2}+\frac{5}{2}i+\frac{5}{2}j+\frac{5}{2}k$. For both proposed Hurwitz signal constellations, the minimum squared Euclidean distance, average energy, and CFM are $1$, $12.5128$, and $0.1598$, respectively. The average energy, minimum squared Euclidean distance and CFM of $\mathcal{H}^{(0)}_{\frac{7}{2}+\frac{7}{2}i+\frac{7}{2}j+\frac{3}{2}k,}$ and $\mathcal{H}^{(0)}_{\frac{9}{2}+\frac{5}{2}i+\frac{5}{2}j+\frac{5}{2}k}$ are $3$, $11.5385$ and $0.5200$, respectively. Therefore, the SNR coding gain of these proposed Hurwitz signal constellations is

$$SNR_{dB}=10\log\left(\frac{0.5200}{0.4643}\right)=0.49\ dB.$$

Consequently, the proposed Hurwitz signal constellations $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}$ and $\mathcal{H}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ have higher CFM, better SNR coding gain, and larger minimum square Euclidean distance. We choose the proposed primitive

Hurwitz integer (encoder Hurwitz integer) $\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k$ to represent in Table 2.

***Example 7*** We consider the proposed Hurwitz signal constellation with $N=3\cdot 29=87$ elements. There exist eight different proposed primitive Hurwitz integers (encoder Hurwitz integers) used to construct proposed Hurwitz signal constellations with $N=87$. These proposed primitive Hurwitz integers (encoder Hurwitz integers) are $\frac{11}{2}+\frac{11}{2}i+\frac{9}{2}j+\frac{5}{2}k$, $\frac{13}{2}+\frac{9}{2}i+\frac{7}{2}j+\frac{7}{2}k$, $\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k$, $\frac{13}{2}+\frac{13}{2}i+\frac{3}{2}j+\frac{1}{2}k$, $\frac{15}{2}+\frac{7}{2}i+\frac{7}{2}j+\frac{5}{2}k$, $\frac{15}{2}+\frac{11}{2}i+\frac{1}{2}j+\frac{1}{2}k$, $\frac{17}{2}+\frac{5}{2}i+\frac{5}{2}j+\frac{3}{2}k$, and $\frac{17}{2}+\frac{7}{2}i+\frac{3}{2}j+\frac{1}{2}k$.

There is no Gaussian signal constellation with $N=87$ elements. The minimum squared Euclidean distance, average energy, and CFM of proposed Hurwitz signal constellations constructed by these proposed primitive Hurwitz integers (encoder Hurwitz integers) are $1$, $28.5057$, and $0.0702$, respectively. These proposed Hurwitz signal constellations are partition the $c=3$ different subsets with each set $d=29$ elements. We consider the Hurwitz signal constellations $\mathcal{H}^{(0)}_{\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k}$, and $\mathcal{H}^{(0)}_{\frac{17}{2}+\frac{7}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ with 29 elements. The minimum square Euclidean distance of these signal constellations is larger than others. The minimum square Euclidean distance of these signal constellations is $9$, but the others are $6$. Also, the average energy and CFM of these signal constellations are $27.5172$ and $0.6541$, respectively, but the others are $27.5172$ and $0.4361$, respectively. The minimum squared Euclidean distance, average energy, and CFM of the Gaussian signal constellation $\mathcal{G}_{5+2i}$ with 29 elements are $1$, $4.8276$, and $0.2071$, respectively. Therefore, the SNR coding gain of Hurwitz signal constellations $\mathcal{H}^{(0)}_{\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k}$ and $\mathcal{H}^{(0)}_{\frac{17}{2}+\frac{7}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ is

$$SNR_{dB}=10\log\left(\frac{0.6541}{0.2071}\right)=4.99\ dB.$$

**Table 1** Table of CFM, energy and SNR coding gain of Hurwitz signal constellations constructed by primitive Hurwitz integers (encoder Hurwitz integer), each component of which is in $\mathbb{Z}+\frac{1}{2}$, ( $d$ : The number of elements in the Hurwitz signal constellation, $\mathcal{G}_\alpha$ : Gaussian signal constellation, $\mathcal{H}_\pi$ : Hurwitz signal constellation )

| $d$ | Primitive Hurwitz Integers $(\pi)$ | Signal Constellations CFM / ENERGY $\mathcal{G}_\alpha$ | $\mathcal{H}_\pi$ | SNR [dB] |
|---|---|---|---|---|
| 5 | $\frac{3}{2}+\frac{3}{2}i+\frac{1}{2}j+\frac{1}{2}k$ | 1.2500 | 1.6667 | 1.25 |
| | | 0.8000 | 1.2000 | |
| 13 | $\frac{5}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k$ | 0.4643 | 0.5200 | 0.49 |
| | | 2.1538 | 3.8462 | |
| 17 | $\frac{5}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{3}{2}k$ | 0.3542 | 0.3864 | 0.38 |
| | | 2.8235 | 5.1765 | |
| 23 | $\frac{9}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{1}{2}k$ | 0.2404 | 0.2551 | 0.26 |
| | | 4.1600 | 7.8400 | |
| 29 | $\frac{9}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k$ | 0.2071 | 0.2180 | 0.22 |
| | | 4.8276 | 9.1724 | |
| 37 | $\frac{11}{2}+\frac{5}{2}i+\frac{1}{2}j+\frac{1}{2}k$ | 0.1623 | 0.1690 | 0.18 |
| | | 6.1622 | 11.8378 | |
| 41 | $\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{3}{2}k$ | 0.1464 | 0.1519 | 0.16 |
| | | 6.8293 | 13.1707 | |
| 53 | $\frac{13}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{3}{2}k$ | 0.1132 | 0.1165 | 0.12 |
| | | 8.8302 | 17.1698 | |
| 61 | $\frac{15}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{1}{2}k$ | 0.0984 | 0.1008 | 0.10 |
| | | 10.1639 | 19.8361 | |
| 65 | $\frac{11}{2}+\frac{11}{2}i+\frac{3}{2}j+\frac{3}{2}k$ | 0.0923 | 0.0945 | 0.10 |
| | | 10.8308 | 21.1692 | |
| 73 | $\frac{17}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k$ | 0.0822 | 0.0839 | 0.09 |
| | | 12.1644 | 23.8356 | |
| 85 | $\frac{13}{2}+\frac{13}{2}i+\frac{1}{2}j+\frac{1}{2}k$ | 0.0706 | 0.0719 | 0.08 |
| | | 14.1467 | 27.8353 | |
| 89 | $\frac{17}{2}+\frac{7}{2}i+\frac{3}{2}j+\frac{3}{2}k$ | 0.0674 | 0.0686 | 0.08 |
| | | 14.8315 | 29.1685 | |
| 97 | $\frac{19}{2}+\frac{5}{2}i+\frac{1}{2}j+\frac{1}{2}k$ | 0.0619 | 0.0628 | 0.06 |
| | | 16.1649 | 31.8351 | |

Consequently, the Hurwitz signal constellations that have higher CFM and larger minimum square Euclidean distance are $\mathcal{H}^{(0)}_{\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k}$ and $\mathcal{H}^{(0)}_{\frac{17}{2}+\frac{7}{2}i+\frac{3}{2}j+\frac{1}{2}k}$. We choose the proposed primitive Hurwitz integer $\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k$ to represent in Table 2.

**Table 2** Table of CFM, energy and SNR coding gain of Hurwitz signal constellations constructed by proposed primitive Hurwitz integers (encoder Hurwitz integers), each component of which is in $\mathbb{Z}+\frac{1}{2}$, ($N$ : The size of Hurwitz signal constellation, $c$ : the number of subsets of the proposed Hurwitz signal constellation, $d$ : the size of subsets of the proposed Hurwitz signal constellation, $\mathcal{G}_\alpha$ : Gauss signal constellation, $\mathcal{H}^{(0)}_\lambda$ : the subset of $\mathcal{H}_\lambda$, where $\mathcal{H}_\lambda$ is the proposed Hurwitz constellation )

| $N=c\cdot d$ | Proposed Primitive Hurwitz Integers ($\lambda$) | Signal Constellations CFM / ENERGY $\mathcal{G}_\alpha$ | Signal Constellations CFM / ENERGY $\mathcal{H}^{(0)}_\lambda$ | $SNR$ $[dB]$ |
|---|---|---|---|---|
| $15=3\cdot5$ | $\frac{7}{2}+\frac{3}{2}i+\frac{1}{2}j+\frac{1}{2}k$ | 1.2500 / 0.8000 | 1.6667 / 3.600 | 1.25 |
| $39=3\cdot13$ | $\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k$ | 0.4643 / 2.1538 | 1.5600 / 11.5385 | 5.26 |
| $51=3\cdot17$ | $\frac{11}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{3}{2}k$ | 0.3542 / 2.8235 | 1.1591 / 15.5294 | 5.15 |
| $75=3\cdot25$ | $\frac{13}{2}+\frac{9}{2}i+\frac{7}{2}j+\frac{1}{2}k$ | 0.2404 / 4.1600 | 0.7653 / 23.5200 | 5.03 |
| $87=3\cdot29$ | $\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k$ | 0.2071 / 4.8276 | 0.6541 / 27.5170 | 4.99 |
| $185=5\cdot37$ | $\frac{21}{2}+\frac{13}{2}i+\frac{9}{2}j+\frac{7}{2}k$ | 0.1623 / 6.1622 | 0.8447 / 59.1892 | 7.16 |
| $205=5\cdot41$ | $\frac{27}{2}+\frac{9}{2}i+\frac{3}{2}j+\frac{1}{2}k$ | 0.1464 / 6.8293 | 0.7593 / 65.8537 | 7.15 |
| $265=5\cdot53$ | $\frac{27}{2}+\frac{15}{2}i+\frac{9}{2}j+\frac{5}{2}k$ | 0.1132 / 8.8302 | 0.5824 / 85.8491 | 7.11 |
| $427=7\cdot61$ | $\frac{33}{2}+\frac{21}{2}i+\frac{13}{2}j+\frac{3}{2}k$ | 0.0984 / 10.1639 | 0.7058 / 138.8520 | 8.56 |
| $455=7\cdot65$ | $\frac{33}{2}+\frac{11}{2}i+\frac{9}{2}j+\frac{3}{2}k$ | 0.0923 / 10.8308 | 0.4724 / 105.8460 | 7.09 |
| $511=7\cdot73$ | $\frac{33}{2}+\frac{21}{2}i+\frac{17}{2}j+\frac{15}{2}k$ | 0.0822 / 12.1644 | 0.5874 / 166.8490 | 8.54 |
| $595=7\cdot85$ | $\frac{33}{2}+\frac{29}{2}i+\frac{21}{2}j+\frac{3}{2}k$ | 0.0706 / 14.1467 | 0.5030 / 194.847 | 8.53 |
| $623=7\cdot89$ | $\frac{35}{2}+\frac{33}{2}i+\frac{13}{2}j+\frac{3}{2}k$ | 0.0674 / 14.8315 | 0.4800 / 204.1800 | 8.53 |
| $873=9\cdot97$ | $\frac{41}{2}+\frac{31}{2}i+\frac{29}{2}j+\frac{3}{2}k$ | 0.0619 / 16.1649 | 0.5654 / 286.5150 | 9.61 |

## 5. CONCLUSION

We showed, with the help of a proposition (Proposition 3.1), some Hurwitz integers are inappropriate for constructing Hurwitz signal constellations with $N(\alpha)$ elements, where $\alpha$ is a primitive Hurwitz integer. To solve this problem, we presented a proposition to help find out the primitive Hurwitz integers that have the division with small remainder (see Proposition 3.1). We also called the set of these integers the "Encoder Hurwitz Integers" set. We showed, with the help of a proposition (see Proposition 3.3), the Euclid division is satisfied by encoder Hurwitz integers. Moreover, we presented new Hurwitz signal constellations constructed by Hurwitz integers, each component of which is in half-integers. We investigated the performances of these signal constellations for transmission over the AWGN channel.

***Funding***
The author has no received any financial support for the research, authorship or publication of this study.

***The Declaration of Conflict of Interest/ Common Interest***
No conflict of interest or common interest has been declared by the author.

***The Declaration of Ethics Committee Approval***
This study does not require ethics committee permission or any special permission.

***The Declaration of Research and Publication Ethics***
The author of the paper declare that I comply with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that I do not make any falsification on the data collected. In addition, I declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

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