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Research Article

Toward the theory of semi-linear Beltrami equations

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ABSTRACT. We study the semi-linear Beltrami equation $\omega_{\bar{z}} - \mu(z)\omega_z = \sigma(z)q(\omega(z))$ and show that it is closely related to the corresponding semi-linear equation of the form $\mathrm{div} A(z)\nabla U(z) = G(z)Q(U(z))$. Applying the theory of completely continuous operators by Ahlfors-Bers and Leray–Schauder, we prove existence of regular solutions both to the semi-linear Beltrami equation and to the given above semi-linear equation in the divergent form, see Theorems 1.1 and 5.2. We also derive their representation through solutions of the semi-linear Vekua type equations and generalized analytic functions with sources. Finally, we apply Theorem 5.2 for several model equations describing physical phenomena in anisotropic and inhomogeneous media.

Keywords: Semi-linear Beltrami equations, generalized analytic functions with sources, semi-linear Poisson type equations, generalized harmonic functions with sources.

2020 Mathematics Subject Classification: 30C62, 35J61, 30G30, 31A35, 35Q35.

1. INTRODUCTION

Let D be a domain in the complex plane $\mathbb{C}$. In this paper, we study semi-linear Beltrami equations of the form

$$\omega_{\bar{z}} - \mu(z)\omega_z = \sigma(z)q(\omega(z)), \quad z \in D, \tag{1.1}$$

where the left hand side is presented by the linear Beltrami operator $\mathcal{L}(\omega) := \omega_{\bar{z}} - \mu\omega_z$ with measurable coefficient $\mu : D \to \mathbb{C}$, satisfying uniform ellipticity condition $|\mu(z)| \leq k < 1$ a.e., $\omega_{\bar{z}} := (\omega_x + i\omega_y)/2$, $\omega_z := (\omega_x - i\omega_y)/2$, $z = x + iy$, ω_x and ω_y are partial derivatives of the function ω in x and y, respectively. The non-linear part of the equation is chosen in such a way that $\sigma : D \to \mathbb{C}$ belongs to class $L_p(D)$, $p > 2$, and $q : \mathbb{C} \to \mathbb{C}$ is a continuous function, satisfying the asymptotic condition

$$\lim_{w\to\infty} \frac{q(w)}{w} = 0. \tag{1.2}$$

One of the main goals of this paper is to establish close links between semi-linear Beltrami equation (1.1) and semi-linear Poisson type equation of the form

$$\mathrm{div}\,[A(z)\,\mathrm{grad}\,U(z)] = G(z)Q(U(z)), \tag{1.3}$$

the diffusion term of which is the divergence form elliptic operator $L(u)$, whereas its reaction term $G(z)Q(U(z))$ is such that $G : D \to \mathbb{R}$ is a function of class $L_{p'}(D)$, $p' > 1$, and $Q : \mathbb{R} \to \mathbb{R}$

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stands for a continuous function such that

$$\lim_{t\to\infty} \frac{Q(t)}{t} \;=\; 0\,. \tag{1.4}$$

From now on, $A(z) = \{a_{ij}(z)\}$ is symmetric matrix function with measurable entries and $\det A(z) = 1$, satisfying the uniform ellipticity condition

$$\frac{1}{K}|\xi|^2 \le \langle A(z)\xi,\,\xi\rangle \le K|\xi|^2 \;\text{ a.e. in } D,\;\; 1\le K<\infty,\quad \forall\,\xi\in\mathbb{R}^2\,. \tag{1.5}$$

The semi-linear Poisson equation, when $A\equiv 1$ in (1.3), was studied in [15], [28] and [29]. A rather comprehensive treatment of the present state of the general theory concerning semi-linear Poisson equations can be found in the excellent books of M. Marcus and L. Véron [23] and L. Véron [34]. For the classic case $A\equiv 1$ and $Q\equiv 1$ of the Poisson equation, see e.g. the recent article [32]. The model case $G\equiv 1$ with general Q and A was first investigated in [12], see also the papers [13]–[14] and [16]-[17].

Links established by us open up new possibilities for the study both of equations (1.1) and (1.3), because one can apply a wide range of effective methods of the potential theory as well as comprehensively developed theory of quasiconformal mappings in the plane, see e.g. [1], [3], [6] and [21]. In particular, it allows us to study in detail both the regularity properties for solutions to the equations (1.1) and (1.3) and the proper representation of such solutions.

Before to formulate the main theorem on semi-linear Beltrami equation (1.1), we need to introduce some definitions. Similarly to [2], see also monograph [1], we assume that the function $\sigma:\mathbb{C}\to\mathbb{C}$ in equation (1.1) belongs to class $L_p(\mathbb{C})$ for some $p>2$ with the condition

$$k\,C_p \;<\; 1\,, \qquad k \;:=\; \|\mu\|_\infty \;<\; 1\,, \tag{1.6}$$

guaranteing the existence of suitable solutions of the equations (1.1), where C_p is the norm of the known operator $T: L_p(\mathbb{C})\to L_p(\mathbb{C})$ defined through the Cauchy principal limit of the singular integral

$$(Tg)(\zeta) \;:=\; \lim_{\varepsilon\to 0}\left\{-\frac{1}{\pi}\int\limits_{|z-\zeta|>\varepsilon} \frac{g(z)}{(z-\zeta)^2}\,dxdy\right\},\;\; z=x+iy\,. \tag{1.7}$$

As known, $\|Tg\|_2 \;=\; \|g\|_2$, i.e., $C_2=1$, and by the Riesz convexity theorem $C_p\to 1$ as $p\to 2$, see e.g. Lemma 2 in [1] and Lemma 4 in [2]. Thus, there are such p, whatever the value of k in (1.6).

Let us denote by B_p the Banach space of functions $\omega:\mathbb{C}\to\mathbb{C}$, which satisfy a Hölder condition of order $1-2/p$, which vanish at the origin, and whose generalized derivatives ω_z and $\omega_{\bar z}$ exist and belong to $L_p(\mathbb{C})$. The norm in B_p is defined by

$$\|\omega\|_{B_p} \;:=\; \sup_{\substack{z_1,z_2\in\mathbb{C},\\ z_1\ne z_2}} \frac{|\omega(z_1)-\omega(z_2)|}{|z_1-z_2|^{1-2/p}} \;+\; \|\omega_z\|_p \;+\; \|\omega_{\bar z}\|_p\,. \tag{1.8}$$

Theorem 1.1. *Let $\mu:\mathbb{C}\to\mathbb{C}$ and $\sigma:\mathbb{C}\to\mathbb{C}$ have compact supports, $\mu\in L_\infty(\mathbb{C})$ with $k:=\|\mu\|_\infty<1$, $\sigma\in L_p(\mathbb{C})$ for some $p>2$ satisfying (1.6). Suppose that $q:\mathbb{C}\to\mathbb{C}$ is a continuous function satisfying condition (1.2). Then the semi-linear Beltrami equation (1.1) has a solution ω of the class $B_p(\mathbb{C})$.*

Moreover, we show that the given solution ω has the representation as a composition $H\circ f$, where f stands for a suitable quasiconformal mapping and H is a generalized analytic function, see Section 2 and Remark 4.3.

Theorem 1.1 together with the standard complexification approach allows us to prove the corresponding existence, representation and regularity result for semi-linear Poisson type equations (1.3), see Theorem 5.2.

The paper is organized as follows. Section 2 contains some definitions and preliminary results. The factoring of solutions for the semi-linear Beltrami equations (1.1) can be found in Section 3. The proof of Theorem 1.1 is given in Section 4. Section 5 includes the statement and the proof of Theorem 5.2. Finally, in Section 6 we apply Theorem 5.2 for several model equations describing some physical phenomena in anisotropic and inhomogeneous media.

2. Definitions and preliminary results

Recall that monograph [33] was devoted to **generalized analytic functions**, i.e., continuous complex valued functions $H(z)$ of one complex variable $z = x + iy$ of class $W^{1,1}_{\rm loc}$ in a domain D satisfying the equations

$$\partial_{\bar z} H \; + \; aH \; + \; b\overline{H} \; = \; S \, , \quad \partial_{\bar z} \; := (\partial_x + i\partial_y)/2 \, , \tag{2.9}$$

with complex valued coefficients $a, b, S \in L_p(D)$, $p > 2$. If $a \equiv 0 \equiv b$, then H will be called **generalized analytic functions with sources** S. Later on, we also need some results on the **semi-linear Vekua type equation**

$$\partial_{\bar z} H(z) \; = \; g(z) \cdot q(H(z)) \tag{2.10}$$

that have been obtained in our preceding papers [17], [18] and [28].

According to the works [15] and [28], a continuous function $h : D \to \mathbb{R}$ of class $W^{2,p}_{\rm loc}$ is also called a **generalized harmonic function with a source** $s \colon D \to \mathbb{R}$ in $L_p(D)$, $p > 2$, if h a.e. satisfies the Poisson equation

$$\triangle h(z) \; = \; s(z) \, , \tag{2.11}$$

where, as usual, $\triangle := \partial^2/\partial x^2 + \partial^2/\partial y^2$, $z = x + iy$, is the Laplacian. Note that by the Sobolev embedding theorem, see Theorem I.10.2 in [31], such functions h belong to the class C^1.

Let H be a generalized analytic function with a complex valued source S. Then we say that a function $h : D \to \mathbb{R}$ is a **weak generalized harmonic function with the source** S, if $h = {\rm Re}\, H$.

It is well known that the homogeneous Beltrami equation

$$f_{\bar z} = \mu(z) f_z \tag{2.12}$$

is the basic equation in analytic theory of quasiconformal and quasiregular mappings in the plane with numerous applications in nonlinear elasticity, gas flow, hydrodynamics and other sections of natural sciences. For the corresponding quasilinear homogeneous Beltrami equations, when the complex coefficient μ depends not only on z but also on f, see the recent papers [10] and [30].

Recall that the equation (2.12) is said to be **nondegenerate** or uniformly elliptic if $||\mu||_\infty < 1$, i.e., if $K_\mu \in L_\infty$,

$$K_\mu(z) \; := \; \frac{1 \; + \; |\mu(z)|}{1 \; - \; |\mu(z)|} \, . \tag{2.13}$$

Homeomorphic solutions f of nondegenerate equation (2.12) of the class $W^{1,2}_{\rm loc}$ are called **quasiconformal mappings** or sometimes $\mu-$**conformal mappings**. Its continuous solutions in $W^{1,2}_{\rm loc}$ are called $\mu-$**conformal functions**. On the corresponding existence theorems for nondegenerate Beltrami equation (2.12), see e.g. [1], [6] and [21].

The **inhomogeneous Beltrami equations**

$$\omega_{\bar{z}} \ = \ \mu(z)\cdot\omega_z \ + \ \sigma(z) \tag{2.14}$$

have been introduced and investigated by L. Ahlfors and L. Bers in paper [2], see also the Ahlfors monograph [1].

One of the principal results in [2], Theorem 1, is the following statement:

Theorem A. *Let $\sigma\in L_p(\mathbb{C})$ for $p>2$, satisfying condition (1.6). Then the equation (2.14) has a unique solution $\omega^{\mu,\sigma}\in B_p$. This is its only solution with $\omega(0)=0$ and $\omega_z\in L_p(\mathbb{C})$.*

As a consequence one deduces, see Theorem 4 and Lemma 8 in [2],

Theorem B. *Let $\mu:\mathbb{C}\to\mathbb{C}$ be in $L_\infty(\mathbb{C})$ with compact support and $\|\mu\|_\infty<1$. Then there exists a unique $\mu-$conformal mapping f^μ in $\mathbb{C}$ which vanishes at the origin and satisfies condition $f^\mu_z-1\in L_p(\mathbb{C})$ for any $p>2$, satisfying (1.6). Moreover, $f^\mu(z)=z+\omega^{\mu,\mu}(z)$.*

3. Factoring of solutions to semi-linear Beltrami equations

Let us start with the following factorization lemma for the linear inhomogeneous Beltrami equations (2.14).

Lemma 3.1. *Let D be a bounded domain in $\mathbb{C}$, $\mu:D\to\mathbb{C}$ be in class $L_\infty(D)$ with $k:=\|\mu\|_\infty<1$, $\sigma:D\to\mathbb{C}$ be in class $L_p(D)$, $p>2$, with condition (1.6). Suppose that $f^\mu:\mathbb{C}\to\mathbb{C}$ is the $\mu-$conformal mapping from Theorem B with an arbitrary extension of μ onto $\mathbb{C}$ keeping compact support and condition (1.6).*

Then each continuous solution ω of equation (2.14) in D of class $W^{1,p}(D)$ has the representation as a composition $H\circ f^\mu|_D$, where H is a generalized analytic function in $D_*:=f^\mu(D)$ with the source $g\in L_{p_*}(D_*)$, $p_*:=p^2/2(p-1)\in(2,p)$,

$$g \ := \ \left(\frac{f^\mu_z}{J^\mu}\cdot\sigma\right)\circ(f^\mu)^{-1}\ , \tag{3.15}$$

where J^μ is the Jacobian of f^μ.

Vice versa, if H is a generalized analytic function with the source $g\in L_{p_*}(D_*)$, $p_*>2$, in (3.15), then $\omega:=H\circ f^\mu$ is a solution of (2.14) of class $C^\alpha_{\rm loc}\cap W^{1,p^*}_{\rm loc}(D)$, where $\alpha=1-2/p^*$ and $p^*:=p_*^2/2(p_*-1)\in(2,p_*)$.

Proof. To be short, let us apply here the notation f instead of f^μ. Let us consider the function $H:=\omega\circ f^{-1}$. First of all, note that by point (iii) of Theorem 5 in [2] $f^*:=f^{-1}|_{D^*}$, $D^*:=f(D)$, is of class $W^{1,p}(D^*)$. Then, arguing in a perfectly similar way as under the proof of Lemma 10 in [2], we obtain that $H\in W^{1,p_*}(D^*)$, where $p_*:=p^2/2(p-1)\in(2,p)$. Hence it has no sense to repeat these arguments here. Since $\omega=H\circ f$, we get also, see e.g. formulas (28) in [2], see also formulas I.C(1) in [1], that

$$\omega_z \ = \ (H_\zeta\circ f)\cdot f_z \ + \ (H_{\bar\zeta}\circ f)\cdot\overline{f_{\bar z}}\ ,$$
$$\omega_{\bar z} \ = \ (H_\zeta\circ f)\cdot f_{\bar z} \ + \ (H_{\bar\zeta}\circ f)\cdot\overline{f_z}\ ,$$

and, thus,

$$\sigma(z) \ = \ \omega_{\bar z} \ - \ \mu(z)\,\omega_z \ = \ (H_{\bar\zeta}\circ f)\,\overline{f_z}\,(1-|\mu(z)|^2) \ = \ (H_{\bar\zeta}\circ f)\,J(z)/f_z\ ,$$

where $J(z)=|f_z|^2-|f_{\bar z}|^2=|f_z|^2(1-|\mu(z)|^2)$ is the Jacobian of f, i.e.,

$$H_{\bar\zeta} \ = \ g(\zeta) \ := \ \left(\frac{f_z}{J}\cdot\sigma\right)\circ f^{-1}(\zeta)\ .$$

Similarly, applying Lemma 10 in [2] and the Sobolev embedding theorem, see Theorem I.10.2 in [31], we come to the inverse conclusion. □

Remark 3.1. *Note that if H is a generalized analytic function with the source g in the domain D_*, then $h = H + \mathcal{A}$ is so for any analytic function $\mathcal{A}$ in D_*, but $|\mathcal{A}'|^{p_*}$ can be integrable only locally in D_*. By Lemma 3.1, the source in (3.15) is always in class $L_{p_*}(D_*)$, $p_* := p^2/2(p-1) \in (2, p)$, in view of Theorem A with σ extended onto $\mathbb{C}$ by zero outside of D. Here we may assume that μ is extended onto $\mathbb{C}$ by zero outside of D. However, any other extension of μ keeping condition (1.6) is suitable here, too. Moreover, we may apply here as f^μ any $\mu-$conformal mappings with different normalizations, in particular, with the hydrodynamic normalization $f^\mu(z) = z + o(1)$ as $z \to \infty$.*

Next statement makes it is possible to reduce the study of the semi-linear Beltrami equations (1.1) to the study of the corresponding semi-linear Vekua type equations (2.10).

Lemma 3.2. *Let D be a bounded domain in $\mathbb{C}$, $\mu : D \to \mathbb{C}$ be measurable with $\|\mu\|_\infty < 1$, $\sigma : D \to \mathbb{C}$ be in class $L_p(D)$, $p > 2$. Suppose that $q : \mathbb{C} \to \mathbb{C}$ is continuous and $f^\mu : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping from Theorem B with an arbitrary extension of μ onto $\mathbb{C}$ keeping compact support and condition (1.6).*

Then each continuous solution ω of equation (1.1) in D of class $W^{1,p}(D)$ has the representation as a composition $H \circ f^\mu|_D$, where H is a continuous solution of (2.10) in class $W^{1,p_*}_{\rm loc}(D_*)$, where $D_* := f^\mu(D)$, $p_* := p^2/2(p-1) \in (2, p)$, with the multiplier g in (2.10) of class $L_{p_*}(D_*)$ defined by formula (3.15).

Vice versa, if H is a continuous solution in class $W^{1,p_*}_{\rm loc}(D_*)$ of (2.10) with multiplier $g \in L_{p_*}(D_*)$, $p_* > 2$, given by (3.15), then $\omega := H \circ f^\mu$ is a solution of (1.1) in class $C^\alpha_{\rm loc} \cap W^{1,p^*}_{\rm loc}(D)$, $\alpha = 1-2/p^*$, where $p^* := p_*^2/2(p_*-1) \in (2, p_*)$.

Proof. Indeed, if ω is a continuous solution of (1.1) in D of class $W^{1,p}(D)$, then ω is a solution of (2.14) in D with the source $\Sigma := \sigma \cdot q \circ \omega$ in the same class. Then by Lemma 3.1 and Remark 3.1 $\omega = H \circ f^\mu$, where H is a generalized analytic function with the source G of class $L_{p_*}(D_*)$ after replacement of σ by Σ in (3.15). Note that $H \in W^{1,p_*}_{\rm loc}(D_*)$, see e.g. Theorems 1.16 and 1.37 in [33]. The proof of the vice versa conclusion of Lemma 3.2 is similar and it is again based on its reduction to Lemma 3.1. □

4. On solutions of semi-linear Beltrami equations

First of all, recall that a **completely continuous** mapping from a metric space M_1 into a metric space M_2 is defined as a continuous mapping on M_1 which takes bounded subsets of M_1 into relatively compact subsets of M_2, i.e., with compact closures in space M_2. When a continuous mapping takes M_1 into a relatively compact subset of M_1, it is nowadays said to be **compact** on M_1. Note that the notion of completely continuous (compact) operators is due essentially to Hilbert in a special space that, in reflexive spaces, is equivalent to Definition VI.5.1 for the Banach spaces in [11], which is due to F. Riesz, see also further comments of Section VI.12 in [11].

Recall some further definitions and one fundamental result of the celebrated paper [22]. Leray and Schauder extend as follows the Brouwer degree, see e.g. [7] and [9], to compact perturbations of the identity I in a Banach space B, i.e., a complete normed linear space. Namely, given an open bounded set $\Omega \subset B$, a compact mapping $F : B \to B$ and $z \notin \Phi(\partial\Omega)$, $\Phi := I - F$, the **(Leray-Schauder) topological degree** $\deg\,[\Phi, \Omega, z]$ of Φ in Ω over z is constructed from the Brouwer degree by approximating the mapping F over Ω by mappings F_ε with range in a finite-dimensional subspace B_ε (containing z) of B. It is showing that the Brouwer degrees $\deg\,[\Phi_\varepsilon, \Omega_\varepsilon, z]$ of $\Phi_\varepsilon := I_\varepsilon - F_\varepsilon$, $I_\varepsilon := I|_{B_\varepsilon}$, in $\Omega_\varepsilon := \Omega \cap B_\varepsilon$ over z stabilize for sufficiently small positive ε to a common value defining $\deg\,[\Phi, \Omega, z]$ of Φ in Ω over z.

This topological degree algebraically counts the number of fixed points of $F(\cdot) - z$ in Ω and conserves the basic properties of the Brouwer degree as additivity and homotopy invariance. Now, let a be an isolated fixed point of F. Then the **local (Leray-Schauder) index** of a is defined by $\mathrm{ind}\,[\Phi, a] := \deg[\Phi, B(a,r), 0]$ for small enough $r > 0$. $\mathrm{ind}\,[\Phi, 0]$ is called by **index** of F. In particular, if $F \equiv 0$, correspondingly, $\Phi \equiv I$, then the index of F is equal to 1. For our goals, we need only the latter fact from the index theory.

Now, let us formulate one of the main results in the Leray-Schauder article [22], Theorem 1, see also the survey [25].

Proposition 4.1. *Let B be a Banach space, and let $F(\cdot,\tau) : B \to B$ be a family of operators with $\tau \in [0,1]$. Suppose that the following hypotheses hold:*

(H1) $F(\cdot,\tau)$ is completely continuous on B for each $\tau \in [0,1]$ and uniformly continuous with respect to the parameter $\tau \in [0,1]$ on each bounded set in B;

(H2) the operator $F := F(\cdot,0)$ has finite collection of fixed points whose total index is not equal to zero;

(H3) the collection of all fixed points of the operators $F(\cdot,\tau)$, $\tau \in [0,1]$, is bounded in B.

Then the collection of all fixed points of the family of operators $F(\cdot,\tau)$ contains a continuum along which τ takes all values in $[0,1]$.

For introduction in the modern fixed point theory, see e.g. survey [20] and monograph [26].

Remark 4.2. *By Lemma 5 in [2] the mapping $\sigma \to \omega^{\mu,\sigma}$ from Theorem A is a bounded linear operator from $L_p(\mathbb{C})$ to $B_p(\mathbb{C})$ with a bound that depends only on k and p in (1.6). In particular, this is a bounded linear operator from $L_p(\mathbb{C})$ to $C(\mathbb{C})$. Namely, by (15) in [2] we have that $\omega^{\mu,\sigma}$ is Hölder continuous:*

$$|\omega^{\mu,\sigma}(z_1) - \omega^{\mu,\sigma}(z_2)| \leq c \cdot \|\sigma\|_p \cdot |z_1 - z_2|^{1-2/p} \quad \forall\, z_1\ z_2 \in \mathbb{C}\,, \tag{4.16}$$

where the constant c may depend only on k and p in (1.6). Moreover, $\omega^{\mu,\sigma}(z)$ is locally bounded because $\omega^{\mu,\sigma}(0) = 0$. Thus, the linear operator $\sigma \to \omega^{\mu,\sigma}|_S$ is completely continuous for each compact set S in $\mathbb{C}$ by Arzela-Ascoli theorem, see e.g. Theorem IV.6.7 in [11].

Finally, we are ready to give a proof of Theorem 1.1.

Proof for Theorem 1.1. If $\|\sigma\|_p = 0$ or $\|q\|_C = 0$, then Theorem A above gives the desired solution $\omega := \omega^{\mu,0}$ of equation (1.1). Thus, we may assume that $\|\sigma\|_p \neq 0$ and $\|q\|_C \neq 0$. Set $q_*(t) = \max\limits_{|w|\leq t} |q(w)|$, $t \in \mathbb{R}^+ := [0,\infty)$. Then the function $q_* : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing and, moreover, by (1.2)

$$\lim_{t\to\infty} \frac{q_*(t)}{t} = 0\,. \tag{4.17}$$

Let us show that the family of operators $F(g;\tau) : L_p^\sigma(\mathbb{C}) \to L_p^\sigma(\mathbb{C})$,

$$F(g;\tau) := \tau\sigma \cdot q(\omega^{\mu,g}) \qquad \forall\, \tau \in [0,1]\,, \tag{4.18}$$

where $L_p^\sigma(\mathbb{C})$ consists of functions $g \in L_p(\mathbb{C})$ with supports in the support S of σ, satisfies hypotheses H1-H3 of Theorem 1 in [22], see Proposition 4.1 above. Indeed:

H1). First of all, the function $F(g;\tau) \in L_p^\sigma(\mathbb{C})$ for all $\tau \in [0,1]$ and $g \in L_p^\sigma(\mathbb{C})$ because the function $q(\omega^{\mu,g})$ is continuous and, furthermore, the operators $F(\cdot;\tau)$ are completely continuous for each $\tau \in [0,1]$ and even uniformly continuous with respect to parameter $\tau \in [0,1]$ by Theorem A and Remark 4.2.

H2). The index of the operator $F(g;0)$ is obviously equal to 1.

H3). Let us assume that the collection of all solutions of the equations $g = F(g;\tau)$, $\tau \in [0,1]$, is not bounded in $L_p^\sigma(\mathbb{C})$, i.e., there is a sequence of functions $g_n \in L_p^\sigma(\mathbb{C})$ with $\|g_n\|_p \to \infty$ as $n \to \infty$ such that $g_n = F(g_n;\tau_n)$ for some $\tau_n \in [0,1]$, $n = 1, 2, \ldots$.

However, then by Remark 4.2, we have that

$$\|g_n\|_p \;\leq\; \|\sigma\|_p \, q_*\left(\|\omega^{\mu,g_n}|_S\|_C\right) \;\leq\; \|\sigma\|_p \, q_*\left(M\,\|g_n\|_p\right)$$

for some constant $M > 0$ and, consequently,

$$\frac{q_*(M\,\|g_n\|_p)}{M\,\|g_n\|_p} \;\geq\; \frac{1}{M\,\|\sigma\|_p} \;>\; 0\,. \tag{4.19}$$

The latter is impossible by condition (4.17). The obtained contradiction disproves the above assumption.

Thus, by Theorem 1 in [22], see Proposition 4.1 above, there is a function $g \in L_p^{\sigma}(\mathbb{C})$ with $F(g;1) = g$, and then by Theorem A the function $\omega := \omega^{\mu,g}$ gives the desired solution of (1.1). □

Remark 4.3. *By Lemma 3.2, ω has the representation as a composition $H \circ f^{\mu}$, where $f^{\mu} : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping from Theorem B and H is a continuous solution of (2.10) in class $W^{1,p_*}_{\rm loc}(\mathbb{C})$, $p_* := p^2/2(p-1) \in (2,p)$, with the multiplier g in (2.10) of class $L_{p_*}(\mathbb{C})$ defined by formula (3.15). Note also that H is a generalized analytic function with a source of the same class.*

Let us also give the following lemma on semi-linear Beltrami equations that may be of independent interest and will be first applied in the next section.

Lemma 4.3. *Let D be a bounded domain in $\mathbb{C}$, $\mu : D \to \mathbb{C}$ in class $L_\infty(D)$, $k := \|\mu\|_\infty < 1$, $G : D \to \mathbb{C}$ be in class $L_{p'}(D)$ for some $p' > 1$ and $\mathcal{L} : L_{p'}(D) \to L_p(D)$ be a linear bounded operator for some $p > 2$ satisfying (1.6).*

Suppose that $q : \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying condition (1.2). Then the semi-linear Beltrami equation of the form

$$\omega_{\bar z} \;=\; \mu(z)\cdot\omega_z \;+\; \mathcal{L}[Gq(\omega)](z)\,, \qquad z\in D, \tag{4.20}$$

has a solution ω of class $C^{\alpha}(D)\cap W^{1,p}(D)$ with $\alpha = 1-2/p$.

Proof. Indeed, arguing perfectly similar to the proof of Theorem 1.1 for

$$F(g;\tau) \;:=\; \mathcal{L}\left[\tau G q(\omega^{\mu,g})\right] \;:\; L_p(D) \to L_p(D)\,, \qquad \tau\in[0,1] \tag{4.21}$$

with μ, G and g extended by zero outside of D, we see that the family of the operators $F(g;\tau)$, $\tau \in [0,1]$, satisfies all the hypotheses of Theorem 1 in [22], see Proposition 4.1 above. Thus, there is $g \in L_p(\mathbb{C})$ with $F(g;1) = g$, and then by Theorem A the function $\omega := \omega^{\mu,g}|_D$ gives the desired solution of (4.20). □

Remark 4.4. *Moreover, arguing similarly to the proofs of Lemmas 3.1 and 3.2 one can show that $\omega = H\circ f^{\mu}|_D$, where $f^{\mu} : \mathbb{C}\to\mathbb{C}$ is a $\mu-$conformal mapping from Theorem B with μ extended onto $\mathbb{C}$ by zero outside of D and $H : D_* \to \mathbb{C}$ is a generalized analytic function in the domain $D_* := f^{\mu}(D)$ with the source*

$$S \;:=\; \left\{ \frac{f^{\mu}_z}{J^{\mu}}\cdot\mathcal{L}\left[Gq(\omega)\right]\right\}\circ\left(f^{\mu}\right)^{-1} \;\in\; L_{p_*}(D_*)\,, \tag{4.22}$$

where J^{μ} is the Jacobian of f^{μ} and $p_* := p^2/2(p-1) \in (2,p)$.

5. Toward semi-linear Poisson type equations

For convenience of presentation, let us denote by $\mathbb{S}^{2\times 2}$ the collection of all 2×2 matrices with real valued elements

$$A \;=\; \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{5.23}$$

which are symmetric, i.e., $a_{12} = a_{21}$, with $\det A = 1$ and **ellipticity condition** $\det(I + A) > 0$, where I is the unit 2×2 matrix. The latter condition means in terms of elements of A that $(1 + a_{11})(1 + a_{22}) > a_{12}a_{21}$.

Now, let us first consider in a domain D of the complex plane $\mathbb{C}$ the linear Poisson type equation

$$\mathrm{div}\,[A(z)\,\nabla\,u(z)] \;=\; g(z)\;, \tag{5.24}$$

where $A : D \to \mathbb{S}^{2\times 2}$ is a measurable matrix valued function whose elements $a_{ij}(z)$, $i, j = 1, 2$ are bounded, $g : D \to \mathbb{R}$ is a scalar function in $L_{1,\mathrm{loc}}$, and here and further ∇ denotes the gradient of the corresponding functions.

Note that (5.24) is one of the main equations of hydromechanics (fluid mechanics) in anisotropic and inhomogeneous media.

We say that a function $u : D \to \mathbb{R}$ is a **generalized A−harmonic function with the source g**, cf. [19], if u is a weak solution of (5.24), i.e., if $u \in W^{1,1}_{\mathrm{loc}}(D)$ and

$$\int\limits_D \langle A(z)\nabla u(z), \nabla\psi(z)\rangle\, d\,m(z) + \int\limits_D g(z)\,\psi(z)\,d\,m(z) = \;0 \tag{5.25}$$

for all $\psi \in C^\infty_0(D)$, where $C^\infty_0(D)$ denotes the collection of all infinitely differentiable functions $\psi : D \to \mathbb{R}$ with compact support in D, $\langle a, b\rangle$ means the scalar product of vectors a and b in $\mathbb{R}^2$, and $d\,m(z)$ corresponds to the Lebesgue measure in the plane $\mathbb{C}$.

Later on, we use the **logarithmic (Newtonian) potential of sources** $g \in L_1(\mathbb{C})$ with compact supports given by the formula:

$$\mathcal{N}^g(z) \;:=\; \frac{1}{2\pi}\int\limits_{\mathbb{C}} \ln|z - w|\, g(w)\, d\,m(w)\;. \tag{5.26}$$

By Lemmas 3 in [14] and in [15], we have its following basic properties.

Remark 5.5. *Let $g : \mathbb{C} \to \mathbb{R}$ has compact support. If $g \in L_1(\mathbb{C})$, then $\mathcal{N}^g \in L_{r,\mathrm{loc}}(\mathbb{C})$ for all $r \in [1, \infty)$, $\mathcal{N}^g \in W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ for all $p \in [1, 2)$, moreover, there exist generalized derivatives by Sobolev $\frac{\partial^2 N^g}{\partial z \partial\bar z}$ and $\frac{\partial^2 N^g}{\partial\bar z\partial z}$ satisfying the equalities, where $\triangle := \partial^2/\partial x^2 + \partial^2/\partial y^2$, $z = x + iy$, is the Laplacian,*

$$4\cdot\frac{\partial^2 N^g}{\partial z\partial\bar z} \;=\; \triangle N^g \;=\; 4\cdot\frac{\partial^2 N^g}{\partial\bar z\partial z} \;=\; g \quad a.e.\,. \tag{5.27}$$

Furthermore, if $g \in L_{p'}(\mathbb{C})$ for some $p' > 1$, then $\mathcal{N}^g \in W^{2,p'}_{\mathrm{loc}}(\mathbb{C})$, moreover, $\mathcal{N}^g \in W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ for some $p > 2$ and, consequently, $\mathcal{N}^g \in C^\alpha_{\mathrm{loc}}(\mathbb{C})$ with $\alpha = 1 - 2/p$. Finally, if $g \in L_{p'}(\mathbb{C})$ for some $p' > 2$, then $\mathcal{N}^g \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{C})$ with $\alpha = 1 - 2/p'$.

Next, we say that a function $v : D \to \mathbb{R}$ is **A−conjugate of a generalized A−harmonic function u with a source** $g : D \to \mathbb{R}$ if $v \in W^{1,1}_{\mathrm{loc}}(D)$ and

$$\nabla v(z) \;=\; \mathbb{H}\,[\,A(z)\nabla u(z) \;-\; \nabla\mathcal{N}^g(z)\,] \quad \text{a.e.}\,, \qquad \mathbb{H} \;:=\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{5.28}$$

Lemma 5.4. *Let D be a bounded domain in $\mathbb{C}$, $g : D \to \mathbb{R}$ be in $L_1(D)$ and let u be a weak solution of equation (5.24) with a matrix function $A : D \to \mathbb{S}^{2\times 2}$ whose elements $a_{ij}(z)$, $i, j = 1, 2$ are bounded and measurable.*

If v is $A-$conjugate of u, then $\omega := u + iv$ satisfies the nondegenerate inhomogeneous Beltrami equation (2.14) with

$$\mu(z) \ := \ \mu_A(z) \ = \frac{1}{\det\,[\,I + A(z)\,]}\ [\,a_{22}(z) - a_{11}(z) \ - \ 2ia_{21}(z)\,]\ , \tag{5.29}$$

$$\sigma(z) \ := \ \mathcal{N}^g_{\bar z}(z) \ + \ \mu(z)\,\overline{\mathcal{N}^g_{\bar z}(z)}\ . \tag{5.30}$$

Conversely, if $\omega \in W^{1,1}_{\mathrm{loc}}(D)$ is a solution of the nondegenerate inhomogeneous Beltrami equation (2.14) with σ given by (5.30), then $u := \mathrm{Re}\,\omega$ is a weak solution of equation (5.24) with the matrix valued function $A : D \to \mathbb{S}^{2\times 2}$,

$$A(z) \ := \left[\begin{array}{cc} \frac{|1-\mu(z)|^2}{1-|\mu(z)|^2} & \frac{-2\mathrm{Im}\,\mu(z)}{1-|\mu(z)|^2} \\ \frac{-2\mathrm{Im}\,\mu(z)}{1-|\mu(z)|^2} & \frac{|1+\mu(z)|^2}{1-|\mu(z)|^2} \end{array}\right]\ , \tag{5.31}$$

whose elements are bounded and measurable.

Remark 5.6. *Hence, in the case $A \equiv I$ and $g \in L_{p'}(D)$, $p' > 2$, we conclude that every generalized harmonic function u with the source g is a weak generalized harmonic function with the same source, see e.g. Theorem 1.16 in [33]. The inverse conclusion is, generally speaking, not true and has no sense at all because in the weak case the source can be complex, not real.*

Proof of Lemma 5.4. Indeed, let u be a weak solution of equation (5.24) with $g : D \to \mathbb{R}$ in $L^1(D)$ and a matrix function $A : D \to \mathbb{S}^{2\times 2}$ whose elements are bounded and measurable. Then by (5.27), because the Laplacian $\triangle = \mathrm{div}\,\mathrm{grad}$, we have that u is a weak solution of the equation

$$\mathrm{div}\,[A(z)\,\nabla\,u(z)] \ = \ \mathrm{div}\,\nabla\mathcal{N}^g(z)\ . \tag{5.32}$$

If v is $A-$conjugate of u, then by Theorem 16.1.6 in [3] the function $\omega := u + iv$ satisfies the nondegenerate inhomogeneous Beltrami equation (2.14) with μ and σ given by (5.29) and (5.30).

Conversely, if $\omega \in W^{1,1}_{\mathrm{loc}}(D)$ is a solution of the nondegenerate inhomogeneous Beltrami equation (2.14) with σ given by (5.30), then, again by Theorem 16.1.6 in [3], the functions $u := \mathrm{Re}\,\omega$ and $v := \mathrm{Im}\,\omega$ satisfy the relation (5.28) with the matrix function $A : D \to \mathbb{S}^{2\times 2}$ given by (5.31) whose elements $a_{ij}(z)$ are measurable in $z \in D$ and bounded because $|a_{ij}| \leq \|K_\mu\|_\infty$. Note that (5.28) is equivalent to the equation

$$A(z)\nabla\,u(z) \ - \ \nabla\mathcal{N}^g(z) \ = \ -\mathbb{H}\nabla\,v(z) \tag{5.33}$$

because $\mathbb{H}^2 = -I$. As known, the curl of any gradient field is zero in the sense of distributions and, moreover, the Hodge operator $\mathbb{H}$ transforms curl-free fields into divergence-free fields, and vice versa, see e.g. 16.1.3 in [3]. Hence u is a weak solution of equation (5.32) as well as of (5.24) in view of (5.27). □

Further we say that a function $u : D \to \mathbb{R}$ is a **weak solution** of (1.3), if $u \in W^{1,1}_{\mathrm{loc}}(D)$ and

$$\int\limits_D \langle A(z)\nabla u(z), \nabla\psi(z)\rangle\, dm(z) \ + \int\limits_D G(z)\, Q(u(z))\, \psi(z)\, dm(z) \ = \ 0 \tag{5.34}$$

for all $\psi \in C_0^\infty(D)$, where $C_0^\infty(D)$ denotes the collection of all infinitely differentiable functions $\psi : D \to \mathbb{R}$ with compact support in D, $\langle a, b\rangle$ means the scalar product of vectors a and b in $\mathbb{R}^2$, and $dm(z)$ corresponds to the Lebesgue measure in the plane $\mathbb{C}$.

Theorem 5.2. *Let D be a bounded domain in $\mathbb{C}$, a scalar function $G : D \to \mathbb{R}$ be in class $L_{p'}(D)$ for some $p' > 1$, a continuous function $Q : \mathbb{R} \to \mathbb{R}$ satisfy condition (1.4) and let $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function whose elements $a_{ij}(z)$, $i, j = 1, 2$ are bounded and measurable.*

Then the semi-linear Poisson type equation (1.3) has a weak solution u of class $C^\alpha(D) \cap W^{1,p}(D)$ with $\alpha = 1 - 2/p$ for some $p > 2$.

Moreover, $u = h \circ f^\mu|_D$, where $f^\mu : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping from Theorem B and h is a weak generalized harmonic function in the domain $D_* := f^\mu(D)$ with the source

$$S \;:=\; \left\{ \frac{f^\mu_z}{J^\mu} \cdot \sigma \right\} \circ (f^\mu)^{-1} \in\; L_{p_*}(D_*)\,, \tag{5.35}$$

where J^μ is the Jacobian of f^μ, $p_* := p^2/2(p-1) \in (2, p)$, μ is defined by formula (5.29) and σ is calculated by formula (5.30) with $g = G\,Q(u)$.

As it is clear from the proof below the degree $p > 2$ and the exponent $\alpha \in (0, 1)$ of the Hölder continuity, correspondingly, cannot be connected with p' in an explicit form.

Proof. With no loss of generality, we may assume here that $p' \in (1, 2]$ and that $g \equiv 0$ outside of D, and then $\mathcal{N}^g \in W^{1,p}(D)$ for all $p \in (1, p^*)$, where $p^* = 2p'/(2 - p') > 2$, see Lemma 3 in [14]. Hence later on, we may also assume that $p > 2$ satisfies condition (1.6) for μ in (5.29). Moreover, again by Lemma 3 in [14], the correspondence $g \to \mathcal{N}^g_{\bar z}$ generates a completely continuous linear operator L acting from real valued $L_{p'}(D)$ to complex valued $L_p(D)$. Thus, the linear operator $\mathcal{L} := L + \mu\bar{L}$ with the multiplier $\mu \in L_\infty(D)$ is bounded. Then by Lemma 4.3, the semi-linear Beltrami equation (4.20) with $q(\omega) := Q(\mathrm{Re}\,\omega)$ has a solution ω of class $C^\alpha(D) \cap W^{1,p}(D)$ with $\alpha = 1 - 2/p$. Moreover, by Lemma 5.4, the function $u := \mathrm{Re}\,\omega$ is a weak solution of equation (5.24) of the given class. Finally, by Lemma 3.1, we conclude that u has the representation as the composition $h \circ f^\mu|_D$, where $f^\mu : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping from Theorem B and h is a weak generalized harmonic function in the domain $D_* := f^\mu(D)$ with the source (5.35). □

6. Some examples of applications

We apply Theorem 5.2 for several model equations describing some physical phenomena in anisotropic and inhomogeneous media.

The first group of such applications is relevant to reaction-diffusion problems. Problems of this type are discussed in [8], p. 4, and, in details, in [4]. A nonlinear system is obtained for the density U and the temperature T of the reactant. Upon eliminating T the system can be reduced to equations of the form

$$\triangle U \;=\; \sigma \cdot Q(U) \tag{6.36}$$

with $\sigma > 0$ and, for isothermal reactions, $Q(U) = U^\lambda$, where $\lambda > 0$ that is called the order of the reaction. It turns out that the density of the reactant U may be zero in a subdomain called a **dead core**. A particularization of results in Chapter 1 of [8] shows that a dead core may exist just if and only if $\beta \in (0, 1)$, see also the corresponding examples in [13].

In the case of anisotropic and inhomogeneous media, we come to the semi-linear Poisson type equations (1.3). In this connection, the following statement may be of independent interest.

Corollary 6.1. *Let D be a bounded domain in $\mathbb{C}$, a scalar function $\sigma : D \to \mathbb{R}$ be in class $L_{p'}(D)$ for some $p' > 1$ and let $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function whose elements $a_{ij}(z)$, $i, j = 1, 2$ are bounded and measurable.*

Then there is a weak solution $u : D \to \mathbb{R}$ of class $C^{\alpha}_{\rm loc} \cap W^{1,p}_{\rm loc}$ with $\alpha = 1 - 2/p$ for some $p > 2$ to the semi-linear Poisson type equation

$$\text{(6.37)} \qquad {\rm div}\,[A(z)\,\nabla\,u(z)] \;=\; \sigma(z)\cdot u^{\lambda}(z)\,, \quad 0\,<\,\lambda\,<\,1\,, \qquad \text{a.e. in } D.$$

Note also that certain mathematical models of a thermal evolution of a heated plasma lead to nonlinear equations of the type (6.36). Indeed, it is known that some of them have the form $\triangle\psi(u) = f(u)$ with $\psi'(0) = \infty$ and $\psi'(u) > 0$ if $u \neq 0$ as, for instance, $\psi(u) = |u|^{q-1}u$ under $0 < q < 1$, see e.g. [8]. With the replacement of the function $U = \psi(u) = |u|^q \cdot {\rm sign}\,u$, we have that $u = |U|^Q \cdot {\rm sign}\,U$, $Q = 1/q$, and, with the choice $f(u) = |u|^{q^2} \cdot {\rm sign}\,u$, we come to the equation $\triangle U = |U|^q \cdot {\rm sign}\,U = \psi(U)$. For anisotropic and inhomogeneous media, we obtain the corresponding equation (6.38) below:

Corollary 6.2. *Let D be a bounded domain in $\mathbb{C}$, a scalar function $\sigma : D \to \mathbb{R}$ be in class $L_{p'}(D)$ for some $p' > 1$ and let $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function whose elements $a_{ij}(z)$, $i, j = 1, 2$ are bounded and measurable.*

Then there is a weak solution $u : D \to \mathbb{R}$ of class $C^{\alpha}_{\rm loc} \cap W^{1,p}_{\rm loc}$ with $\alpha = 1 - 2/p$ for some $p > 2$ to the semi-linear Poisson type equation

$$\text{(6.38)} \qquad {\rm div}\,[A(z)\,\nabla\,u(z)] \;=\; \sigma(z)\cdot|u(z)|^{\lambda-1}u(z)\,, \quad 0\,<\,\lambda\,<\,1\,, \qquad \text{a.e. in } D.$$

Finally, we recall that in the combustion theory, see e.g. [5] and [27] and the references therein, the following model equation

$$\text{(6.39)} \qquad \frac{\partial u(z,t)}{\partial t} \;=\; \frac{1}{\delta}\cdot\triangle u \;+\; e^{u}\,, \quad \delta\,>\,0\,,\; t \geq 0,\; z \in D$$

takes a special part. Here $u \geq 0$ is the temperature of the medium. We restrict ourselves here by the stationary case, although our approach makes it possible to study the parabolic equation (6.39), see [13]. The corresponding equation of the type (1.3), see (6.40) below, appears in anisotropic and inhomogeneous media with the function $Q(u) = e^{-|u|}$ that is uniformly bounded at all.

Corollary 6.3. *Let D be a bounded domain in $\mathbb{C}$, a scalar function $\sigma : D \to \mathbb{R}$ be in class $L_{p'}(D)$ for some $p' > 1$ and let $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function whose elements $a_{ij}(z)$, $i, j = 1, 2$ are bounded and measurable.*

Then there is a weak solution $u : D \to \mathbb{R}$ of class $C^{\alpha}_{\rm loc} \cap W^{1,p}_{\rm loc}$ with $\alpha = 1 - 2/p$ for some $p > 2$ to the semi-linear Poisson type equation

$$\text{(6.40)} \qquad {\rm div}\,[A(z)\,\nabla\,u(z)] \;=\; \sigma(z)\cdot e^{-|u(z)|} \qquad \text{a.e. in } D.$$

Remark 6.7. *Such solutions u in Corollaries 6.1, 6.2, 6.3 have the representation as the composition $h \circ f^{\mu}|_D$, where $f^{\mu} : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping in Theorem B with μ extended onto $\mathbb{C}$ by zero outside of D, and all h are weak generalized harmonic functions with sources of class $L_{p_*}(D_*)$, $D_* := f^{\mu}(D)$ and $p_* := p^2/2(p-1) \in (2, p)$.*

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