http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 37–63 (2024)
DOI:10.31801/cfsuasmas.1249576
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: February 9, 2023; Accepted: September 19, 2023

GENERALIZED BIVARIATE CONDITIONAL FIBONACCI AND LUCAS HYBRINOMIALS

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Abstract. The Hybrid numbers are generalizations of complex, hyperbolic and dual numbers. In recent years, studies related with hybrid numbers have been increased significantly. In this paper, we introduce the generalized bivariate conditional Fibonacci and Lucas hybrinomials. Also, we present the Binet formula, generating functions, some significant identities, Catalan's identities and Cassini's identities of the generalized bivariate conditional Fibonacci and Lucas hybrinomials. Finally, we give more general results compared to the previous works.

1. Introduction

The Fibonacci and Lucas numbers are defined by

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geqslant 2 \end{cases} \quad \text{and } L_n = \begin{cases} 2 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ L_{n-1} + L_{n-2} & \text{if } n \geqslant 2 \end{cases}, \quad (1)$$

respectively. For more information about the Fibonacci and Lucas numbers, we refer to book [9]. Until now, there have been interesting generalizations and applications of the Fibonacci and Lucas numbers [5–7, 12, 16]. For example, Falcon and Plaza found the general $k-$Fibonacci sequence $\{F_{k,n}\}_{n=0}^{\infty}$ by studying the recursive application of two geometrical transformations used in the well-known $4-$triangle longest-edge (4TLE) partition [7]. Furthermore, Edson and Yayenie [6] proposed the bi-periodic Fibonacci sequence. Also they gave generating function, the generalized Binet formula and some basic identities for q_n. By analogy to the studies [6] and [16], Bilgici [5] defined the bi-periodic Lucas numbers and he gave generating functions, the Binet formulas and some special identities for these sequences. Later,

2020 *Mathematics Subject Classification.* 05A19, 11B37.
Keywords. Bivariate conditional polynomials, hybrid numbers, Binet formula's, generating function, Catalan's identities, Cassini's identities.
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Yılmaz et al. [18] presented generalized of Fibonacci and Lucas polynomials. Also they obtained some new algebraic properties on these numbers and polynomials. Yazlık et al. introduced a novel extension of the Fibonacci and Lucas $p-$numbers and demonstrated that these generalized Fibonacci and Lucas $p-$sequences can be simplified into various other number sequences [17]. Ait-Amrane and Belbachir presented the bi-periodic $r-$Fibonacci sequence and its related family of companion sequences. They also explored the bi-periodic $r-$Lucas sequence of type s, where s ranges from 1 to r, extending the classical Fibonacci and Lucas sequences. [1]. Belbachir and Bencherif [4] have generalized to bivariate polynomials of the Fibonacci and Lucas, properties obtained for Chebyshev polynomials. Ait-Amrane et al. presented a novel extension of hybrid polynomials, which combine elements of both Fibonacci and Lucas polynomials and studied various fundamental characteristics of these polynomials, including recurrence relations, generating functions, Binet formulas, summation formulas, and a matrix representation [2]. Panwar and Singh [11] introduced a generalized bivariate Fibonacci-Like polynomials sequence. Bala and Verma [15] presented the generalized Bivariate bi-periodic Fibonacci polynomials.

For any nonzero real numbers a, b, c and d, the generalization of bivariate bi-periodic Fibonacci polynomial is defined as [15],

$$B_n(x,y) = \begin{cases} axB_{n-1}(x,y) + cyB_{n-2}(x,y), & \text{if } n \text{ is even} \\ bxB_{n-1}(x,y) + dyB_{n-2}(x,y), & \text{if } n \text{ is odd} \end{cases}, n \geq 2 \tag{2}$$

where $B_0(x,y) = 0, B_1(x,y) = 1$. Also, the authors obtained Catalan's identity, Cassini's identity, d'Ocagne identity and Gelin Cesaro identity along with Generating function and Binet's formula for the bivariate bi-periodic Fibonacci polynomial. The authors presented the generating function of the bivariate bi-periodic Fibonacci polynomial as:

$$G(t) = \frac{t + axt^2 - cyt^3}{1 - (abx^2 + (c+d)y)t^2 + cdy^2t^4}. \tag{3}$$

Moreover, they obtained Binet's formula for the bivariate bi-periodic Fibonacci polynomial as:

$$B_n(x,y) = \frac{(ax)^{1-\xi(n)}}{(abx^2)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\beta_1^{\lfloor \frac{n}{2} \rfloor}(\beta_1 + (d-c)y)^{n-\lfloor \frac{n}{2} \rfloor} - \beta_2^{\lfloor \frac{n}{2} \rfloor}(\beta_2 + (d-c)y)^{n-\lfloor \frac{n}{2} \rfloor}}{\beta_1 - \beta_2} \right). \tag{4}$$

Then, Bala and Verma [3] defined the bivariate bi-periodic Lucas polynomials as follows:

For any nonzero real numbers a_1 and a_2, the generalization of bivariate bi-periodic Lucas polynomial is defined as [3],

$$l_n(x,y) = \begin{cases} a_1xl_{n-1}(x,y) + yl_{n-2}(x,y), & \text{if } n \text{ is even} \\ a_2xl_{n-1}(x,y) + yl_{n-2}(x,y), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2 \tag{5}$$

where, $l_0(x,y) = 2, l_1(x,y) = a_2x$.

Özdemir [10] introduced the hybrid numbers as a new generalization of complex, hyperbolic and dual numbers. The set of hybrid numbers, denoted by $\mathbb{K}$, is defined as

$$\mathbb{K} = \left\{ a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h} : \quad a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \boldsymbol{\varepsilon}^2 = 0, \mathbf{h}^2 = 1, \mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i} \right\}. \quad (6)$$

The following table presents products of $\mathbf{i}$, $\boldsymbol{\varepsilon}$, and $\mathbf{h}$.

TABLE 1. Products of $\mathbf{i}$, $\boldsymbol{\varepsilon}$, and $\mathbf{h}$

$\times$	1	$\mathbf{i}$	ε	$\mathbf{h}$
1	1	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	ε	$\mathbf{h}+1$	0	$-\varepsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	1

This table shows that the multiplication operation in the hybrid numbers is not commutative, but associative. Liana [13] presented the special kind of hybrid numbers, namely Horadam hybrid numbers. Then, Kızılateş [8] obtained a new generalization of Fibonacci hybrid and Lucas hybrid numbers. He gave some algebraic properties of $q-$Fibonacci hybrid numbers and the $q-$Lucas hybrid numbers. Finally, Liana and Wloch [14] introduced the Fibonacci and Lucas hybrinomials, which can be considered as a generalization of the Fibonacci hybrid numbers and the Lucas hybrid numbers. Sevgi [12] defined the generalized Lucas hybrinomials with two variables. Also, he obtained the Binet formula, generating function and some properties for the generalized Lucas hybrinomials.

In the light of the above–cited recent works, some natural questions are that: can we define the bivariate conditional Fibonacci and Lucas Hybrinomials? Moreover, can we find the generating function, Binet formulas and some important identities for the bivariate conditional Fibonacci and Lucas Hybrinomials? In this study, we will investigate the answer to these questions.

This paper is structured in four section. First section includes preliminaries and literature review. In the second section, we define bivariate conditional Fibonacci hybrinomials and we give generating functions, Binet formulas and some important identities of these hybrinomials. In the third section, we discuss bivariate conditional Lucas polynomials and the bivariate conditional Lucas hybrinomials.

2. Generalized Bivariate Conditional Fibonacci Hybrinomials

In this section we give some identities of the generalized bivariate conditional Fibonacci hybrinomials. The next definition presents the bivariate conditional Fibonacci Hybrinomials.

Definition 1. *For any variables x, y and nonzero real numbers a, b, c and d, we have*

$$BH_n(x,y) = B_n(x,y) + \boldsymbol{i}B_{n+1}(x,y) + \varepsilon B_{n+2}(x,y) + \boldsymbol{h}B_{n+3}(x,y), \quad (7)$$

where $B_n(x,y)$ was given in (2) and the initial conditions are $BH_0(x,y) = \boldsymbol{i} + \varepsilon ax + \boldsymbol{h}(abx^2 + dy)$ and $BH_1(x,y) = 1 + \boldsymbol{i}ax + \varepsilon(abx^2 + dy) + \boldsymbol{h}(a^2bx^3 + adxy + acxy)$.

We can see from the following table that the generalized bivariate conditional Fibonacci hybrinomials are the generalization of many works for different values of a, b, c and d.

TABLE 2. The generalized bivariate conditional Fibonacci hybrinomials

a	***b***	***c***	***d***	***Generalized Bivariate Conditional Fibonacci Hybrinomials***
1	1	1	1	*Bivariate Fibonacci Hybrinomials*
a	b	1	1	*Bivariate Conditional Fibonacci Hybrinomials*
2	2	1	1	*Bivariate Pell Hybrinomials*
1	1	2	2	*Bivariate Jacobsthal Hybrinomials*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Lemma 1. *For the generalized bivariate conditional Fibonacci hybrinomials $\{BH_n(x,y)\}_{n=0}^{\infty}$, we have*

$$BH_{2n}(x,y) = \left(abx^2 + (c+d)y\right) BH_{2n-2}(x,y) - cdy^2 BH_{2n-4}(x,y)$$
$$BH_{2n+1}(x,y) = \left(abx^2 + (c+d)y\right) BH_{2n-1}(x,y) - cdy^2 BH_{2n-3}(x,y).$$

Proof. By using the definition of the generalized bivariate conditional Fibonacci hybrinomials, we obtain

$$\begin{aligned}
BH_{2n}(x,y) &= B_{2n}(x,y) + \mathbf{i}B_{2n+1}(x,y) + \varepsilon B_{2n+2}(x,y) + \mathbf{h}B_{2n+3}(x,y) \\
&= (axB_{2n-1}(x,y) + cyB_{2n-2}(x,y)) + \mathbf{i}\,(bxB_{2n}(x,y) + dyB_{2n-1}(x,y)) \\
&\quad + \varepsilon(axB_{2n+1}(x,y) + cyB_{2n}(x,y)) \\
&\quad + \mathbf{h}(bxB_{2n+2}(x,y) + dyB_{2n+1}(x,y)) \\
&= [ax\,(bxB_{2n-2}(x,y) + dyB_{2n-3}(x,y)) + cyB_{2n-2}(x,y)] \\
&\quad + \mathbf{i}\,[bx\,(axB_{2n-1}(x,y) + cyB_{2n-2}(x,y)) + dyB_{2n-1}(x,y)] \\
&\quad +\varepsilon\,[ax\,(bxB_{2n}(x,y) + dyB_{2n-1}(x,y)) + cyB_{2n}(x,y))] \\
&\quad + \mathbf{h}\,[bx\,(axB_{2n+1}(x,y) + cyB_{2n}(x,y)) + dyB_{2n+1}(x,y)] \\
&= [\left(abx^2 + cy\right) B_{2n-2}(x,y) + dy\,(axB_{2n-3}(x,y))] \\
&\quad + \mathbf{i}[\left(abx^2 + dy\right) B_{2n-1}(x,y) + cy\,(bxB_{2n-2}(x,y))] \\
&\quad + \varepsilon[\left(abx^2 + cy\right) B_{2n}(x,y) + dy\,(axB_{2n-1}(x,y))] \\
&\quad + \mathbf{h}[\left(abx^2 + dy\right) B_{2n+1}(x,y) + cy\,(bxB_{2n}(x,y))]
\end{aligned}$$

$$
\begin{aligned}
&= [\left(abx^2+cy\right)B_{2n-2}(x,y)+dy\left(B_{2n-2}(x,y)-cyB_{2n-4}(x,y)\right)] \\
&\quad + \mathbf{i}[\left(abx^2+dy\right)B_{2n-1}(x,y)+cy\left(B_{2n-1}(x,y)-dyB_{2n-3}(x,y)\right)] \\
&\quad + \varepsilon[\left(abx^2+cy\right)B_{2n}(x,y)+dy\left(B_{2n}(x,y)-cyB_{2n-2}(x,y)\right)] \\
&\quad + \mathbf{h}[\left(abx^2+dy\right)B_{2n+1}(x,y)+cy\left(B_{2n+1}(x,y)-dyB_{2n-1}(x,y)\right)] \\
&= [\left(abx^2+(c+d)\,y\right)B_{2n-2}(x,y)-cdy^2B_{2n-4}(x,y)] \\
&\quad + \mathbf{i}[\left(abx^2+(c+d)\,y\right)B_{2n-1}(x,y)-cdy^2B_{2n-3}(x,y)] \\
&\quad + \varepsilon[\left(abx^2+(c+d)\,y\right)B_{2n}(x,y)-cdy^2B_{2n-2}(x,y)] \\
&\quad + \mathbf{h}[\left(abx^2+(c+d)\,y\right)B_{2n+1}(x,y)-cdy^2B_{2n-1}(x,y)] \\
&= \left(abx^2+(c+d)\,y\right)[B_{2n-2}(x,y)+\mathbf{i}B_{2n-1}(x,y)+\varepsilon B_{2n}(x,y)+\mathbf{h}B_{2n+1}(x,y)] \\
&\quad - cdy^2[B_{2n-4}(x,y)+\mathbf{i}B_{2n-3}(x,y)+\varepsilon B_{2n-2}(x,y)+\mathbf{h}B_{2n-1}(x,y)] \\
&= \left(abx^2+(c+d)\,y\right)BH_{2n-2}(x,y)-cdy^2BH_{2n-4}(x,y).
\end{aligned}
$$

Similar to the above steps, we can obtain

$$BH_{2n+1}(x,y)=\left(abx^2+(c+d)y\right)BH_{2n-1}(x,y)-cdy^2BH_{2n-3}(x,y).$$

Thus, the proof is completed. □

Next, we give the generating function of the bivariate conditional Fibonacci hybrinomial $BH_n(x,y)$.

Theorem 1. *The generating function for the bivariate conditional Fibonacci hybrinomial $BH_n(x,y)$ is*

$$
\begin{aligned}
\mathfrak{G}(t)=\sum_{n=0}^{\infty}BH_n(x,y)t^n &= \frac{BH_0(x,y)+BH_1(x,y)t}{1-\left(abx^2+(c+d)\,y\right)t^2+cdy^2t^4} \\
&\quad + \frac{\left[BH_2(x,y)-\left(abx^2+(c+d)y\right)BH_0(x,y)\right]t^2}{1-\left(abx^2+(c+d)\,y\right)t^2+cdy^2t^4} \\
&\quad + \frac{\left[BH_3(x,y)-\left(abx^2+(c+d)\,y\right)BH_1(x,y)\right]t^3}{1-\left(abx^2+(c+d)\,y\right)t^2+cdy^2t^4}.
\end{aligned}
\tag{8}
$$

Proof. We define

$$
\begin{aligned}
\mathfrak{G}_0(t) &= \sum_{n=0}^{\infty}BH_{2n}(x,y)t^{2n} \\
\mathfrak{G}_1(t) &= \sum_{n=0}^{\infty}BH_{2n+1}(x,y)t^{2n+1}.
\end{aligned}
$$

So that

$$\mathfrak{G}(t)=\mathfrak{G}_0(t)+\mathfrak{G}_1(t).$$

We have

$$\begin{aligned}
\mathfrak{G}_0(t) &= \sum_{n=0}^{\infty} BH_{2n}(x,y)t^{2n} \\
&= \sum_{n=0}^{\infty} BH_{2n}(x,y)t^{2n} = BH_0(x,y)t^0 + BH_2(x,y)t^2 + \sum_{n=2}^{\infty} BH_{2n}(x,y)t^{2n} \\
&= BH_0(x,y) + BH_2(x,y)t^2 \\
&\quad + \sum_{n=2}^{\infty} \left[\left(abx^2 + (c+d)y\right) BH_{2n-2}(x,y) - cdy^2 BH_{2n-4}(x,y)\right] t^{2n} \\
&= BH_0(x,y) + BH_2(x,y)t^2 + \left(abx^2 + (c+d)y\right) t^2 \sum_{n=2}^{\infty} BH_{2n-2}(x,y)t^{2n-2} \\
&\quad - cdy^2 t^4 \sum_{n=2}^{\infty} BH_{2n-4}(x,y)t^{2n-4} \\
&= BH_0(x,y) + BH_2(x,y)t^2 \\
&\quad + \left(abx^2 + (c+d)y\right) t^2 \\
&\qquad \times \left[\sum_{n=2}^{\infty} BH_{2n-2}(x,y)t^{2n-2} + BH_0(x,y)t^0 - BH_0(x,y)t^0\right] \\
&\quad - cdy^2 t^4 \mathfrak{G}_0(t) \\
&= BH_0(x,y) + BH_2(x,y)t^2 + \left(abx^2 + (c+d)y\right) t^2 \mathfrak{G}_0(t) \\
&\quad - \left(abx^2 + (c+d)y\right) t^2 BH_0(x,y) - cdy^2 t^4 \mathfrak{G}_0(t).
\end{aligned}$$

Thus, we get

$$\mathfrak{G}_0(t) = \frac{BH_0(x,y) + \left(BH_2(x,y) - \left(abx^2 + (c+d)y\right) BH_0(x,y)\right) t^2}{1 - \left(abx^2 + (c+d)y\right) t^2 + cdy^2 t^4}. \tag{9}$$

Similarly, we find

$$\begin{aligned}
\mathfrak{G}_1(t) &= \sum_{n=0}^{\infty} BH_{2n+1}(x,y)t^{2n+1} \\
&= \sum_{n=0}^{\infty} BH_{2n+1}(x,y)t^{2n+1} \\
&= BH_1(x,y)t + BH_3(x,y)t^3 + \sum_{n=2}^{\infty} BH_{2n+1}(x,y)t^{2n+1}
\end{aligned}$$

$$
\begin{aligned}
&= BH_1(x,y)t + BH_3(x,y)t^3 \\
&\quad + \sum_{n=2}^{\infty} \left[\left(abx^2+(c+d)y\right) BH_{2n-1}(x,y) - cdy^2 BH_{2n-3}(x,y)\right] t^{2n+1} \\
&= BH_1(x,y)t + BH_3(x,y)t^3 + \left(abx^2+(c+d)y\right) t^2 \sum_{n=2}^{\infty} BH_{2n-1}(x,y)t^{2n-1} \\
&\quad - cdy^2t^4 \sum_{n=2}^{\infty} BH_{2n-3}(x,y)t^{2n-3} \\
&= BH_1(x,y)t + BH_3(x,y)t^3 \\
&\quad + \left(abx^2+(c+d)y\right) t^2 \left[\sum_{n=2}^{\infty} BH_{2n-1}(x,y)t^{2n-1} + BH_1(x,y)t - BH_1(x,y)t\right] \\
&\quad - cdy^2t^4 \mathfrak{G}_1(t) \\
&= BH_1(x,y)t + BH_3(x,y)t^3 + \left(abx^2+(c+d)y\right) t^2 \mathfrak{G}_1(t) \\
&\quad - \left(abx^2+(c+d)y\right) t^3 BH_1(x,y) - cdy^2t^4 \mathfrak{G}_1(t).
\end{aligned}
$$

Therefore, we get

$$
\mathfrak{G}_1(t) = \frac{BH_1(x,y)t + \left(BH_3(x,y) - \left(abx^2+(c+d)y\right) BH_1(x,y)\right) t^3}{1-\left(abx^2+(c+d)y\right)t^2 + cdy^2t^4}. \tag{10}
$$

By virtue of (9) and (10), we can obtain

$$
\begin{aligned}
\mathfrak{G}(t) &= \mathfrak{G}_0(t) + \mathfrak{G}_1(t) \\
&= \sum_{n=0}^{\infty} BH_n(x,y)t^n \\
&= \frac{BH_0(x,y) + BH_1(x,y)t + \left[BH_2(x,y) - \left(abx^2+(c+d)y\right) BH_0(x,y)\right] t^2}{1-\left(abx^2+(c+d)\,y\right)t^2 + cdy^2t^4} \\
&\quad + \frac{\left[BH_3(x,y) - \left(abx^2+(c+d)\,y\right) BH_1(x,y)\right] t^3}{1-\left(abx^2+(c+d)\,y\right)t^2 + cdy^2t^4}.
\end{aligned}
$$

Hence, the proof is completed. □

Now we give the Binet formula of the bivariate conditional Fibonacci hybrinomial $BH_n(x,y)$.

Theorem 2. *The n^{th} term of the generalized bivariate conditional Fibonacci hybrinomial $BH_n(x,y)$ is*

$$
BH_n(x,y) = \frac{\widehat{\alpha}_{\xi(n)}\beta_1^{\lfloor \frac{n}{2} \rfloor}\left(\beta_1+(d-c)y\right)^{\lfloor \frac{n}{2} \rfloor+\xi(n)} - \widehat{\gamma}_{\xi(n)}\beta_2^{\lfloor \frac{n}{2} \rfloor}\left(\beta_2+(d-c)y\right)^{\lfloor \frac{n}{2} \rfloor+\xi(n)}}{\left(abx^2\right)^{\lfloor \frac{n}{2} \rfloor}\left(\beta_1-\beta_2\right)}. \tag{11}
$$

where β_1 *and* β_2 *roots of the characteristic equation* $\lambda^2-(abx^2+(c-d)y)\lambda-abdx^2y=0$. *Also,*

$$\begin{aligned}\widehat{\alpha}_{\xi(n)} &= (ax)^{\xi(n+1)} + \boldsymbol{i}\frac{(ax)^{\xi(n)}\beta_1^{\xi(n)}}{(abx^2)^{\xi(n)}}(\beta_1+(d-c)y)^{\xi(n+1)} \\ &+ \varepsilon\frac{(ax)^{\xi(n+1)}\beta_1}{(abx^2)}(\beta_1+(d-c)y) \\ &+ \boldsymbol{h}\frac{(ax)^{\xi(n)}\beta_1^{\xi(n)+1}}{(abx^2)^{\xi(n)+1}}(\beta_1+(d-c)y)^{\xi(n+1)+1}\end{aligned}$$

and

$$\begin{aligned}\widehat{\gamma}_{\xi(n)} &= (ax)^{\xi(n+1)} + \boldsymbol{i}\frac{(ax)^{\xi(n)}\beta_2^{\xi(n)}}{(abx^2)^{\xi(n)}}(\beta_2+(d-c)y)^{\xi(n+1)} \\ &+ \varepsilon\frac{(ax)^{\xi(n+1)}\beta_2}{(abx^2)}(\beta_2+(d-c)y) \\ &+ \boldsymbol{h}\frac{(ax)^{\xi(n)}\beta_2^{\xi(n)+1}}{(abx^2)^{\xi(n)+1}}(\beta_2+(d-c)y)^{\xi(n+1)+1}.\end{aligned}$$

Proof. We use the following properties throughout the proof:

- $\beta_1+\beta_2=abx^2+(c-d)y$
- $\beta_1\cdot\beta_2=-abdx^2y$
- $(\beta_1+dy)(\beta_2+dy)=cdy^2$
- $(\beta_1+dy)\left(abx^2\right)=\beta_1(\beta_1+(d-c)y)$
- $(\beta_2+dy)\left(abx^2\right)=\beta_2(\beta_2+(d-c)y)$.

Note that $\beta_1(x,y)=\beta_1$ and $\beta_2(x,y)=\beta_2$. By using (4), we have

$$\begin{aligned}BH_{2n}(x,y) &= B_{2n}(x,y)+\mathbf{i}B_{2n+1}(x,y)+\varepsilon B_{2n+2}(x,y)+\mathbf{h}B_{2n+3}(x,y) \\ &= \frac{(ax)}{(abx^2)^n}\left[\frac{\beta_1^n(\beta_1+(d-c)y)^n-\beta_2^n(\beta_2+(d-c)y)^n}{\beta_1-\beta_2}\right] \\ &+ \mathbf{i}\frac{1}{(abx^2)^n}\left[\frac{\beta_1^n(\beta_1+(d-c)y)^{n+1}-\beta_2^n(\beta_2+(d-c)y)^{n+1}}{\beta_1-\beta_2}\right] \\ &+ \varepsilon\frac{(ax)}{(abx^2)^{n+1}}\left[\frac{\beta_1^{n+1}(\beta_1+(d-c)y)^{n+1}-\beta_2^{n+1}(\beta_2+(d-c)y)^{n+1}}{\beta_1-\beta_2}\right] \\ &+ \mathbf{h}\frac{1}{(abx^2)^{n+1}}\left[\frac{\beta_1^{n+1}(\beta_1+(d-c)y)^{n+2}-\beta_2^{n+1}(\beta_2+(d-c)y)^{n+2}}{\beta_1-\beta_2}\right]\end{aligned}$$

$$\begin{aligned}
&= \frac{\beta_1^n (\beta_1 + (d-c)y)^n}{(abx^2)^n (\beta_1 - \beta_2)} \\
&\quad \times \left[ax + \mathbf{i} (\beta_1 + (d-c)y) + \varepsilon \frac{ax}{abx^2} \beta_1 (\beta_1 + (d-c)y) + \mathbf{h} \frac{1}{abx^2} \beta_1 (\beta_1 + (d-c)y)^2 \right] \\
&\quad - \frac{\beta_2^n (\beta_2 + (d-c)y)^n}{(abx^2)^n (\beta_1 - \beta_2)} \\
&\quad \times \left[ax + \mathbf{i} (\beta_2 + (d-c)y) + \varepsilon \frac{ax}{abx^2} \beta_2 (\beta_2 + (d-c)y) + \mathbf{h} \frac{1}{abx^2} \beta_2 (\beta_2 + (d-c)y)^2 \right]
\end{aligned}$$

Here, we choose the $\widehat{\alpha}_0$ and $\widehat{\gamma}_0$ as follows:

$$\begin{aligned}
\widehat{\alpha}_0 &= \left[ax + \mathbf{i} (\beta_1 + (d-c)y) + \varepsilon \frac{ax}{abx^2} \beta_1 (\beta_1 + (d-c)y) + \mathbf{h} \frac{1}{abx^2} \beta_1 (\beta_1 + (d-c)y)^2 \right] \\
\widehat{\gamma}_0 &= \left[ax + \mathbf{i} (\beta_2 + (d-c)y) + \varepsilon \frac{ax}{abx^2} \beta_2 (\beta_2 + (d-c)y) + \mathbf{h} \frac{1}{abx^2} \beta_2 (\beta_2 + (d-c)y)^2 \right].
\end{aligned}$$

Finally, the following equation is obtained:

$$BH_{2n}(x,y) = \frac{\widehat{\alpha}_0 \beta_1^n (\beta_1 + (d-c)y)^n - \widehat{\gamma}_0 \beta_2^n (\beta_2 + (d-c)y)^n}{(abx^2)^n (\beta_1 - \beta_2)}. \tag{12}$$

In a similar way, by using (4), we have

$$\begin{aligned}
BH_{2n+1}(x,y) &= B_{2n+1}(x,y) + \mathbf{i} B_{2n+2}(x,y) + \varepsilon B_{2n+3}(x,y) + \mathbf{h} B_{2n+4}(x,y) \\
&= \frac{1}{(abx^2)^n} \left[\frac{\beta_1^n (\beta_1 + (d-c)y)^{n+1} - \beta_2^n (\beta_2 + (d-c)y)^{n+1}}{\beta_1 - \beta_2} \right] \\
&\quad + \mathbf{i} \frac{(ax)}{(abx^2)^{n+1}} \left[\frac{\beta_1^{n+1} (\beta_1 + (d-c)y)^{n+1} - \beta_2^{n+1} (\beta_2 + (d-c)y)^{n+1}}{\beta_1 - \beta_2} \right] \\
&\quad + \varepsilon \frac{1}{(abx^2)^{n+1}} \left[\frac{\beta_1^{n+1} (\beta_1 + (d-c)y)^{n+2} - \beta_2^{n+1} (\beta_2 + (d-c)y)^{n+2}}{\beta_1 - \beta_2} \right] \\
&\quad + \mathbf{h} \frac{(ax)}{(abx^2)^{n+2}} \left[\frac{\beta_1^{n+2} (\beta_1 + (d-c)y)^{n+2} - \beta_2^{n+2} (\beta_2 + (d-c)y)^{n+2}}{\beta_1 - \beta_2} \right] \\
&= \frac{\beta_1^n (\beta_1 + (d-c)y)^{n+1}}{(abx^2)^n (\beta_1 - \beta_2)} \\
&\quad \times \left[1 + \mathbf{i} \frac{ax}{abx^2} \beta_1 + \varepsilon \frac{1}{abx^2} \beta_1 (\beta_1 + (d-c)y) + \mathbf{h} \frac{ax}{(abx^2)^2} \beta_1^2 (\beta_1 + (d-c)y) \right] \\
&\quad - \frac{\beta_2^n (\beta_2 + (d-c)y)^{n+1}}{(abx^2)^n (\beta_2 - \beta_2)} \\
&\quad \times \left[1 + \mathbf{i} \frac{ax}{abx^2} \beta_2 + \varepsilon \frac{1}{abx^2} \beta_2 (\beta_2 + (d-c)y) + \mathbf{h} \frac{ax}{(abx^2)^2} \beta_2^2 (\beta_2 + (d-c)y) \right].
\end{aligned}$$

Here, we choose the $\widehat{\alpha}_1$ and $\widehat{\gamma}_1$ as follows;

$$\widehat{\alpha}_1 = \left[1 + \mathbf{i}\frac{ax}{abx^2}\beta_1 + \varepsilon\frac{1}{abx^2}\beta_1\left(\beta_1 + (d-c)y\right) + \mathbf{h}\frac{ax}{(abx^2)^2}\beta_1^2\left(\beta_1 + (d-c)y\right)\right]$$
$$\widehat{\gamma}_1 = \left[1 + \mathbf{i}\frac{ax}{abx^2}\beta_2 + \varepsilon\frac{1}{abx^2}\beta_2\left(\beta_2 + (d-c)y\right) + \mathbf{h}\frac{ax}{(abx^2)^2}\beta_2^2\left(\beta_2 + (d-c)y\right)\right].$$

Finally, the following equation is obtained.

$$BH_{2n+1}(x,y) = \frac{\widehat{\alpha}_1\beta_1^n\left(\beta_1 + (d-c)y\right)^{n+1} - \widehat{\gamma}_1\beta_2^n\left(\beta_2 + (d-c)y\right)^{n+1}}{(abx^2)^n\left(\beta_1 - \beta_2\right)} \tag{13}$$

By virtue of (12) and (13), we can obtain the following equation.

$$BH_n(x,y) = \frac{\widehat{\alpha}_{\xi(n)}\beta_1^{\lfloor \frac{n}{2}\rfloor}\left(\beta_1 + (d-c)y\right)^{\lfloor \frac{n}{2}\rfloor + \xi(n)} - \widehat{\gamma}_{\xi(n)}\beta_2^{\lfloor \frac{n}{2}\rfloor}\left(\beta_2 + (d-c)y\right)^{\lfloor \frac{n}{2}\rfloor + \xi(n)}}{(abx^2)^{\lfloor \frac{n}{2}\rfloor}\left(\beta_1 - \beta_2\right)}.$$

where β_1 and β_2 roots of the characteristic equation $\lambda^2 - (abx^2 + (c-d)y)\lambda - abdx^2y = 0$. Also,

$$\begin{aligned}\widehat{\alpha}_{\xi(n)} = (ax)^{\xi(n+1)} &+ \mathbf{i}\frac{(ax)^{\xi(n)}\beta_1^{\xi(n)}}{(abx^2)^{\xi(n)}}\left(\beta_1 + (d-c)y\right)^{\xi(n+1)} \\ &+ \varepsilon\frac{(ax)^{\xi(n+1)}\beta_1}{(abx^2)}\left(\beta_1 + (d-c)y\right) \\ &+ \mathbf{h}\frac{(ax)^{\xi(n)}\beta_1^{\xi(n)+1}}{(abx^2)^{\xi(n)+1}}\left(\beta_1 + (d-c)y\right)^{\xi(n+1)+1}\end{aligned}$$

and

$$\begin{aligned}\widehat{\gamma}_{\xi(n)} = (ax)^{\xi(n+1)} &+ \mathbf{i}\frac{(ax)^{\xi(n)}\beta_2^{\xi(n)}}{(abx^2)^{\xi(n)}}\left(\beta_2 + (d-c)y\right)^{\xi(n+1)} \\ &+ \varepsilon\frac{(ax)^{\xi(n+1)}\beta_2}{(abx^2)}\left(\beta_2 + (d-c)y\right) \\ &+ \mathbf{h}\frac{(ax)^{\xi(n)}\beta_2^{\xi(n)+1}}{(abx^2)^{\xi(n)+1}}\left(\beta_2 + (d-c)y\right)^{\xi(n+1)+1}.\end{aligned}$$

□

Now, we give the Catalan's identity of the bivariate conditional Fibonacci hybrinomial $BH_n(x,y)$.

Theorem 3. *For any integers n and r and $n \geqslant r \geqslant 0$, we have*

$$
\begin{aligned}
&BH_{2(n+r)+\xi(i)}(x,y)BH_{2(n-r)+\xi(i)}(x,y)-\left(BH_{2n+\xi(i)}(x,y)\right)^2\\
&=\frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}\\
&\quad\times\left[\widehat{\alpha}_{\xi(i)}\widehat{\gamma}_{\xi(i)}{\beta_1}^n{\beta_2}^n(\beta_1+(d-c)y)^{n+\xi(i)}(\beta_2+(d-c)y)^{n+\xi(i)}\left[1-\left(\frac{\beta_1(\beta_1+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right]\\
&\quad+\frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}\\
&\quad\times\left[\widehat{\gamma}_{\xi(i)}\widehat{\alpha}_{\xi(i)}{\beta_2}^n{\beta_1}^n(\beta_2+(d-c)y)^{n+\xi(i)}(\beta_1+(d-c)y)^{n+\xi(i)}\left[1-\left(\frac{\beta_2(\beta_2+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right],
\end{aligned}
$$

where $\widehat{\alpha}_{\xi(i)}$ and $\widehat{\gamma}_{\xi(i)}$ are defined in Theorem(2) and $i\in\{0,1\}$.

Proof. In order to prove Catalan's identity, we will examine in two different cases.
Case $i=0$:

$$
\begin{aligned}
BH_{2(n+r)}(x,y)&=\frac{\widehat{\alpha}_{\xi(2n+2r)}\beta_1^{\lfloor\frac{2n+2r}{2}\rfloor}(\beta_1+(d-c)y)^{\lfloor\frac{2n+2r}{2}\rfloor+\xi(2n+2r)}}{(abx^2)^{\lfloor\frac{2n+2r}{2}\rfloor}(\beta_1-\beta_2)}\\
&\quad-\frac{\widehat{\gamma}_{\xi(2n+2r)}\beta_2^{\lfloor\frac{2n+2r}{2}\rfloor}(\beta_2+(d-c)y)^{\lfloor\frac{2n+2r}{2}\rfloor+\xi(2n+2r)}}{(abx^2)^{\lfloor\frac{2n+2r}{2}\rfloor}(\beta_1-\beta_2)}\\
&=\frac{\widehat{\alpha}_0\beta_1^{n+r}(\beta_1+(d-c)y)^{n+r}-\widehat{\gamma}_0\beta_2^{n+r}(\beta_2+(d-c)y)^{n+r}}{(abx^2)^{n+r}(\beta_1-\beta_2)}
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
BH_{2(n-r)}(x,y)&=\frac{\widehat{\alpha}_{\xi(2n-2r)}\beta_1^{\lfloor\frac{2n-2r}{2}\rfloor}(\beta_1+(d-c)y)^{\lfloor\frac{2n-2r}{2}\rfloor+\xi(2n-2r)}}{(abx^2)^{\lfloor\frac{2n-2r}{2}\rfloor}(\beta_1-\beta_2)}\\
&\quad-\frac{\widehat{\gamma}_{\xi(2n-2r)}\beta_2^{\lfloor\frac{2n-2r}{2}\rfloor}(\beta_2+(d-c)y)^{\lfloor\frac{2n-2r}{2}\rfloor+\xi(2n-2r)}}{(abx^2)^{\lfloor\frac{2n-2r}{2}\rfloor}(\beta_1-\beta_2)}\\
&=\frac{\widehat{\alpha}_0\beta_1^{n-r}(\beta_1+(d-c)y)^{n-r}-\widehat{\gamma}_0\beta_2^{n-r}(\beta_2+(d-c)y)^{n-r}}{(abx^2)^{n-r}(\beta_1-\beta_2)}
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
BH_{2n}(x,y)&=\frac{\widehat{\alpha}_{\xi(2n)}\beta_1^{\lfloor\frac{2n}{2}\rfloor}(\beta_1+(d-c)y)^{\lfloor\frac{2n}{2}\rfloor+\xi(2n-2r)}}{(abx^2)^{\lfloor\frac{2n}{2}\rfloor}(\beta_1-\beta_2)}\\
&\quad-\frac{\widehat{\gamma}_{\xi(2n)}\beta_2^{\lfloor\frac{2n}{2}\rfloor}(\beta_2+(d-c)y)^{\lfloor\frac{2n}{2}\rfloor+\xi(2n)}}{(abx^2)^{\lfloor\frac{2n}{2}\rfloor}(\beta_1-\beta_2)}\\
&=\frac{\widehat{\alpha}_0\beta_1^{n}(\beta_1+(d-c)y)^{n}-\widehat{\gamma}_0\beta_2^{n}(\beta_2+(d-c)y)^{n}}{(abx^2)^{n}(\beta_1-\beta_2)}.
\end{aligned}
\tag{16}
$$

By virtue of (14), (15) and (16), we have

$$\begin{aligned}
&BH_{2(n+r)}(x,y)BH_{2(n-r)}(x,y)-(BH_{2n}(x,y))^2\\
&=\frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}\\
&\quad\times\left[\widehat{\alpha}_0\widehat{\gamma}_0{\beta_1}^n{\beta_2}^n(\beta_1+(d-c)y)^n(\beta_2+(d-c)y)^n\left[1-\left(\frac{\beta_1(\beta_1+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right]\\
&\quad+\frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}\\
&\quad\times\left[\widehat{\gamma}_0\widehat{\alpha}_0{\beta_2}^n{\beta_1}^n(\beta_2+(d-c)y)^n(\beta_1+(d-c)y)^n\left[1-\left(\frac{\beta_2(\beta_2+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right].
\end{aligned}$$

Case $i=1$

$$BH_{2(n+r)+1}(x,y)=\frac{\widehat{\alpha}_1\beta_1^{n+r}(\beta_1+(d-c)y)^{n+r+1}-\widehat{\gamma}_1\beta_2^{n+r}(\beta_2+(d-c)y)^{n+r+1}}{(abx^2)^{n+r}(\beta_1-\beta_2)} \tag{17}$$

$$BH_{2(n-r)+1}(x,y)=\frac{\widehat{\alpha}_1\beta_1^{n-r}(\beta_1+(d-c)y)^{n-r+1}-\widehat{\gamma}_1\beta_2^{n-r}(\beta_2+(d-c)y)^{n-r+1}}{(abx^2)^{n-r}(\beta_1-\beta_2)} \tag{18}$$

$$BH_{2n+1}(x,y)=\frac{\widehat{\alpha}_1\beta_1^{n}(\beta_1+(d-c)y)^{n+1}-\widehat{\gamma}_1\beta_2^{n}(\beta_2+(d-c)y)^{n+1}}{(abx^2)^{n}(\beta_1-\beta_2)}. \tag{19}$$

By virtue of (17), (18) and (19), we have

$$\begin{aligned}
&BH_{2(n+r)+1}(x,y)BH_{2(n-r)+1}(x,y)-(BH_{2n+1}(x,y))^2\\
&=\frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}\\
&\quad\times\left[\widehat{\alpha}_1\widehat{\gamma}_1{\beta_1}^n{\beta_2}^n(\beta_1+(d-c)y)^{n+1}(\beta_2+(d-c)y)^{n+1}\left[1-\left(\frac{\beta_1(\beta_1+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right]\\
&\quad+\frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}\\
&\quad\times\left[\widehat{\gamma}_1\widehat{\alpha}_1{\beta_2}^n{\beta_1}^n(\beta_2+(d-c)y)^{n+1}(\beta_1+(d-c)y)^{n+1}\left[1-\left(\frac{\beta_2(\beta_2+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right].
\end{aligned}$$

Finally, we get

$$\begin{aligned}
&BH_{2(n+r)+\xi(i)}(x,y)BH_{2(n-r)+\xi(i)}(x,y)-\left(BH_{2n+\xi(i)}(x,y)\right)^2\\
&=\frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}\\
&\quad\times\left[\widehat{\alpha}_{\xi(i)}\widehat{\gamma}_{\xi(i)}{\beta_1}^n{\beta_2}^n(\beta_1+(d-c)y)^{n+\xi(i)}(\beta_2+(d-c)y)^{n+\xi(i)}\left[1-\left(\frac{\beta_1(\beta_1+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right]\\
&\quad+\frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}\\
&\quad\times\left[\widehat{\gamma}_{\xi(i)}\widehat{\alpha}_{\xi(i)}{\beta_2}^n{\beta_1}^n(\beta_2+(d-c)y)^{n+\xi(i)}(\beta_1+(d-c)y)^{n+\xi(i)}\left[1-\left(\frac{\beta_2(\beta_2+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)^r\right]\right].
\end{aligned}$$

□

Now, we give the Cassini's identity of the bivariate conditional Fibonacci hybrinomial $BH_n(x,y)$.

Corollary 1. *For $n \geq 0$, we get*

$$BH_{2(n+1)+\xi(i)}(x,y)BH_{2(n-1)+\xi(i)}(x,y) - \left(BH_{2n+\xi(i)}(x,y)\right)^2$$
$$= \frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}$$
$$\times\left[\widehat{\alpha}_{\xi(i)}\widehat{\gamma}_{\xi(i)}{\beta_1}^n{\beta_2}^n(\beta_1+(d-c)y)^{n+\xi(i)}(\beta_2+(d-c)y)^{n+\xi(i)}\left[1-\left(\frac{\beta_1(\beta_1+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)\right]\right]$$
$$+\frac{1}{(abx^2)^{2n}(\beta_1-\beta_2)^2}$$
$$\times\left[\widehat{\gamma}_{\xi(i)}\widehat{\alpha}_{\xi(i)}{\beta_2}^n{\beta_1}^n(\beta_2+(d-c)y)^{n+\xi(i)}(\beta_1+(d-c)y)^{n+\xi(i)}\left[1-\left(\frac{\beta_2(\beta_2+(d-c)y)}{\beta_2(\beta_2+(d-c)y)}\right)\right]\right].$$

where $\widehat{\alpha}_{\xi(i)}$ and $\widehat{\gamma}_{\xi(i)}$ are defined in Theorem(2) and $i \in \{0,1\}$.

Proof. Taking $r = 1$ in Catalan's identity the proof is completed. □

3. Generalized Bivariate Conditional Lucas Hybrinomials

In this section we give some identities of the generalized bivariate conditional Lucas hybrinomials. We start with the following definition.

Definition 2. *For any four numbers a, b, c and d belonging to $\mathbb{R} - \{0\}$, the generalization of bivariate conditional Fibonacci polynomial is defined as,*

$$L_n(x,y) = \begin{cases} bxL_{n-1}(x,y) + dyL_{n-2}(x,y), & \text{if } n \text{ is even} \\ axL_{n-1}(x,y) + cyL_{n-2}(x,y), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2, \tag{20}$$

where $L_0(x,y) = 2, L_1(x,y) = ax$.

Lemma 2. *For the generalized bivariate conditional Lucas polynomials $\{L_n(x,y)\}_{n=0}^{\infty}$, we have*

$$L_{2n}(x,y) = \left(abx^2 + (c+d)y\right)L_{2n-2}(x,y) - cdy^2L_{2n-4}(x,y)$$
$$L_{2n+1}(x,y) = \left(abx^2 + (c+d)y\right)L_{2n-1}(x,y) - cdy^2L_{2n-3}(x,y).$$

Proof. By using the definition of the generalized bivariate conditional Lucas polynomials, we have

$$\begin{aligned} L_{2n}(x,y) &= (bxL_{2n-1}(x,y) + dyL_{2n-2}(x,y)) \\ &= [bx(axL_{2n-2}(x,y) + cyL_{2n-3}(x,y)) + dyL_{2n-2}(x,y)] \\ &= [\left(abx^2 + dy\right)L_{2n-2}(x,y) + cy(bxL_{2n-3}(x,y))] \\ &= [\left(abx^2 + dy\right)L_{2n-2}(x,y) + cy(L_{2n-2}(x,y) - dyL_{2n-4}(x,y))] \\ &= [\left(abx^2 + (c+d)y\right)L_{2n-2}(x,y) - cdy^2L_{2n-4}(x,y)]. \end{aligned}$$

Similar to above steps, we can obtain

$$L_{2n+1}(x,y) = \left(abx^2 + (c+d)y\right)L_{2n-1}(x,y) - cdy^2L_{2n-3}(x,y).$$

Thus, the proof is completed. □

Next we give the generating function for the bivariate conditional Lucas polynomial $L_n(x,y)$.

Theorem 4. *The generating function for the bivariate conditional Lucas polynomial $L_n(x,y)$ is*

$$E(t)=\sum_{n=0}^{\infty}L_n(x,y)t^n=\frac{2+axt-\left(abx^2+2cy\right)t^2+adxyt^3}{1-\left(abx^2+\left(c+d\right)y\right)t^2+cdy^2t^4}. \tag{21}$$

Proof. We define

$$E_0(t)=\sum_{n=0}^{\infty}L_{2n}(x,y)t^{2n}$$
$$E_1(t)=\sum_{n=0}^{\infty}L_{2n+1}(x,y)t^{2n+1}.$$

So that

$$E(t)=E_0(t)+E_1(t).$$

We have

$$\begin{aligned}
E_0(t)&=\sum_{n=0}^{\infty}L_{2n}(x,y)t^{2n}\\
&=\sum_{n=0}^{\infty}L_{2n}(x,y)t^{2n}=L_0(x,y)t^0+L_2(x,y)t^2+\sum_{n=2}^{\infty}L_{2n}(x,y)t^{2n}\\
&=L_0(x,y)+L_2(x,y)t^2\\
&\quad+\sum_{n=2}^{\infty}\left[\left(abx^2+(c+d)y\right)L_{2n-2}(x,y)-cdy^2L_{2n-4}(x,y)\right]t^{2n}\\
&=L_0(x,y)+L_2(x,y)t^2+\left(abx^2+(c+d)y\right)t^2\sum_{n=2}^{\infty}L_{2n-2}(x,y)t^{2n-2}\\
&\quad-cdy^2t^4\sum_{n=2}^{\infty}L_{2n-4}(x,y)t^{2n-4}\\
&=2+\left(abx^2+2dy\right)t^2\\
&\quad+\left(abx^2+(c+d)y\right)t^2\left[\sum_{n=2}^{\infty}L_{2n-2}(x,y)t^{2n-2}+L_0(x,y)t^0-L_0(x,y)t^0\right]\\
&\quad-cdy^2t^4E_0(t)
\end{aligned}$$

$$
\begin{aligned}
&= 2 + \left(abx^2 + 2dy\right) t^2 + \left(abx^2 + (c+d)y\right) t^2 E_0(t) \\
&\quad - 2\left(abx^2 + (c+d)y\right) t^2 - cdy^2 t^4 E_0(t) \\
E_0(t)&\left[1 - \left(abx^2 + (c+d)y\right) t^2 + cdy^2 t^4\right] = 2 - \left(abx^2 + 2cy\right) t^2.
\end{aligned}
$$

Thus, we get

$$
E_0(t) = \frac{2 - \left(abx^2 + 2cy\right) t^2}{1 - \left(abx^2 + (c+d)y\right) t^2 + cdy^2 t^4}. \tag{22}
$$

Similarly, we find

$$
\begin{aligned}
E_1(t) &= \sum_{n=0}^{\infty} L_{2n+1}(x,y) t^{2n+1} \\
&= \sum_{n=0}^{\infty} L_{2n+1}(x,y) t^{2n+1} = L_1(x,y)t^1 + L_3(x,y)t^3 + \sum_{n=2}^{\infty} L_{2n+1}(x,y) t^{2n+1} \\
&= L_1(x,y)t + L_3(x,y)t^3 \\
&\quad + \sum_{n=2}^{\infty} \left[\left(abx^2 + (c+d)y\right) L_{2n-1}(x,y) - cdy^2 L_{2n-3}(x,y)\right] t^{2n+1} \\
&= L_1(x,y)t + L_3(x,y)t^3 \\
&\quad + \left(abx^2 + (c+d)y\right) t^2 \sum_{n=2}^{\infty} L_{2n-1}(x,y) t^{2n-1} \\
&\quad - cdy^2 t^4 \sum_{n=2}^{\infty} L_{2n-3}(x,y) t^{2n-3} \\
&= axt + \left(a^2bx^3 + 2adxy + acxy\right) t^3 \\
&\quad + \left(abx^2 + (c+d)y\right) t^2 \left[\sum_{n=2}^{\infty} L_{2n-1}(x,y) t^{2n-1} + L_1(x,y)t - L_1(x,y)t\right] \\
&\quad - cdy^2 t^4 E_1(t) \\
&= axt + \left(a^2bx^3 + 2adxy + acxy\right) t^3 + \left(abx^2 + (c+d)y\right) t^2 E_1(t) \\
&\quad - ax\left(abx^2 + (c+d)y\right) t^3 - cdy^2 t^4 E_1(t) \\
E_1(t)&\left[1 - \left(abx^2 + (c+d)y\right) t^2 + cdy^2 t^4\right] = axt + adxyt^3.
\end{aligned}
$$

Therefore, we get

$$
E_1(t) = \frac{axt + adxyt^3}{1 - \left(abx^2 + (c+d)y\right) t^2 + cdy^2 t^4}. \tag{23}
$$

By virtue of (22) and (23), we can obtain

$$\begin{aligned} E(t) &= E_0(t) + E_1(t) \\ &= \sum_{n=0}^{\infty} L_n(x,y)t^n = \frac{2 + axt - \left(abx^2 + 2cy\right)t^2 + adxyt^3}{1 - \left(abx^2 + (c+d)\,y\right)t^2 + cdy^2t^4}. \end{aligned}$$

Hence, the proof is completed. □

Now we give the Binet formula of the bivariate conditional Lucas polynomial $L_n(x,y)$.

Theorem 5. *The n^{th} term of the generalized of bivariate conditional Lucas polynomial $L_n(x,y)$ is*

$$\begin{aligned} L_n(x,y) = \frac{(-ax)^{\xi(n)}}{\beta_1-\beta_2} & \\ \times \Bigg[& \left(\xi(n+1)\beta_1 + (-1)^{\xi(n+1)}\beta_2\right)(\beta_2+dy)^{\lfloor \frac{n}{2} \rfloor} \\ & + \left((-1)^{\xi(n+1)}(2)^{\xi(n)}dy + \xi(n+1)cy\right)(\beta_2+dy)^{\lfloor \frac{n}{2} \rfloor} \\ & - \left(\xi(n+1)\beta_2 + (-1)^{\xi(n+1)}\beta_1\right)(\beta_1+dy)^{\lfloor \frac{n}{2} \rfloor} \\ & - \left((-1)^{\xi(n+1)}(2)^{\xi(n)}dy + \xi(n+1)cy\right)(\beta_1+dy)^{\lfloor \frac{n}{2} \rfloor} \Bigg] \end{aligned} \tag{24}$$

where β_1 and β_2 roots of the characteristic equation $\lambda^2-(abx^2+(c-d)y)\lambda-abdx^2y = 0$.

Proof. We use the following properties throughout the proof:

- $\beta_1 + \beta_2 = abx^2 + (c-d)y$
- $\beta_1 \cdot \beta_2 = -abdx^2y$
- $(\beta_1 + dy)(\beta_2 + dy) = cdy^2$
- $(\beta_1 + dy)\left(abx^2\right) = \beta_1\left(\beta_1 + (d-c)y\right)$
- $(\beta_2 + dy)\left(abx^2\right) = \beta_2\left(\beta_2 + (d-c)y\right)$

Note that $\beta_1(x,y) = \beta_1$ and $\beta_2(x,y) = \beta_2$. Since $\frac{\beta_1+dy}{cdy^2}$ and $\frac{\beta_2+dy}{cdy^2}$ are roots of

$$1 - (abx^2 + (c+d)y)t^2 + cdy^2t^4 = 0.$$

If we assume

$$\begin{aligned} E_0(t) &= \sum_{n=0}^{\infty} L_{2n}(x,y)t^{2n} \\ E_1(t) &= \sum_{n=0}^{\infty} L_{2n+1}(x,y)t^{2n+1}. \end{aligned}$$

Then,

$$E(t) = E_0(t) + E_1(t).$$

By using Maclaurin's series expansion

$$\frac{Az - B}{z^2 - C} = \sum_{n=0}^{\infty} BC^{-n-1} Z^{2n} - \sum_{n=0}^{\infty} AC^{-n-1} Z^{2n+1}$$

and above-mentioned identities, we simplify both $E_0(t)$ and $E_1(t)$ as follows:

$$\begin{aligned}
E_0(t) &= \frac{1}{cdy^2(\beta_1 - \beta_2)} \left[\frac{2cdy^2 - \left(abx^2 + 2cy\right) \cdot (\beta_1 + dy)}{t^2 - \left(\frac{\beta_1 + dy}{cdy^2}\right)} \right. \\
&\qquad \left. - \frac{2cdy^2 - \left(abx^2 + 2cy\right)(\beta_2 + y)}{t^2 - \left(\frac{\beta_2 + dy}{cdy^2}\right)} \right] \\
&= \frac{1}{cdy^2(\beta_1 - \beta_2)} \sum_{n=0}^{\infty} \left[\left((abx^2 + 2cy)(\beta_1 + dy) - 2cdy^2\right) \left(\frac{\beta_1 + dy}{cdy^2}\right)^{-n-1} \right] t^{2n} \\
&\quad - \frac{1}{cdy^2(\beta_1 - \beta_2)} \sum_{n=0}^{\infty} \left[\left((abx^2 + 2cy)(\beta_2 + dy) - 2cdy^2\right) \left(\frac{\beta_2 + dy}{cdy^2}\right)^{-n-1} \right] t^{2n} \\
&= \frac{1}{cdy^2(\beta_1 - \beta_2)} \sum_{n=0}^{\infty} \left[\left((abx^2 + 2cy)(\beta_1 + dy)(\beta_2 + dy) \right. \right. \\
&\qquad \left. \left. - 2cdy^2(\beta_2 + dy) \right) (\beta_2 + dy)^n \right] t^{2n} \\
&\quad - \frac{1}{cdy^2(\beta_1 - \beta_2)} \sum_{n=0}^{\infty} \left[\left((abx^2 + 2cy)(\beta_2 + dy)(\beta_1 + dy) \right. \right. \\
&\qquad \left. \left. - 2cdy^2(\beta_1 + dy) \right) (\beta_1 + dy)^n \right] t^{2n} \\
&= \frac{1}{(\beta_1 - \beta_2)} \sum_{n=0}^{\infty} \left[\left(abx^2 - 2\beta_2 + 2cy - 2dy\right)(\beta_2 + dy)^n \right. \\
&\qquad \left. - \left(abx^2 - 2\beta_1 + 2cy - 2dy\right)(\beta_1 + dy)^n \right] t^{2n} \\
&= \frac{1}{(\beta_1 - \beta_2)} \sum_{n=0}^{\infty} \left[\left(\beta_1 - \beta_2 - (d - c)y\right)(\beta_2 + dy)^n \right. \\
&\qquad \left. - \left(\beta_2 - \beta_1 - (d - c)y\right)(\beta_1 + dy)^n \right] t^{2n}.
\end{aligned}$$

We solve $E_1(t)$ with the same approach used in $E_0(t)$ and we get the value of

$$E_1(t) = \frac{-ax}{(\beta_1 - \beta_2)} \sum_{n=0}^{\infty} \left[(\beta_2 + 2dy)(\beta_2 + dy)^n - (\beta_1 + 2dy)(\beta_1 + dy)^n \right] t^{2n+1}.$$

We know that $E(t) = E_0(t) + E_1(t)$. So we find

$$\begin{aligned} E(t) = \sum_{n=0}^{\infty} \frac{(-ax)^{\xi(n)}}{\beta_1 - \beta_2} \Bigg[\Bigg(& \xi(n+1)\beta_1 + (-1)^{\xi(n+1)}\beta_2 \\ & + (-1)^{\xi(n+1)}(2)^{\xi(n)}dy + \xi(n+1)cy \Bigg) (\beta_2 + dy)^{\lfloor \frac{n}{2} \rfloor} \\ - \Bigg(& \xi(n+1)\beta_2 + (-1)^{\xi(n+1)}\beta_1 \\ & + (-1)^{\xi(n+1)}(2)^{\xi(n)}dy + \xi(n+1)cy \Bigg) (\beta_1 + dy)^{\lfloor \frac{n}{2} \rfloor} \Bigg]. \end{aligned}$$

Thus,

$$\begin{aligned} L_n(x,y) = \frac{(-ax)^{\xi(n)}}{\beta_1 - \beta_2} \Bigg[\Bigg(& \xi(n+1)\beta_1 + (-1)^{\xi(n+1)}\beta_2 \\ & + (-1)^{\xi(n+1)}(2)^{\xi(n)}dy + \xi(n+1)cy \Bigg) (\beta_2 + dy)^{\lfloor \frac{n}{2} \rfloor} \\ - \Bigg(& \xi(n+1)\beta_2 + (-1)^{\xi(n+1)}\beta_1 \\ & + (-1)^{\xi(n+1)}(2)^{\xi(n)}dy + \xi(n+1)cy \Bigg) (\beta_1 + dy)^{\lfloor \frac{n}{2} \rfloor} \Bigg]. \end{aligned}$$

□

In the following definition, we give bivariate conditional Lucas Hybrinomials.

Definition 3. *For any variable x, y and nonzero real numbers a, b, c and d, we have*

$$LH_n(x,y) = L_n(x,y) + \boldsymbol{i}L_{n+1}(x,y) + \varepsilon L_{n+2}(x,y) + \boldsymbol{h}L_{n+3}(x,y) \tag{25}$$

where $L_n(x,y)$ was given in (20) *and the initial conditions are with $LH_0(x,y) = 2 + \boldsymbol{i}ax + \varepsilon(abx^2 + 2dy) + \boldsymbol{h}(a^2bx^3 + 2adxy + acxy)$ and $LH_1(x,y) = ax + \boldsymbol{i}(abx^2 + 2dy) + +\varepsilon(a^2bx^3 + 2adxy + acxy) + \boldsymbol{h}(a^2b^2x^4 + 2bcdx^2y + abcx^2y + abdx^2y + 2d^2y^2)$.*

We can see from the following table that the generalized bivariate conditional Fibonacci hybrinomials are the generalization of many works for different values of a, b, c and d.

TABLE 3. The generalized bivariate conditional Lucas hybrinomials

a	***b***	***c***	***d***	***Generalized Bivariate Conditional Lucas Hybrinomials***
1	1	1	1	*Bivariate Lucas Hybrinomials*
a	b	1	1	*Bivariate Conditional Lucas Hybrinomials*
2	2	1	1	*Bivariate Pell Lucas Hybrinomials*
1	1	2	2	*Bivariate Jacobsthal Lucas Hybrinomials*
⋮	⋮	⋮	⋮	⋮

Lemma 3. *For the generalized bivariate conditional Lucas hybrinomials* $\{LH_n(x,y)\}_{n=0}^{\infty}$*, we have*

$$LH_{2n}(x,y) = \left(abx^2 + (c+d)y\right) LH_{2n-2}(x,y) - cdy^2 LH_{2n-4}(x,y)$$
$$LH_{2n+1}(x,y) = \left(abx^2 + (c+d)y\right) LH_{2n-1}(x,y) - cdy^2 LH_{2n-3}(x,y).$$

Proof. By using the definition of the generalized bivariate conditional Lucas hybrinomials, we obtain

$$\begin{aligned}
LH_{2n}(x,y) &= L_{2n}(x,y) + \mathbf{i}L_{2n+1}(x,y) + \varepsilon L_{2n+2}(x,y) + \mathbf{h}L_{2n+3}(x,y) \\
&= (bxL_{2n-1}(x,y) + dyL_{2n-2}(x,y)) + \mathbf{i}\,(axL_{2n}(x,y) + cyL_{2n-1}(x,y)) \\
&\quad + \varepsilon(bxL_{2n+1}(x,y) + dyL_{2n}(x,y)) + \mathbf{h}(axL_{2n+2}(x,y) + cyL_{2n+1}(x,y)) \\
&= [bx\,(axL_{2n-2}(x,y) + cyL_{2n-3}(x,y)) + dyL_{2n-2}(x,y)] \\
&\quad + \mathbf{i}\,[ax\,(bxL_{2n-1}(x,y) + dyL_{2n-2}(x,y)) + cyL_{2n-1}(x,y)] \\
&\quad + \varepsilon\,[bx\,(axL_{2n}(x,y) + cyL_{2n-1}(x,y)) + dyL_{2n}(x,y))] \\
&\quad + \mathbf{h}\,[ax\,(bxL_{2n+1}(x,y) + dyL_{2n}(x,y)) + cyL_{2n+1}(x,y)] \\
&= [\left(abx^2 + dy\right) L_{2n-2}(x,y) + cy\,(bxL_{2n-3}(x,y))] \\
&\quad + \mathbf{i}[\left(abx^2 + cy\right) L_{2n-1}(x,y) + dy\,(axL_{2n-2}(x,y))] \\
&\quad + \varepsilon[\left(abx^2 + dy\right) L_{2n}(x,y) + cy\,(bxL_{2n-1}(x,y))] \\
&\quad + \mathbf{h}[\left(abx^2 + cy\right) L_{2n+1}(x,y) + dy\,(axL_{2n}(x,y))] \\
&= [\left(abx^2 + dy\right) L_{2n-2}(x,y) + cy\,(L_{2n-2}(x,y) - dyL_{2n-4}(x,y))] \\
&\quad + \mathbf{i}[\left(abx^2 + cy\right) L_{2n-1}(x,y) + dy\,(L_{2n-1}(x,y) - cyL_{2n-3}(x,y))] \\
&\quad + \varepsilon[\left(abx^2 + dy\right) L_{2n}(x,y) + cy\,(L_{2n}(x,y) - dyL_{2n-2}(x,y))] \\
&\quad + \mathbf{h}[\left(abx^2 + cy\right) L_{2n+1}(x,y) + dy\,(L_{2n+1}(x,y) - cyL_{2n-1}(x,y))] \\
&= [\left(abx^2 + (c+d)\,y\right) L_{2n-2}(x,y) - cdy^2 L_{2n-4}(x,y)] \\
&\quad + \mathbf{i}[\left(abx^2 + (c+d)\,y\right) L_{2n-1}(x,y) - cdy^2 L_{2n-3}(x,y)] \\
&\quad + \varepsilon[\left(abx^2 + (c+d)\,y\right) L_{2n}(x,y) - cdy^2 L_{2n-2}(x,y)] \\
&\quad + \mathbf{h}[\left(abx^2 + (c+d)\,y\right) L_{2n+1}(x,y) - cdy^2 L_{2n-1}(x,y)]
\end{aligned}$$

$$= \left(abx^2 + (c+d)\,y\right)[L_{2n-2}(x,y) + \mathbf{i}L_{2n-1}(x,y) + \varepsilon L_{2n}(x,y) + \mathbf{h}L_{2n+1}(x,y)]$$
$$- cdy^2[L_{2n-4}(x,y) + \mathbf{i}L_{2n-3}(x,y) + \varepsilon L_{2n-2}(x,y) + \mathbf{h}L_{2n-1}(x,y)]$$
$$= \left(abx^2 + (c+d)\,y\right) LH_{2n-2}(x,y) - cdy^2 LH_{2n-4}(x,y)$$

Similar to above, we can obtain

$$LH_{2n+1}(x,y) = \left(abx^2 + (c+d)y\right) LH_{2n-1}(x,y) - cdy^2 LH_{2n-3}(x,y).$$

Thus, the proof is completed. □

Next we give the generating function of the bivariate conditional Lucas hybrinomial $LH_n(x,y)$.

Theorem 6. *The generating function for the bivariate conditional Lucas hybrinomial* $LH_n(x,y)$ *is*

$$\begin{aligned}\Omega(t) &= \sum_{n=0}^{\infty} LH_n(x,y)t^n \\ &= \frac{LH_0(x,y) + LH_1(x,y)t + \left[LH_2(x,y) - \left(abx^2 + (c+d)y\right) LH_0(x,y)\right] t^2}{1 - \left(abx^2 + (c+d)\,y\right)t^2 + cdy^2t^4} \\ &\quad + \frac{\left[LH_3(x,y) - \left(abx^2 + (c+d)\,y\right) LH_1(x,y)\right] t^3}{1 - \left(abx^2 + (c+d)\,y\right)t^2 + cdy^2t^4}.\end{aligned}$$

Proof. We define

$$\Omega_0(t) = \sum_{n=0}^{\infty} LH_{2n}(x,y)t^{2n}$$
$$\Omega_1(t) = \sum_{n=0}^{\infty} LH_{2n+1}(x,y)t^{2n+1}.$$

So that

$$\Omega(t) = \Omega_0(t) + \Omega_1(t).$$

We have

$$\begin{aligned}
\Omega_0(t) &= \sum_{n=0}^{\infty} LH_{2n}(x,y)t^{2n} \\
&= \sum_{n=0}^{\infty} LH_{2n}(x,y)t^{2n} = LH_0(x,y)t^0 + LH_2(x,y)t^2 + \sum_{n=2}^{\infty} LH_{2n}(x,y)t^{2n} \\
&= LH_0(x,y) + LH_2(x,y)t^2 \\
&\quad + \sum_{n=2}^{\infty} \left[\left(abx^2 + (c+d)y\right) LH_{2n-2}(x,y) - cdy^2 LH_{2n-4}(x,y) \right] t^{2n} \\
&= LH_0(x,y) + LH_2(x,y)t^2 + \left(abx^2 + (c+d)y\right) t^2 \sum_{n=2}^{\infty} LH_{2n-2}(x,y)t^{2n-2} \\
&\quad - cdy^2 t^4 \sum_{n=2}^{\infty} LH_{2n-4}(x,y)t^{2n-4} \\
&= LH_0(x,y) + LH_2(x,y)t^2 \\
&\quad + \left(abx^2 + (c+d)y\right) t^2 \left[\sum_{n=2}^{\infty} LH_{2n-2}(x,y)t^{2n-2} + LH_0(x,y)t^0 - LH_0(x,y)t^0 \right] \\
&\quad - cdy^2 t^4 \Omega_0(t) \\
&= LH_0(x,y) + LH_2(x,y)t^2 + \left(abx^2 + (c+d)y\right) t^2 \Omega_0(t) \\
&\quad - \left(abx^2 + (c+d)y\right) t^2 LH_0(x,y) - cdy^2 t^4 \Omega_0(t).
\end{aligned}$$

Thus, we get

$$\Omega_0(t) = \frac{LH_0(x,y) + \left(LH_2(x,y) - \left(abx^2 + (c+d)y\right) LH_0(x,y)\right) t^2}{1 - \left(abx^2 + (c+d)y\right) t^2 + cdy^2 t^4}. \tag{26}$$

Similarly, we find

$$\begin{aligned}
\Omega_1(t) &= \sum_{n=0}^{\infty} LH_{2n+1}(x,y)t^{2n+1} \\
&= \sum_{n=0}^{\infty} LH_{2n+1}(x,y)t^{2n+1} = LH_1(x,y)t + LH_3(x,y)t^3 \\
&\quad + \sum_{n=2}^{\infty} LH_{2n+1}(x,y)t^{2n+1} \\
&= LH_1(x,y)t + LH_3(x,y)t^3 \\
&\quad + \sum_{n=2}^{\infty} \left[\left(abx^2 + (c+d)y\right) LH_{2n-1}(x,y) - cdy^2 LH_{2n-3}(x,y) \right] t^{2n+1}
\end{aligned}$$

$$\begin{aligned}
&= LH_1(x,y)t + LH_3(x,y)t^3 \\
&\quad + \left(abx^2 + (c+d)y\right) t^2 \sum_{n=2}^{\infty} LH_{2n-1}(x,y)t^{2n-1} - cdy^2t^4 \sum_{n=2}^{\infty} LH_{2n-3}(x,y)t^{2n-3} \\
&= LH_1(x,y)t + LH_3(x,y)t^3 \\
&\quad + \left(abx^2 + (c+d)y\right) t^2 \left[\sum_{n=2}^{\infty} LH_{2n-1}(x,y)t^{2n-1} + LH_1(x,y)t - LH_1(x,y)t\right] \\
&\quad - cdy^2t^4\Omega_1(t) \\
&= LH_1(x,y)t + LH_3(x,y)t^3 + \left(abx^2 + (c+d)y\right) t^2\Omega_1(t) \\
&\quad - \left(abx^2 + (c+d)y\right) t^3 LH_1(x,y) - cdy^2t^4\Omega_1(t).
\end{aligned}$$

Therefore, we get

$$\Omega_1(t) = \frac{LH_1(x,y)t + \left(LH_3(x,y) - \left(abx^2 + (c+d)y\right) LH_1(x,y)\right) t^3}{1 - \left(abx^2 + (c+d)y\right) t^2 + cdy^2t^4}. \tag{27}$$

By virtue of (26) and (27), we can obtain

$$\begin{aligned}
\Omega(t) &= \Omega_0(t) + \Omega_1(t) \\
&= \sum_{n=0}^{\infty} LH_n(x,y)t^n \\
&= \frac{LH_0(x,y) + LH_1(x,y)t + \left[LH_2(x,y) - \left(abx^2 + (c+d)y\right) LH_0(x,y)\right] t^2}{1 - \left(abx^2 + (c+d)\,y\right) t^2 + cdy^2t^4} \\
&\quad + \frac{\left[LH_3(x,y) - \left(abx^2 + (c+d)\,y\right) LH_1(x,y)\right] t^3}{1 - \left(abx^2 + (c+d)\,y\right) t^2 + cdy^2t^4}.
\end{aligned}$$

Hence, the proof is completed. □

Now we give the Binet formula of the bivariate conditional Lucas hybrinomial $LH_n(x,y)$.

Theorem 7. *The n^{th} term of the generalized of bivariate conditional Lucas hybrinomial $LH_n(x,y)$ is*

$$LH_n(x,y) = \frac{\widehat{\omega}_{\xi(n)} \left(\beta_2 + dy\right)^{\lfloor \frac{n}{2} \rfloor} - \widehat{\sigma}_{\xi(n)} \left(\beta_1 + dy\right)^{\lfloor \frac{n}{2} \rfloor}}{\beta_1 - \beta_2}. \tag{28}$$

where β_1 *and* β_2 *roots of the characteristic equation* $\lambda^2-(abx^2+(c-d)y)\lambda-abdx^2y=0$. *Also,*

$$\begin{aligned}
\widehat{\omega}_{\xi(n)} &= (-ax)^{\xi(n)}\left(\xi(n+1)\beta_1+(-1)^{\xi(n+1)}\beta_2+(-1)^{\xi(n+1)}(2)^{\xi(n)}dy+\xi(n+1)cy\right)\\
&+\boldsymbol{i}(-ax)^{\xi(n+1)}\left(\xi(n)\beta_1+(-1)^{\xi(n)}\beta_2+(-1)^{\xi(n)}(2)^{\xi(n+1)}dy+\xi(n)cy\right)(\beta_2+dy)^{\xi(n)}\\
&+\varepsilon(-ax)^{\xi(n)}\left(\xi(n+1)\beta_1+(-1)^{\xi(n+1)}\beta_2+(-1)^{\xi(n+1)}(2)^{\xi(n)}dy+\xi(n+1)cy\right)(\beta_2+dy)\\
&+\boldsymbol{h}(-ax)^{\xi(n+1)}\left(\xi(n)\beta_1+(-1)^{\xi(n)}\beta_2+(-1)^{\xi(n)}(2)^{\xi(n+1)}dy+\xi(n)cy\right)(\beta_2+dy)^{\xi(n)+1}
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\sigma}_{\xi(n)} &= (-ax)^{\xi(n)}\left(\xi(n+1)\beta_2+(-1)^{\xi(n+1)}\beta_1+(-1)^{\xi(n+1)}(2)^{\xi(n)}dy+\xi(n+1)cy\right)\\
&+\boldsymbol{i}(-ax)^{\xi(n+1)}\left(\xi(n)\beta_2+(-1)^{\xi(n)}\beta_1+(-1)^{\xi(n)}(2)^{\xi(n+1)}dy+\xi(n)cy\right)(\beta_1+dy)^{\xi(n)}\\
&+\varepsilon(-ax)^{\xi(n)}\left(\xi(n+1)\beta_2+(-1)^{\xi(n+1)}\beta_1+(-1)^{\xi(n+1)}(2)^{\xi(n)}dy+\xi(n+1)cy\right)(\beta_1+dy)\\
&+\boldsymbol{h}(-ax)^{\xi(n+1)}\left(\xi(n)\beta_2+(-1)^{\xi(n)}\beta_1+(-1)^{\xi(n)}(2)^{\xi(n+1)}dy+\xi(n)cy\right)(\beta_1+dy)^{\xi(n)+1}.
\end{aligned}$$

Proof. Firstly, by using (24), we have

$$\begin{aligned}
LH_{2n}(x,y) &= L_{2n}(x,y)+\mathbf{i}L_{2n+1}(x,y)+\varepsilon L_{2n+2}(x,y)+\mathbf{h}L_{2n+3}(x,y)\\
&= \frac{(\beta_2+dy)^n}{\beta_1-\beta_2}\Big[(\beta_1-\beta_2-dy+cy)+\mathbf{i}(-ax)(\beta_2+2dy)\\
&\qquad+\varepsilon(\beta_1-\beta_2-dy+cy)(\beta_2+dy)\\
&\qquad+\mathbf{h}(-ax)(\beta_2+2dy)(\beta_2+dy)\Big]\\
&\quad-\frac{(\beta_1+dy)^n}{\beta_1-\beta_2}\Big[(\beta_2-\beta_1-dy+cy)+\mathbf{i}(-ax)(\beta_1+2dy)\\
&\qquad+\varepsilon(\beta_2-\beta_1-dy+cy)(\beta_1+dy)\\
&\qquad+\mathbf{h}(-ax)(\beta_1+2dy)(\beta_1+dy)\Big].
\end{aligned}$$

Here, we choose the $\widehat{\omega}_0$ and $\widehat{\sigma}_0$ as follows;

$$\begin{aligned}
\widehat{\omega}_0 &= \Big[(\beta_1-\beta_2-dy+cy)+\mathbf{i}(-ax)(\beta_2+2dy)+\varepsilon(\beta_1-\beta_2-dy+cy)(\beta_2+dy)\\
&\quad+\mathbf{h}(-ax)(\beta_2+2dy)(\beta_2+dy)\Big]\\
\widehat{\sigma}_0 &= \Big[(\beta_2-\beta_1-dy+cy)+\mathbf{i}(-ax)(\beta_1+2dy)+\varepsilon(\beta_2-\beta_1-dy+cy)(\beta_1+dy)\\
&\quad+\mathbf{h}(-ax)(\beta_1+2dy)(\beta_1+dy)\Big].
\end{aligned}$$

Finally, the following equation is obtained:

$$LH_{2n}(x,y)=\frac{\widehat{\omega}_0(\beta_2+dy)^n-\widehat{\sigma}_0(\beta_1+dy)^n}{\beta_1-\beta_2}. \tag{29}$$

In similar way, by using (24), we have

$$\begin{aligned} LH_{2n+1}(x,y) &= L_{2n+1}(x,y) + \mathbf{i}L_{2n+2}(x,y) + \varepsilon L_{2n+3}(x,y) + \mathbf{h}L_{2n+4}(x,y) \\ &= \frac{(\beta_2+dy)^n}{\beta_1-\beta_2}\big[(-ax)(\beta_2+2dy) + \mathbf{i}(\beta_1-\beta_2-dy+cy)(\beta_2+dy) \\ &\qquad + \varepsilon(-ax)(\beta_2+2dy)(\beta_2+dy) \\ &\qquad + \mathbf{h}(\beta_1-\beta_2-dy+cy)(\beta_2+dy)^2\big] \\ &\quad - \frac{(\beta_1+dy)^n}{\beta_1-\beta_2}\big[(-ax)(\beta_1+2dy) + \mathbf{i}(\beta_2-\beta_1-dy+cy)(\beta_1+dy) \\ &\qquad + \varepsilon(-ax)(\beta_1+2dy)(\beta_1+dy) \\ &\qquad + \mathbf{h}(\beta_2-\beta_1-dy+cy)(\beta_1+dy)^2\big]. \end{aligned}$$

Here, we choose the $\widehat{\omega}_1$ and $\widehat{\sigma}_1$ as follows;

$$\begin{aligned} \widehat{\omega}_1 &= \big[(-ax)(\beta_2+2dy) + \mathbf{i}(\beta_1-\beta_2-dy+cy)(\beta_2+dy) \\ &\quad + \varepsilon(-ax)(\beta_2+2dy)(\beta_2+dy) + \mathbf{h}(\beta_1-\beta_2-dy+cy)(\beta_2+dy)^2\big] \\ \widehat{\sigma}_1 &= \big[(-ax)(\beta_1+2dy) + \mathbf{i}(\beta_2-\beta_1-dy+cy)(\beta_1+dy) \\ &\quad + \varepsilon(-ax)(\beta_1+2dy)(\beta_1+dy) + \mathbf{h}(\beta_2-\beta_1-dy+cy)(\beta_1+dy)^2\big]. \end{aligned}$$

Finally, the following equation is obtained.

$$LH_{2n+1}(x,y) = \frac{\widehat{\omega}_1(\beta_2+dy)^n - \widehat{\sigma}_1(\beta_1+dy)^n}{\beta_1-\beta_2}. \tag{30}$$

By virtue of (29) and (30), we can obtain the following equation

$$LH_n(x,y) = \frac{\widehat{\omega}_{\xi(n)}(\beta_2+dy)^{\lfloor\frac{n}{2}\rfloor} - \widehat{\sigma}_{\xi(n)}(\beta_1+dy)^{\lfloor\frac{n}{2}\rfloor}}{\beta_1-\beta_2}.$$

where β_1 and β_2 roots of the characteristic equation $\lambda^2 - (abx^2 + (c-d)y)\lambda - abdx^2y = 0$. □

Now, we give the Catalan's identity of the bivariate conditional Lucas hybrinomial $LH_n(x,y)$.

Theorem 8. *For any integers n and r and $n \geqslant r \geqslant 0$, $r \geq 0$, we have*

$$\begin{aligned} &LH_{2(n+r)+\xi(i)}(x,y)LH_{2(n-r)+\xi(i)}(x,y) - \left(LH_{2n+\xi(i)}(x,y)\right)^2 \\ &\qquad = \frac{\widehat{\omega}_{\xi(i)}\widehat{\sigma}_{\xi(i)}(\beta_2+dy)^n(\beta_1+dy)^n\left[1-\left(\frac{\beta_2+dy}{\beta_1+dy}\right)^r\right]}{(\beta_1-\beta_2)^2} \\ &\qquad\quad + \frac{\widehat{\sigma}_{\xi(i)}\widehat{\omega}_{\xi(i)}(\beta_1+dy)^n(\beta_2+dy)^n\left[1-\left(\frac{\beta_1+dy}{\beta_2+dy}\right)^r\right]}{(\beta_1-\beta_2)^2}. \end{aligned}$$

where $\widehat{\omega}_{\xi(i)}$ and $\widehat{\sigma}_{\xi(i)}$ are defined in Theorem(7) and $i \in \{0,1\}$.

Proof. In order to prove Catalan's identity, we will examine two different cases.
Case $i=0$:

$$\begin{aligned} LH_{2(n+r)}(x,y) &= \frac{\widehat{\omega}_{\xi(2n+2r)}\left(\beta_2+dy\right)^{\lfloor\frac{2n+2r}{2}\rfloor}-\widehat{\sigma}_{\xi(2n+2r)}\left(\beta_1+dy\right)^{\lfloor\frac{2n+2r}{2}\rfloor}}{\beta_1-\beta_2} \\ &= \frac{\widehat{\omega}_0\left(\beta_2+dy\right)^{n+r}-\widehat{\sigma}_0\left(\beta_1+dy\right)^{n+r}}{\beta_1-\beta_2} \end{aligned} \tag{31}$$

$$\begin{aligned} LH_{2(n-r)}(x,y) &= \frac{\widehat{\omega}_{\xi(2n-2r)}\left(\beta_2+dy\right)^{\lfloor\frac{2n-2r}{2}\rfloor}-\widehat{\sigma}_{\xi(2n-2r)}\left(\beta_1+dy\right)^{\lfloor\frac{2n-2r}{2}\rfloor}}{\beta_1-\beta_2} \\ &= \frac{\widehat{\omega}_0\left(\beta_2+dy\right)^{n-r}-\widehat{\sigma}_0\left(\beta_1+dy\right)^{n-r}}{\beta_1-\beta_2} \end{aligned} \tag{32}$$

$$\begin{aligned} LH_{2n}(x,y) &= \frac{\widehat{\omega}_{\xi(2n)}\left(\beta_2+dy\right)^{\lfloor\frac{2n}{2}\rfloor}-\widehat{\sigma}_{\xi(2n)}\left(\beta_1+dy\right)^{\lfloor\frac{2n}{2}\rfloor}}{\beta_1-\beta_2} \\ &= \frac{\widehat{\omega}_0\left(\beta_2+dy\right)^{n}-\widehat{\sigma}_0\left(\beta_1+dy\right)^{n}}{\beta_1-\beta_2} \end{aligned} \tag{33}$$

By virtue of (31), (32) and (33), we have

$$\begin{aligned} & LH_{2(n+r)}(x,y)LH_{2(n-r)}(x,y)-\left(LH_{2n}(x,y)\right)^2 \\ &\qquad = \frac{\widehat{\omega}_0\widehat{\sigma}_0\left(\beta_2+dy\right)^n\left(\beta_1+dy\right)^n\left[1-\left(\frac{\beta_2+dy}{\beta_1+dy}\right)^r\right]}{\left(\beta_1-\beta_2\right)^2} \\ &\qquad\quad + \frac{\widehat{\sigma}_0\widehat{\omega}_0\left(\beta_1+dy\right)^n\left(\beta_2+dy\right)^n\left[1-\left(\frac{\beta_1+dy}{\beta_2+dy}\right)^r\right]}{\left(\beta_1-\beta_2\right)^2}. \end{aligned}$$

Case $i=1$:

$$LH_{2(n+r)+1}(x,y) = \frac{\widehat{\omega}_1\left(\beta_2+dy\right)^{n+r}-\widehat{\sigma}_1\left(\beta_1+dy\right)^{n+r}}{\beta_1-\beta_2} \tag{34}$$

$$LH_{2(n-r)+1}(x,y) = \frac{\widehat{\omega}_1\left(\beta_2+dy\right)^{n-r}-\widehat{\sigma}_1\left(\beta_1+dy\right)^{n-r}}{\beta_1-\beta_2} \tag{35}$$

$$LH_{2n+1}(x,y) = \frac{\widehat{\omega}_1\left(\beta_2+dy\right)^{n}-\widehat{\sigma}_1\left(\beta_1+dy\right)^{n}}{\beta_1-\beta_2} \tag{36}$$

By virtue of (34), (35) and (36), we have

$$LH_{2(n+r)+1}(x,y)LH_{2(n-r)+1}(x,y) - (LH_{2n+1}(x,y))^2$$
$$= \frac{\widehat{\omega}_1\widehat{\sigma}_1 (\beta_2+dy)^n (\beta_1+dy)^n \left[1-\left(\frac{\beta_2+dy}{\beta_1+dy}\right)^r\right]}{(\beta_1-\beta_2)^2}$$
$$+ \frac{\widehat{\sigma}_1\widehat{\omega}_1 (\beta_1+dy)^n (\beta_2+dy)^n \left[1-\left(\frac{\beta_1+dy}{\beta_2+dy}\right)^r\right]}{(\beta_1-\beta_2)^2}.$$

Thus, the proof is completed. □

Now, we give the Cassini's identity of the bivariate conditional Lucas hybrinomial $LH_n(x,y)$.

Corollary 2. *For $n \geq 0$, we get*

$$LH_{2(n+1)+\xi(i)}(x,y)LH_{2(n-1)+\xi(i)}(x,y) - \left(LH_{2n+\xi(i)}(x,y)\right)^2$$
$$= \frac{\widehat{\sigma}_{\xi(i)}\widehat{\omega}_{\xi(i)} (\beta_1+dy)^n (\beta_2+dy)^n \left[1-\left(\frac{\beta_1+dy}{\beta_2+dy}\right)\right]}{(\beta_1-\beta_2)^2}$$
$$+ \frac{\widehat{\sigma}_{\xi(i)}\widehat{\omega}_{\xi(i)} (\beta_1+dy)^n (\beta_2+dy)^n \left[1-\left(\frac{\beta_1+dy}{\beta_2+dy}\right)\right]}{(\beta_1-\beta_2)^2}$$

where $\widehat{\omega}_{\xi(i)}$ and $\widehat{\sigma}_{\xi(i)}$ are defined in Theorem(7) and $i \in \{0,1\}$.

Proof. Taking $r = 1$ in Catalan's identity the proof is completed. □

4. Conclusion

The Fibonacci and Lucas numbers are well–known numbers, which have been studied by many researchers for years. These numbers arise in the applications of mathematics, computer science, physics, biology and statistics [9]. In this paper, by combining the Fibonacci and Lucas numbers with hybrid numbers, we present the generalized bivariate conditional Fibonacci and Lucas hybrinomials which are generalization of many works in the literature. Moreover, we derive many properties of generalized bivariate conditional Fibonacci and Lucas hybrinomials such as Binet's formulas, Catalan's identity, Cassini's identity of the hybrinomials.

Author Contribution Statements All authors contributed equally to this manuscript and they read and approved the final manuscript.

Declaration of Competing Interests The authors declare that they have no competing interest.

Acknowledgment This study is a part of the second author's Master Thesis.

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