Research Article GU J Sci 37(2): 979-1002 (2024) DOI: 10.35378/gujs.1249968

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Classical and Bayesian Inference for the Length Biased Weighted Lomax Distribution under Progressive Censoring Scheme

Amal HASSAN, Samah ATIA*, Hiba MUHAMMED

Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt

Highlights

- This work studies reliability estimation for weighted Lomax model with progressive censored samples.
- A comparative study of Bayesian and non-Bayesian is employed for the estimation problem.
- For illustration, a simulation study as well as a real data set is given.

Article Info

Received: 15 Feb 2023
Accepted: 23 July 2023

Keywords

Length biased weighted Lomax, Maximum likelihood estimation, Squared error loss function, Informative prior, Markov chain Monte Carlo

Abstract

In this study, the length biassed weighted Lomax (LBWLo) distribution's reliability and hazard functions, as well as the population characteristics, are evaluated using progressively Type II censored samples. The proposed estimators are obtained by combining the maximum likelihood and Bayesian approaches. The posterior distribution of the LBWLo distribution is derived from the Gamma and Jeffery's priors, which, respectively, act as informative and non-informative priors. The Metropolis-Hasting (MH) algorithm is also utilized to get the Bayesian estimates. Based on the Fisher information matrix, we derive asymptotic confidence intervals. We create the intervals with the highest posterior density using the sample the MH technique generated. Numerical simulation research is done to evaluate the effectiveness of the approaches. Through Monte Carlo simulation, we compare the proposed estimates in terms of mean squared error. It is possible to get coverage probability and average interval lengths of 95%. The study's findings supported the idea that, in the majority of the cases, Bayes estimates with an informative prior are more appropriate than other estimates. Additionally, one set of actual data supported the findings of the study.

1. INTRODUCTION

Among other areas, the field of life testing has given the Lomax (Pareto II) distribution a great deal of attention [1]. Data on income and wealth were modelled using the Lomax distribution (see [2] and [3]). In accordance with information in [4], it has also found use in biological science. In reliability and life testing investigations, it has been employed (see [5–9]).

The length biassed weighted Lomax (LBWLo) distribution was proposed as a more flexible option for modelling data in a range of domains, including lifetime analysis, engineering, and biomedical sciences. When observations from a sample are recorded with uneven probability, the LBWLo distribution manifests in practise and provides a unifying solution for the problems that arise when the observations fall into the non-experimental, non-replicated, and non-random categories. The LBWLo distribution was proposed in [10] and some of its statistical features were covered. The following is the formula for the probability density function (PDF) of the LBWLo distribution with shape parameter $\theta > 1$ and scale parameter $\lambda > 0$:

$$f(x) = \frac{\theta(\theta-1)x}{\lambda^2}\left[1+\frac{x}{\lambda}\right]^{-(\theta+1)} \qquad x, \lambda > 0, \theta > 1. \tag{1}$$

The cumulative distribution function (CDF) of the LBWLo distribution is given by:

*Corresponding author, e-mail: 12422019446259@pg.cu.edu.eg

$$F(x)=1-\left[1+\frac{\theta x}{\lambda}\right]\left[1+\frac{x}{\lambda}\right]^{-\theta}, \qquad x,\lambda>0,\theta>1. \tag{2}$$

The LBWLo distribution's reliability function (RF) and hazard rate function (HRF) are described as follows:

$$S(x)=\left[1+\frac{\theta x}{\lambda}\right]\left[1+\frac{x}{\lambda}\right]^{-\theta}, \qquad x,\lambda>0,\theta>1,$$

and,

$$h(x)=\frac{\theta x(\theta-1)}{(\lambda+x)(\lambda+\theta x)}, \qquad x,\lambda>0,\theta>1.$$

Plots of the PDF and HRF are represented in Figure 1. As seen, the PDF and HRF of the LBWLo distribution take different shapes. There are several helpful forms for the PDF in Figure 1. The HRF of the LBWLo distribution can take on a number of shapes, including expanding, decreasing, and upside-down, as shown graphically in Figure 1.

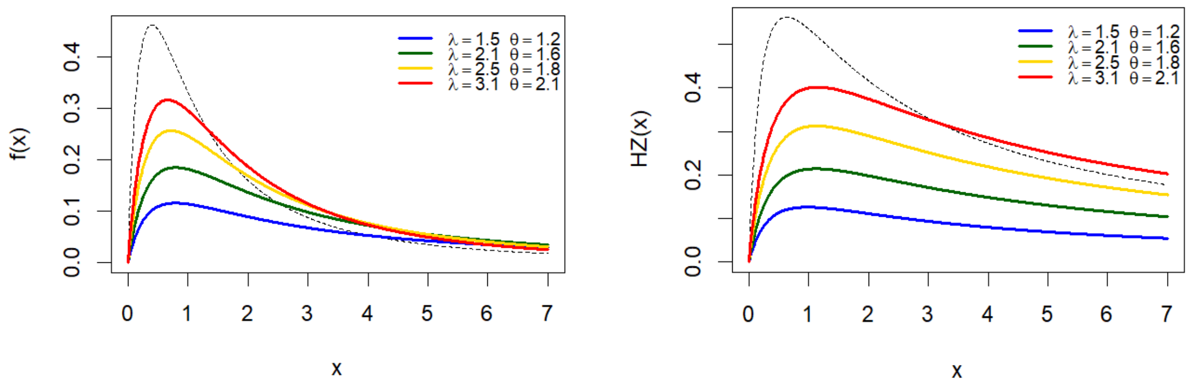


***Figure 1.** The distribution's LBWLo PDF and HRF plots*

[11] examined the LBWLo distribution's stress strength reliability estimator in the presence of outliers. [12] discussed Bayesian estimator of accelerated life tests for LBWLo distribution. Therefore, this study was carried out considering that in [11] and [12], the LBWLo distribution is suitable for reliability and life tests and the progressive Type II censoring (PT2C) scheme is very important in these areas. Suppose n identical units are placed on a life testing experiment and the progressive censoring scheme $\underline{R}=(R_1,R_2,...,R_m)$ is pre-fixed such that after the first failure R_1 surviving items are extracted from remaining $(n-1)$ live items and after the second failure R_2 surviving items are eliminated from remaining $(n-R_1-2)$ live items, and so on. After m^{th} failure, this process is repeated until all $R_m=n-m-R_1-...R_{m-1}$ remaining items are eliminated (see [13]). Consequently, a PT2C scheme includes m and $R_1,R_2,...,R_m$; such that $\sum_{i=1}^{m}R_i+m=n$. Take note of the fact that the PT2C scheme is reduced to complete sampling scheme, for $R_1=R_2=...=R_m=0$. Additionally, PT2C offers the type II censoring method for $R_1=R_2=...=R_{m-1}=0$ and $R_m=n-m$ (see [14]). For more studies and application for progressive censoring, the reader can refer to [15–19].

The estimation of the LBWLo distribution is considered in the current work utilising PT2C data. The parameters, RF, and HRF are estimated using maximum likelihood (ML) and Bayesian methods. ACIs and BCIs, also known as approximate confidence intervals, are created. Here is a suggested structure for this essay. In section 2, you will find the ML estimators and ACIs. We present Bayesian estimation using uniform and gamma priors, together with their BCIs, in sections 3 and 4, respectively. Studies and findings related to numbers are covered in section 5. The study of one actual data set is discussed in section 6. A few concluding remarks are made in section 7.

2. MAXIMUM LIKELIHOOD ESTIMATION

The ML estimator of the population parameters, RF, and HRF of the LBWLo distribution are derived in this section.

Let $x_{(1)},x_{(2)},...,x_{(m)}$ be an ordered PT2C sample, according to [20], the likelihood function of the observed sample is given by:

$$l(\underline{x}|\theta)=C\prod_{i=1}^{m} f(x_{(i)})\left[1-F(x_{(i)})\right]^{R_i}, \tag{3}$$

where $C=n(n-R_1-1)\,...\,(n-R_1-R_2-\cdots-R_{m-1}-m+1).$ Consider a random sample of size n from PDF (1) and CDF (2), the likelihood function of the LBWLo distribution under PT2C using (3) is as follows:

$$l(\underline{x}|\lambda,\theta)=C\prod_{i=1}^{m}\frac{\theta(\theta-1)x_i}{\lambda^2}\left(1+\frac{x_i}{\lambda}\right)^{-(\theta+1)}\left[\left(1+\frac{x_i}{\lambda}\right)^{-\theta}\left(1+\frac{\theta x_i}{\lambda}\right)\right]^{R_i}, \tag{4}$$

for simplicity write x_i instead of $x_{(i.)}$. The log likelihood function, denoted by $\ln l$, is given by:

$$\begin{aligned}\ln l &= \ln C + m\ln\theta + m\ln(\theta-1) - 2m\ln\lambda + \sum_{i=1}^{m}\ln x_i - (\theta+1)\sum_{i=1}^{m}\ln\left(1+\frac{x_i}{\lambda}\right)\\ &+\sum_{i=1}^{m}R_i\ln\left(1+\frac{\theta x_i}{\lambda}\right)-\theta\sum_{i=1}^{m}R_i\ln\left(1+\frac{x_i}{\lambda}\right).\end{aligned} \tag{5}$$

The following non-linear equations must be simultaneously solved to provide the ML estimators, say $\hat{\lambda}$ and $\hat{\theta}$,

$$\frac{\partial \ln l}{\partial\lambda}=\frac{-2m}{\lambda}+(\theta+1)\sum_{i=1}^{m}\frac{x_i}{(\lambda^2+\lambda x_i)}-\sum_{i=1}^{m}\frac{\theta R_i x_i}{(\lambda^2+\lambda\theta x_i)}+\sum_{i=1}^{m}\frac{\theta R_i x_i}{(\lambda^2+\lambda x_i)},$$

and,

$$\frac{\partial \ln l}{\partial\theta}=\frac{m}{\theta}+\frac{m}{\theta-1}-\sum_{i=1}^{m}\ln\left(1+\frac{x_i}{\lambda}\right)-\sum_{i=1}^{m}R_i\ln\left(1+\frac{x_i}{\lambda}\right)+\sum_{i=1}^{m}R_i\frac{x_i}{(\lambda+\theta x_i)}.$$

By simultaneously solving the likelihood equations, the ML estimators $\hat{\lambda}$ and $\hat{\theta}$ for the PT2C sample are obtained:

$$\frac{\partial \ln l}{\partial\lambda}\Big|_{\lambda=\hat{\lambda}}=0 \text{ and } \frac{\partial \ln l}{\partial\theta}\Big|_{\theta=\hat{\theta}}=0.$$

The estimators will be numerically derived using non-linear optimization software because the equations lack a closed form solution. The RF and HRF estimators are assessed based on the invariance property of the ML estimators, as illustrated below:

$$\hat{S}(t)=\left[1+\frac{t}{\hat{\lambda}}\right]^{-\hat{\theta}}\left[1+\frac{\hat{\theta}t}{\hat{\lambda}}\right],$$

and,

$$\hat{h}(t)=\frac{\hat{\theta}t\,(\hat{\theta}-1)}{(\hat{\lambda}+t)(\hat{\lambda}+\hat{\theta}t)}.$$

Additionally, the type II censoring method is used to derive the ML estimators of $\lambda, \theta, S(t)$ and $h(t)$ for $R_1 = R_2 = ... = R_{m-1} = 0$ and $R_m = n - m$. The ML estimators of $\lambda, \theta, S(t)$ and $h(t)$ are produced from complete sample for $R_1 = R_2 = ... = R_{m-1} = 0$ and $R_m = 0$ as well.

Additionally, the asymptotic variance-covariance matrix (VCM) for the estimators $\hat{\lambda}$ and $\hat{\theta}$ can be obtained by inverting Fisher information matrix (FIM) where its elements are the negative of second order derivatives of the natural logarithm of likelihood function (5)

$$\hat{I}(\hat{\lambda},\hat{\theta}) = -E\begin{bmatrix} \frac{\partial^2 \ln l}{\partial \lambda^2} & \frac{\partial^2 \ln l}{\partial \lambda\, \partial \theta} \\ \frac{\partial^2 \ln l}{\partial \theta\, \partial \lambda} & \frac{\partial^2 \ln l}{\partial \theta^2} \end{bmatrix}_{(\lambda=\hat{\lambda}, \theta=\hat{\theta})}.$$

Unfortunately, it is challenging to find exact closed forms for the aforementioned requirements. Consequently, the observed FIM $\hat{I}(\hat{\lambda},\hat{\theta})$, which is derived by removing the expectation operator E, will be used to create confidence intervals of the parameters (see [21]). The second partial derivatives of the log-likelihood function, which are simple to construct, are the entries in the observed FIM. Consequently, the observed FIM is represented by:

$$\hat{I}(\hat{\lambda},\hat{\theta}) = -\begin{bmatrix} \frac{\partial^2 \ln l}{\partial \lambda^2} & \frac{\partial^2 \ln l}{\partial \lambda\, \partial \theta} \\ \frac{\partial^2 \ln l}{\partial \theta\, \partial \lambda} & \frac{\partial^2 \ln l}{\partial \theta^2} \end{bmatrix}_{(\lambda=\hat{\lambda}, \theta=\hat{\theta})}.$$

The elements of the FIM are obtained as follows:

$$I_{11} = \frac{\partial^2 \ln l}{\partial \lambda^2} = \frac{2m}{\lambda^2} - (\theta+1)\sum_{i=1}^{m} \frac{2\lambda x_i + x_i^2}{(\lambda^2 + \lambda x_i)^2} + \sum_{i=1}^{m} \frac{\theta x_i R_i (2\lambda + \theta x_i)}{(\lambda^2 + \lambda\, \theta x_i)^2} - \sum_{i=1}^{m} \frac{\theta x_i R_i (2\lambda + x_i)}{(\lambda^2 + \lambda x_i)^2},$$

$$I_{22} = \frac{\partial^2 \ln l}{\partial \theta^2} = \frac{-m}{\theta^2} - \frac{m}{(\theta-1)^2} - \sum_{i=1}^{m} R_i \frac{x_i^2}{(\lambda + \theta x_i)^2},$$

and

$$I_{12} = I_{21} = \frac{\partial^2 \ln l}{\partial \theta\, \partial \lambda} = \sum_{i=1}^{m} \frac{x_i (1+R_i)}{(\lambda^2 + \lambda x_i)} - \sum_{i=1}^{m} \frac{R_i x_i}{(\lambda + \theta x_i)^2}.$$

In order to construct the asymptotic VCM $[\hat{V}]$, for the ML estimators, the observed FIM $\hat{I}(\hat{\lambda},\hat{\theta})$, is inverted as follows:

$$[\hat{V}] = \hat{I}^{-1}(\hat{\lambda},\hat{\theta}) = -\begin{bmatrix} \mathrm{var}(\hat{\lambda}) & \mathrm{cov}(\hat{\lambda},\hat{\theta}) \\ \mathrm{cov}(\hat{\lambda},\hat{\theta}) & \mathrm{var}(\hat{\theta}) \end{bmatrix}. \quad (6)$$

Again, a numerical technique using R and computer facilities are used to obtain the VCM. According to [22], under particular regularity conditions, $(\hat{\lambda},\hat{\theta})$ roughly follow a bivariate normal distribution with their mean of $(\hat{\lambda},\hat{\theta})$ and VCM $\hat{I}^{-1}(\lambda,\theta)$. Hence, the two-sided $100(1-\omega)\%$ ACI for λ and θ can be constructed based on the asymptotic normality conditions of the ML estimators as:

$\mathrm{ACI\_UL}(\hat{\lambda}) = \hat{\lambda} + Z_{\omega/2}\sqrt{\mathrm{var}(\hat{\lambda})}$, $\mathrm{ACI\_LL}(\hat{\lambda}) = \hat{\lambda} - Z_{\omega/2}\sqrt{\mathrm{var}(\hat{\lambda})}$,

$\mathrm{ACI\_UL}(\hat{\theta}) = \hat{\theta} + Z_{\omega/2}\sqrt{\mathrm{var}(\hat{\theta})}$, $\mathrm{ACI\_LL}(\hat{\theta}) = \hat{\theta} - Z_{\omega/2}\sqrt{\mathrm{var}(\hat{\theta})}$, where AIL= ACI_UL − ACI_LL, and AIL is average width, the first and the second elements on the main diagonal of the asymptotic VCM (6) are $\mathrm{var}(\hat{\lambda})$ and $\mathrm{var}(\hat{\theta})$ respectively, and the right tail probability $\omega/2$ when $Z_{\omega/2}$ is the percentile of the standard normal distribution.

Additionally, we must ascertain their variations in order to derive the ACI for RF and HRF. We employ the delta approach described in [23] to derive a rough estimate of $S(t)$ and $h(t)$, This methodology allows us to approximate the variance of $S(t)$ and $h(t)$, respectively, as follows:

$\mathrm{var}(\hat{S}(t)) = [\nabla_1 \hat{S}(t)]^T [\hat{V}][\nabla_1 \hat{S}(t)]$, and $\mathrm{var}(\hat{h}(t)) = [\nabla_2 \hat{h}(t)]^T [\hat{V}][\nabla_2 \hat{h}(t)]$,

where $\nabla_1 \hat{S}(t) = \left(\frac{\partial S(t)}{\partial \lambda}, \frac{\partial S(t)}{\partial \theta}\right)$, and $\nabla_2 \hat{h}(t) = \left(\frac{\partial h(t)}{\partial \lambda}, \frac{\partial h(t)}{\partial \theta}\right)$. Thus, the two-sided $100(1-\omega)\%$ ACI of $\hat{S}(t)$ and $\hat{h}(t)$ can be constructed as follows:

$$\mathrm{ACI\_UL}(\hat{S}(t)) = \hat{S}(t) + Z_{\omega/2}\sqrt{\mathrm{var}(\hat{S}(t))}, \qquad \mathrm{ACI\_UL}(\hat{S}(t)) = \hat{S}(t) + Z_{\omega/2}\sqrt{\mathrm{var}(\hat{S}(t))}$$

$$\mathrm{ACI\_UL}(\hat{h}(t)) = \hat{h}(t) + Z_{\omega/2}\sqrt{\mathrm{var}(\hat{h}(t))}, \qquad \mathrm{ACI\_UL}(\hat{h}(t)) = \hat{h}(t) + Z_{\omega/2}\sqrt{\mathrm{var}(\hat{h}(t))}$$

3. BAYESIAN ESTIMATORS USING DIFFERENT PRIORS

The squared error loss function (SELF) will be used in this section to analyses Bayesian estimators (BAEs) under the assumption that λ and θ have informative prior (IFP) and non-informative prior (NIFP) distributions.

First, the gamma prior distributions of the LBWLo distribution parameters based on PT2C are assumed to be true in order to obtain the BAEs. We assumed that the gamma distribution of λ and θ exists. If λ and θ are separately distributed, then the following formula gives the combined prior distribution of parameters:

$$g_1(\lambda,\theta) \propto \lambda^{a_1-1}\theta^{a_2-1}e^{-b_1\lambda-b_2\theta}. \tag{7}$$

Given the data $\underline{x}$ based on the likelihood function (4) and the joint prior distribution (7), the joint posterior density of λ and θ is calculated as follows:

$$\pi(\lambda,\theta|\underline{x}) = \frac{l(\underline{x}|\lambda,\theta)\, g_1(\lambda,\theta)}{\int_\theta\int_\lambda l(\underline{x}|\lambda,\theta)\, g_1(\lambda,\theta)\, d\lambda\, d\theta}$$

$$\propto \lambda^{a_1-2m-1}\theta^{a_2+m-1}(\theta-1)^m e^{-b_1\lambda-b_2\theta}\prod_{i=1}^{m} x_i\,\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i},$$

where $\eta_i = \left(1+\frac{x_i}{\lambda}\right)$ and $\tau_i = \left(1+\frac{\theta x_i}{\lambda}\right)$. The following are the forms of λ and θ marginal posterior distributions:

$$g_{11}(\lambda|\underline{x}) \propto \lambda^{a_1-2m-1}e^{-b_1\lambda}\int_0^\infty \theta^{a_2+m-1}(\theta-1)^m e^{-b_2\theta}\prod_{i=1}^{m} x_i\,\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta,$$

and

$$g_{12}(\theta|\underline{x}) \propto \theta^{a_2+m-1}(\theta-1)^m e^{-b_2\theta}\int_0^\infty \lambda^{a_1-2m-1}e^{-b_1\lambda}\prod_{i=1}^{m} x_i\,\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\lambda.$$

The following steps are taken to obtain the BAEs for λ and θ, under SELF, symbolised by $\hat{\lambda}_1$ and $\hat{\theta}_1$,

$$\hat{\lambda}_1 = D_1^{-1}\int_0^\infty\int_0^\infty \lambda^{a_1-2m}\theta^{a_2+m-1}(\theta-1)^m e^{-b_1\lambda-b_2\theta}\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\lambda d\theta,$$

and

$$\hat{\theta}_1 = D_1^{-1}\int_0^\infty\int_0^\infty \theta^{a_2+m}\lambda^{a_1-2m-1}(\theta-1)^m e^{-b_1\lambda-b_2\theta}\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta d\lambda,$$

where, $D_1 = \int_0^\infty\int_0^\infty \lambda^{a_1-2m-1}\theta^{a_2+m-1}(\theta-1)^m e^{-b_1\lambda-b_2\theta}\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\lambda d\theta.$

The BAEs of $S(t)$ and $h(t)$ are given by:

$$\hat{S}_1(t) = D_1^{-1}\int_0^\infty\int_0^\infty \lambda^{a_1-2m-1}\theta^{a_2+m-1}(\theta-1)^m e^{-b_1\lambda-b_2\theta}\left(1+\frac{\theta t}{\lambda}\right)\left(1+\frac{t}{\lambda}\right)^{-\theta}\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta d\lambda,$$

and

$$\hat{h}_1(t) = D_1^{-1}\int_0^\infty\int_0^\infty \frac{\theta^{a_2+m}\, t\,(\theta-1)^{m+1}}{(\lambda+t)(\lambda+\theta t)}\lambda^{a_1-2m-1}e^{-b_1\lambda-b_2\theta}\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta d\lambda.$$

The above BAEs $\hat{\lambda}_1, \hat{\theta}_1, \hat{S}_1(t)$ and $\hat{h}_1(t)$ can be quantitatively assessed for the provided values of $a_1, b_1, a_2, b_2, n, m, \underline{x}$ and $\underline{R}$ even though they are not in closed forms.

Second, assuming the hyper-parameters to be zero on Equation (7), the joint prior distribution for λ and θ in the case of NIPF is defined as follows:

$$g_2(\lambda,\theta) \propto \frac{1}{\lambda\,\theta} \qquad 0<\lambda,\ \theta<1.$$

The joint posterior distribution of parameters λ and θ in this case is given by:

$$\pi_1(\lambda,\theta|\underline{x}) \propto \theta^{m-1}\lambda^{-2m-1}(\theta-1)^m\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i}. \tag{8}$$

As a result, the λ and θ marginal posterior distributions have the following shapes:

$$g_{21}(\lambda|\underline{x}) \propto \int_0^\infty \theta^{m-1}\lambda^{-2m-1}(\theta-1)^m\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta,$$

and

$$g_{22}(\theta|\underline{x}) \propto \int_0^\infty \theta^{m-1}\lambda^{-2m-1}(\theta-1)^m\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\lambda.$$

The BAEs of λ and θ denoted by $\hat{\lambda}_1$ and $\hat{\theta}_1$, under SELF are obtained as follows:

$$\hat{\lambda}_2 = D_2^{-1}\int_0^\infty\int_0^\infty \theta^{m-1}\lambda^{-2m}(\theta-1)^m\prod_{i=1}^{m} x_i\,\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta d\lambda,$$

and,

$$\hat{\theta}_2 = D_2^{-1}\int_0^\infty\int_0^\infty \theta^{m}\lambda^{-2m-1}(\theta-1)^m\prod_{i=1}^{m} x_i\,\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta d\lambda,$$

where $D_2 = \int_0^\infty\int_0^\infty \theta^{m-1}\lambda^{-2m-1}(\theta-1)^m\prod_{i=1}^{m} x_i\,\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta d\lambda.$

The BAEs of $S(t)$ and $h(t)$ are given by:

$$\hat{S}_2(t) = D_2^{-1}\int_0^\infty\int_0^\infty \theta^{m-1}\lambda^{-2m-1}(\theta-1)^m\left(1+\frac{\theta t}{\lambda}\right)\left(1+\frac{t}{\lambda}\right)^{-\theta}\prod_{i=1}^{m} t_i\,\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta d\lambda,$$

and,

$$\widehat{h}_2(t) = D_2^{-1} \int_0^{\infty}\int_0^{\infty} \frac{\theta^m\, t\,(\theta-1)^{m+1}}{(\lambda+t)(\lambda+\theta t)} \lambda^{-2m-1} \prod_{i=1}^{m} t_i\, \eta_i^{-(\theta+1)} \left[\tau_i \eta_i^{-\theta}\right]^{R_i} d\theta d\lambda\,.$$

The above BAEs $\widehat{\lambda}_2, \widehat{\theta}_2, \widehat{S}_2(t)$, and $\widehat{h}_2(t)$ have no closed forms but can be evaluated numerically for the given values of $n, m, \underline{x}$ and $\underline{R}$.

4. BAYESIAN CREDIBLE INTERVALS

The BCI is an interval with a specific subjective probability that an unobserved parameter value will fall within. It is a range inside the scope of either a predictive or a posterior probability distribution. The BCIs of λ and θ denoted by, $\dddot{\lambda}_1$ and $\ddot{\theta}_1$ are acquired under IFP as follows:

$$\dddot{\lambda}_1 = D_1^{-1} \int_L^U \int_0^{\infty} \lambda^{a_1-2m-1}\, \theta^{a_2+m-1}\, (\theta-1)^m\, e^{-b_1\lambda-b_2\theta} \prod_{i=1}^{m} x_i\, \eta_i^{-(\theta+1)} \left[\tau_i \eta_i^{-\theta}\right]^{R_i} d\theta\, d\lambda = 0.95,$$

and,

$$\ddot{\theta}_1 = D_1^{-1} \int_L^U \int_0^{\infty} \lambda^{a_1-2m-1}\, \theta^{a_2+m-1}\, (\theta-1)^m\, e^{-b_1\lambda-b_2\theta} \prod_{i=1}^{m} x_i\, \eta_i^{-(\theta+1)} \left[\tau_i \eta_i^{-\theta}\right]^{R_i} d\lambda d\theta = 0.95.$$

These integrals are extremely difficult to resolve analytically, hence the Metropolis-Hasting (MH) technique will be used to do so. Additionally, the NIFP is used to obtain the BCIs of λ and θ symbolized by $\dddot{\lambda}_2$ and $\ddot{\theta}_2$ as follows:

$$\dddot{\lambda}_2 = D_2^{-1} \int_L^U \int_0^{\infty} \theta^{m-1}\, \lambda^{-2m-1}\, (\theta-1)^m \prod_{i=1}^{m} x_i\, \eta_i^{-(\theta+1)} \left[\tau_i \eta_i^{-\theta}\right]^{R_i} d\theta d\lambda = 0.95,$$

and,

$$\ddot{\theta}_2 = D_2^{-1} \int_L^U \int_0^{\infty} \theta^{m-1}\, \lambda^{-2m-1}\, (\theta-1)^m \prod_{i=1}^{m} x_i\, \eta_i^{-(\theta+1)} \left[\tau_i \eta_i^{-\theta}\right]^{R_i} d\lambda\, d\theta = 0.95\,.$$

Since it is quite challenging to evaluate these integrals analytically, the MH technique will be used to solve them.

5. NUMERICAL STUDIES & RESULTS

In this section, a simulation is run to get the maximum likelihood estimates (MLEs) and Bayesian estimates (BEs) of $\lambda, \theta, S(t)$ and $h(t)$ for the LBWLo distribution under two PT2C-based schemes. We see the strategies as follows:

- Scheme 1 (Sc. 1): $R_1 = n-m,\ R_2 = R_3 = ... = R_m = 0$.
- Scheme 2 (Sc. 2): $R_1 = R_2 = (n-m)/2,\ R_3 = R_4 = ... = R_m = 0$.

The following steps are done via R 3.6.1 program.

Step 1: Using the same algorithm offered in [24], a random sample $x_1, x_2, ..., x_n$ is generated from the LBWLo distribution with the following steps:

1. Generate *m* independent and identically (*iid*) random numbers $P_1, P_2, ..., P_m$ from uniform distribution *U* (0,1).

2. Set $V_i = P_i^{(1/i+R_m+R_{m-1}+...+R_{m-i+1})}$ for *i* = 1, 2,…, *m*.

3. Set $U_i = 1 - V_m V_{m-1} ... V_{m-i+1}$ and for *i* = 1, 2,…, *m*. Then $U_1, U_2, ..., U_m$ is the PT2C sample from uniform (0,1) distribution.

4. Finally, set $x_i = F^{-1}(U_i)$ for *i* = 1, 2,…, *m* , where $F^{-1}(.)$ is the inverse LBWLo CDF. Then $x_1, x_2, ..., x_n$ is the required PT2C from LBWLo distribution with censoring scheme $\underline{R} = (R_1, R_2, ..., R_m)$.

The MH is one of the Markov chain Monte Carlo methods that is most frequently used in real-world applications today to simulate deviations from the posterior density and generate accurate approximations. The MH algorithm's proposed distribution $q(\alpha'|\alpha)$ and beginning values $\alpha^{(0)}$ for the unidentified parameters were defined. Consider a bivariate normal distribution for the proposal distribution. To do this, choose $q(\alpha'|\alpha)=N_2(\alpha,S_\alpha^\bullet)$ where $\alpha=(\lambda,\theta)$ and $S_\alpha^\bullet$ denotes the VCM. It should be highlighted that the bivariate normal distribution could result in undesirable negative findings (see [25]). The MH algorithm's steps for selecting a sample from the posterior density are as follows:

a. Set beginning values of α as $\alpha=\alpha^{(0)}$.

b. For i = 1, 2,…, M repeat the following steps:

a) Set $\alpha=\alpha^{(i-1)}$.

b) Generate a new candidate parameter value δ from $N_2(\ln\alpha, I_0^{-1}(\lambda,\theta))$.

c) Set $\alpha'=\exp(\delta)$.

d) Calculate $\varepsilon=\min\left[1,\dfrac{\pi(\alpha'|x)\,\alpha'}{\pi(\alpha|x)\,\alpha}\right]$.

e) Update $\alpha^{(i)}=\alpha'$ with probability ε; otherwise set $\alpha^{(i)}=\alpha$.

The choice of $I^{-1}(\lambda,\theta)$ in the MH algorithm, where the acceptance rate depends on it, is a crucial decision. In the case where $I(.)$ is the FIM, it is thought to be the asymptotic VCM $\hat{I}^{-1}(\alpha)$. The MLE of α is regarded as the beginning value for α.

Finally, some of the initial samples (burn-in) can be removed from the random samples of size M derived from the posterior density, and remaining samples can be used to calculate BEs. The BE is given by the SELF and depends on:

$$\tilde{g}_{MH}(\alpha)=\frac{1}{M-\iota_B}\sum_{i=\iota_B}^{M} g(\alpha),$$

where ι_B reflects the quantity of samples used for burn-in.

Step 2: For the propose of generating random samples, the following parameter values are chosen:

(i) $\lambda=0.5$, $\theta=1.5$ and (ii) $\lambda=2$, $\theta=1.5$.

Step 3: For each set of parameters, the MLEs and BEs of λ and θ are calculated using different censoring schemes $\underline{R}=(R_1,R_2,...,R_m)$ and stage counts (m) for each sample size of n = 50, 100, and 150. We use t = 0.8 for the estimates of the HRF and RF.

Step 4: According to [26], the mean and variance of $\hat{\lambda}^i$ and $\hat{\theta}^i$ are equated with the mean and variance of the gamma distributions to determine the hyper-parameters for gamma priors,

$$a_1=\frac{\left(N^{-1}\sum_{i=1}^{N}\hat{\lambda}^i\right)^2}{(N-1)^{-1}\left[\sum_{i=1}^{N}\left(\hat{\lambda}^i-N^{-1}\sum_{i=1}^{N}\hat{\lambda}^i\right)^2\right]},\quad b_1=\frac{N^{-1}\sum_{i=1}^{N}\hat{\lambda}^i}{(N-1)^{-1}\left[\sum_{i=1}^{N}\left(\hat{\lambda}^i-N^{-1}\sum_{i=1}^{N}\hat{\lambda}^i\right)^2\right]},$$

and,

$$a_2=\frac{\left(N^{-1}\sum_{i=1}^{N}\hat{\theta}^i\right)^2}{(N-1)^{-1}\left[\sum_{i=1}^{N}\left(\hat{\theta}^i-N^{-1}\sum_{i=1}^{N}\hat{\theta}^i\right)^2\right]},\quad b_2=\frac{N^{-1}\sum_{i=1}^{N}\hat{\theta}^i}{(N-1)^{-1}\left[\sum_{i=1}^{N}\left(\hat{\theta}^i-N^{-1}\sum_{i=1}^{N}\hat{\theta}^i\right)^2\right]},$$

where N is the number of samples and $i = 1, 2, ..., N$.

Step 5: Using the MH approach, the deviations from the posterior density are simulated (see [26]).

Step 6: The bias, mean squared error (MSE), AIL, and coverage probability (CP) for two schemes are used to compare results for different sample sizes, with 10000 repeated samples**.**

The numerical outcomes are displayed in Figures 2–7 and reported in Tables A1–A4 in the Appendix. The effectiveness of different estimates is revealed in the following observations.

- In the majority of the cases, the MSEs, biases, and AILs for all estimates (ML and Bayesian) decrease with *n* and *m*.
- The CPs of HRF estimates increase as *n* and *m* increase.
- The MSEs of parameter estimates at true value $\lambda = 0.5,\ \theta = 1.5$ get the largest values in Sc.1and Sc.2, while the MSEs of $\hat{\lambda}_1$ and $\hat{\theta}_1$ get the smallest value under Sc.1and Sc.2 (see Figure 2).

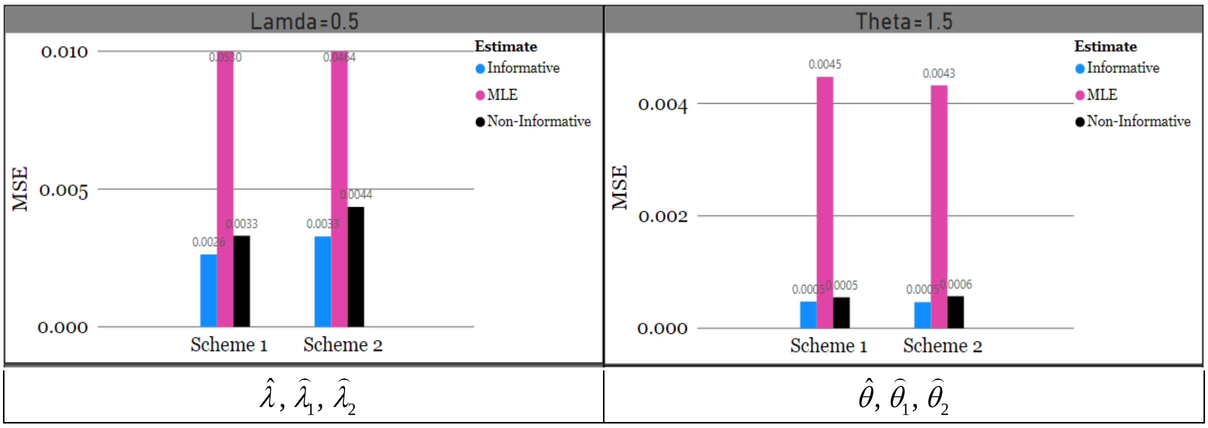


Figure 2. *MSEs for parameter estimates at* $\lambda = 0.5,\ \theta = 1.5$ *for all values of m*

- For both schemes, the MSEs of $\hat{\lambda}$ and $\hat{\theta}$ at $\lambda = 2,\ \theta = 1.5$ have the biggest value, whereas the MSEs of $\hat{\lambda}_1$ and $\hat{\theta}_1$ take the least value (see Figure 3).

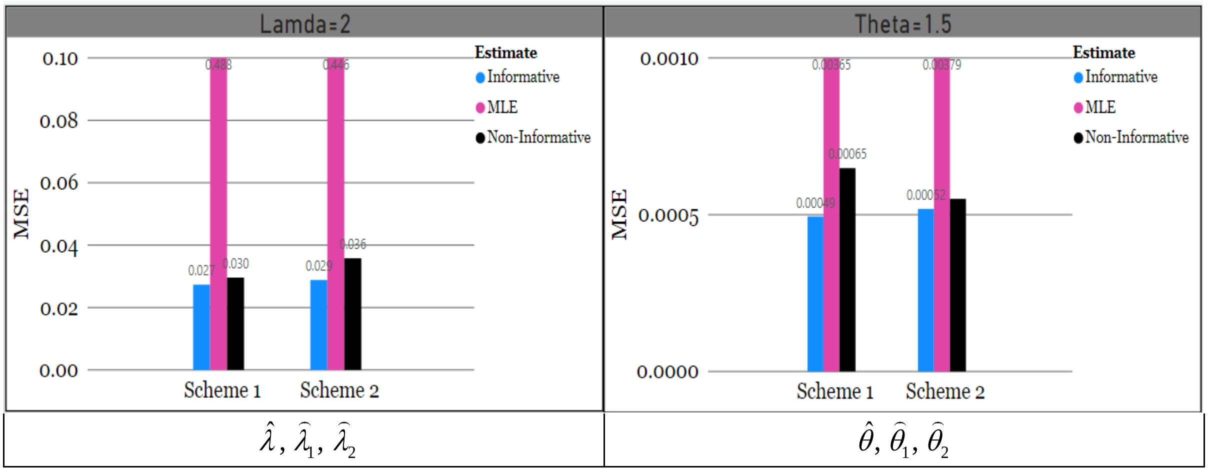


Figure 3. *MSEs for parameter estimates at* $\lambda = 2,\ \theta = 1.5$ *for all values of m*

- As shown in Figure 4, the MSEs of $\hat{S}_1(t)$ and $\hat{h}_1(t)$ have the least values for the two schemes, where as $\hat{S}(t)$ and $h(t)$ have the biggest values for Sc. 1 and Sc. 2 at $\lambda = 0.5,\ \theta = 1.5$.

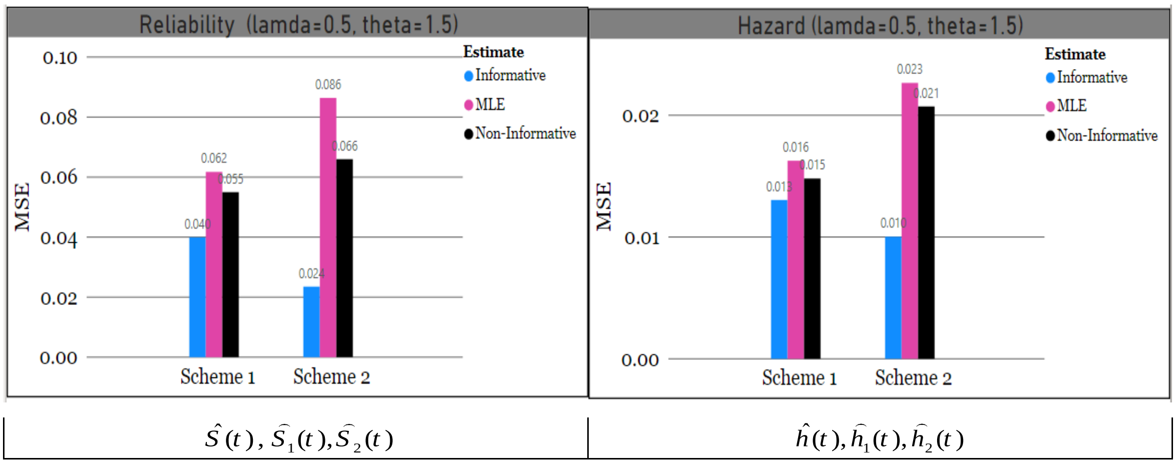


Figure 4. *MSEs at* $\lambda = 0.5$, $\theta = 1.5$ *for all values of m for RF and HRF estimates*

- The values of $\hat{S}_1(t)$ and $\hat{h}_2(t)$ MSEs are the least in Sc.1 and Sc.2, respectively. The biggest values in both schemes' MSEs are found for $\hat{S}(t)$ and $\hat{h}(t)$ (see Figure 5).

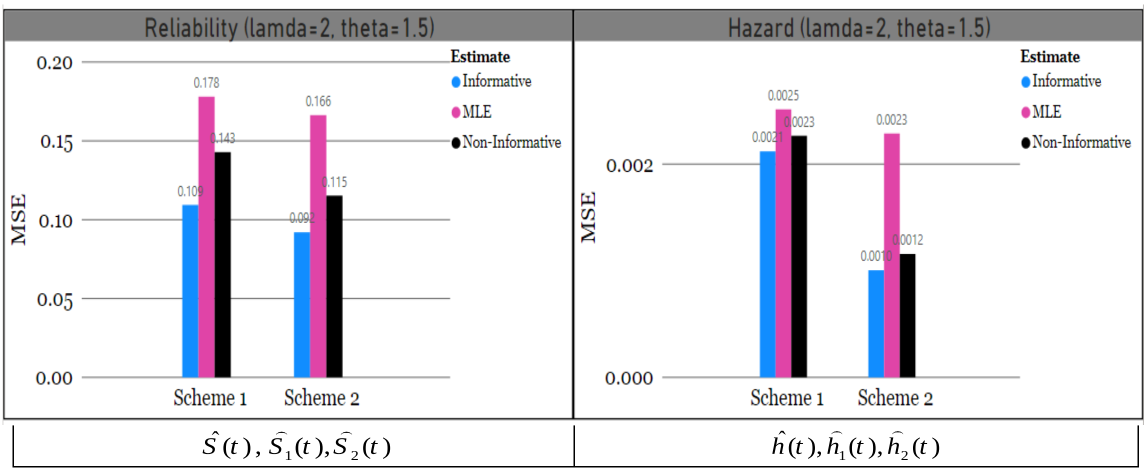
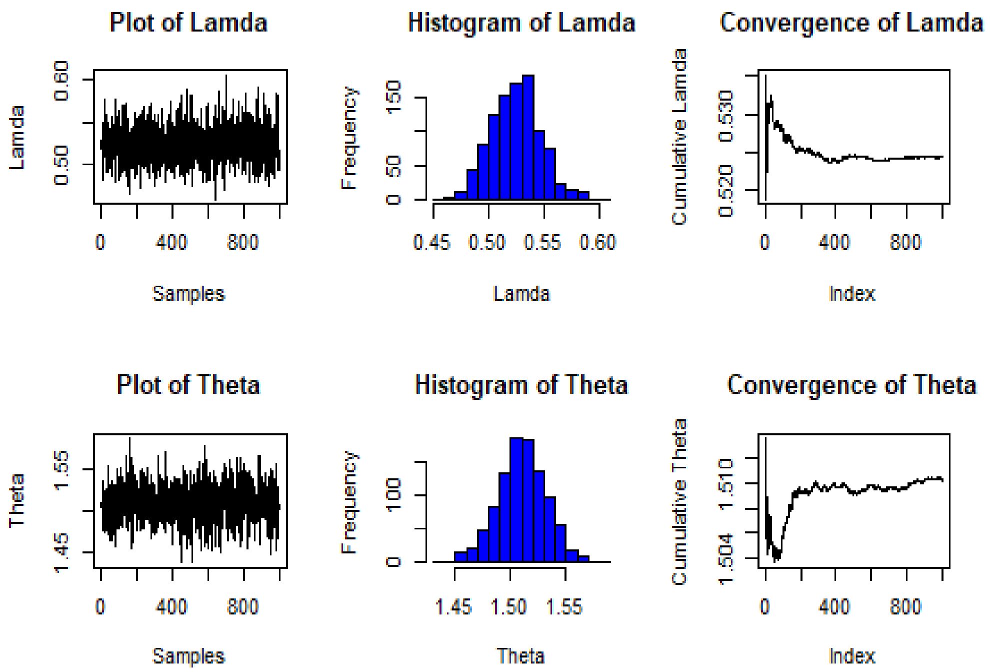
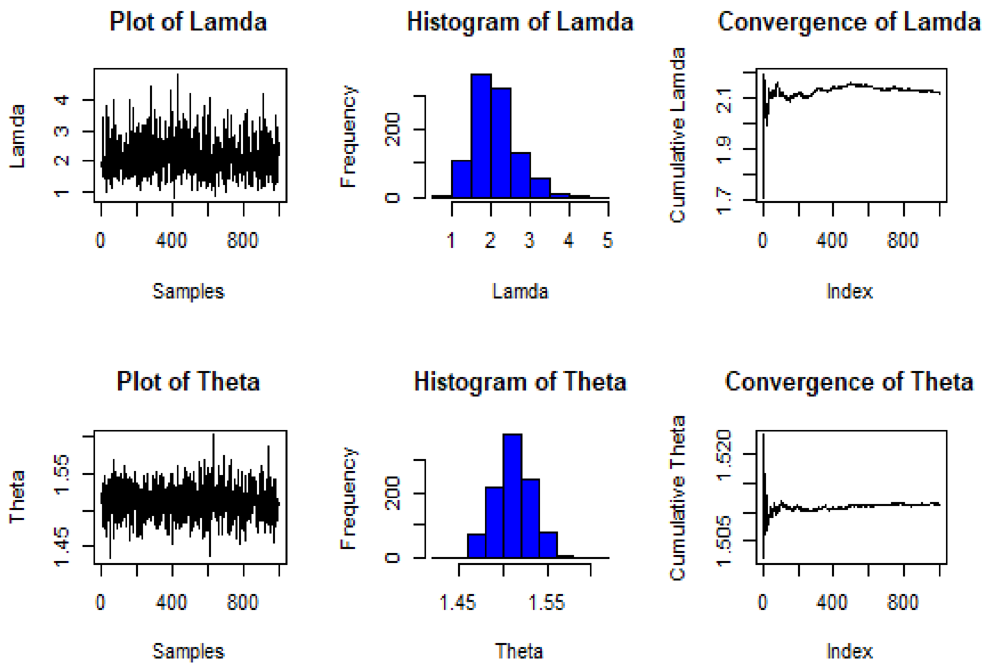


Figure 5. *MSEs at* $\lambda = 2$, $\theta = 1.5$ *for all values of m for RF and HRF estimates*

- The MSEs of parameters under IFP generally take the smallest values in all schemes in roughly the majority of circumstances, it may be said.
- The biases of $\hat{\lambda}_1$ and $\hat{\theta}_1$ for LBWLo distribution are smaller than that the corresponding of $\hat{\lambda}$, $\hat{\theta}$, $\hat{\lambda}_2$ and $\hat{\theta}_2$.
- The AILs for BCI estimates of the LBWLo distribution under IFP are smaller than that the corresponding of the MLEs and BEs under NIFP.
- The CPs of the BEs for the LBWLo distribution under gamma priors are greater than that the corresponding of the MLEs and BEs under IFPs.
- Figure 6 shows history graphs for various λ and θ estimaties in NIFP situations. The plots of the parameters of the chains λ and θ resemble a horizontal band without any obvious extended upward or downward trends, which are evidence of convergence.



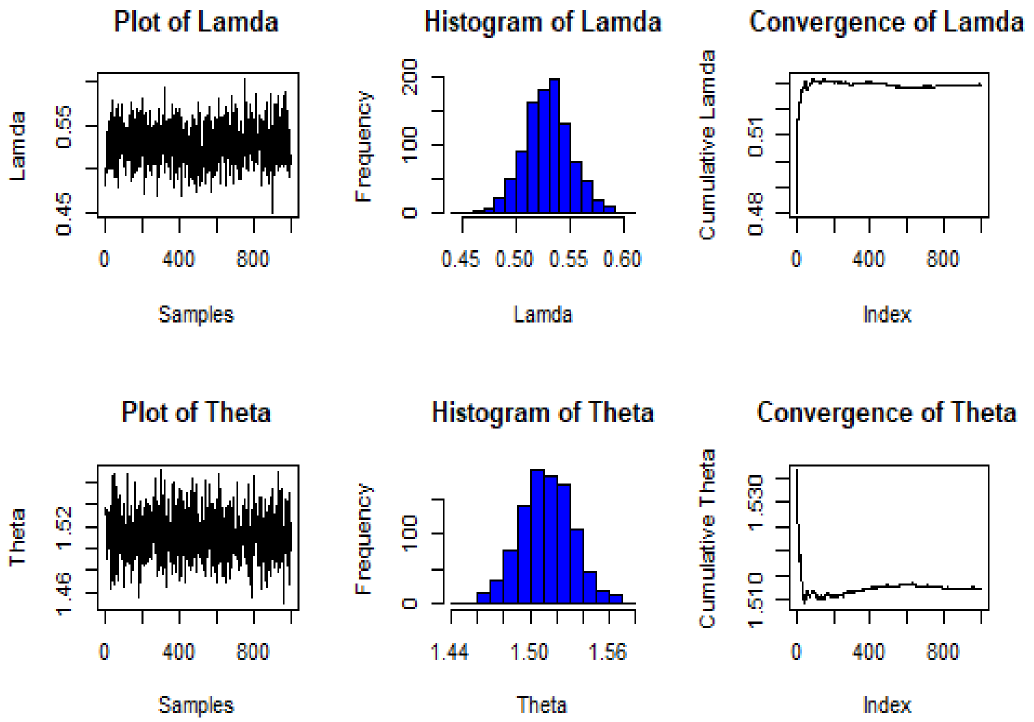
(***a***) $\hat{\lambda}_2$ and $\hat{\theta}_2$ at *n*=100, *m*=50 for $\lambda = 0.5$, $\theta = 1.5$



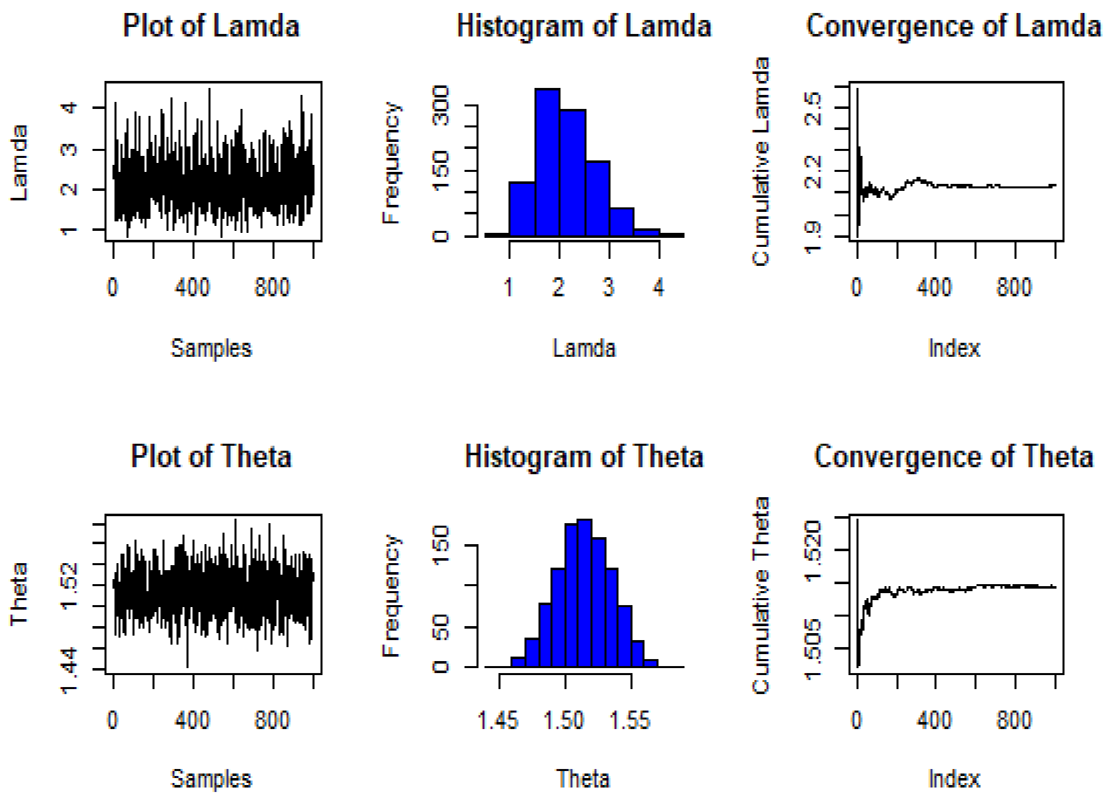
(***b***) $\hat{\lambda}_2$ and $\hat{\theta}_2$ at *n*=100, *m*=50 for λ =2, $\theta = 1.5$

Figure 6. *Different BEs for* λ *and* θ *under uniform priors*

- For IFPs, Figure 7 displays history plots for several estimates of λ and θ. Without any discernible long-term rising or falling trends, which are indications of convergence, the plots of the chains for the parameters create a horizontal band.



(a) *$\hat{\lambda}_1$ and $\hat{\theta}_1$ at n=100, m=50 for* $\lambda = 0.5$, $\theta = 1.5$



(b) *$\hat{\lambda}_1$ and $\hat{\theta}_1$ at n=100, m=50 for* $\lambda = 2$, $\theta = 1.5$

Figure 7. *Different BEs for λ and θ under gamma priors*

6. DATA ANALYSIS

The data set is connected to the period between failures for three repairable objects and is presented below. The data set are provided in [27] and explored later by [28].

1.43	0.11	0.71	0.77	2.63	1.49	3.46	2.46	0.59	0.74
1.23	0.94	4.36	0.40	1.74	4.73	2.23	0.45	0.70	1.06
1.46	0.30	1.82	2.37	0.63	1.23	1.24	1.97	1.86	1.17

To assess if the data distribution is suitable for the LBWLo distribution or not, the Kolmogorov-Smirnov (K-S) test was applied. The K-S distance is calculated to be 0.065072, and the P-value is 0.996. The estimated PDF, CDF, and probability-probability (PP) plots are displayed in Figure 8.

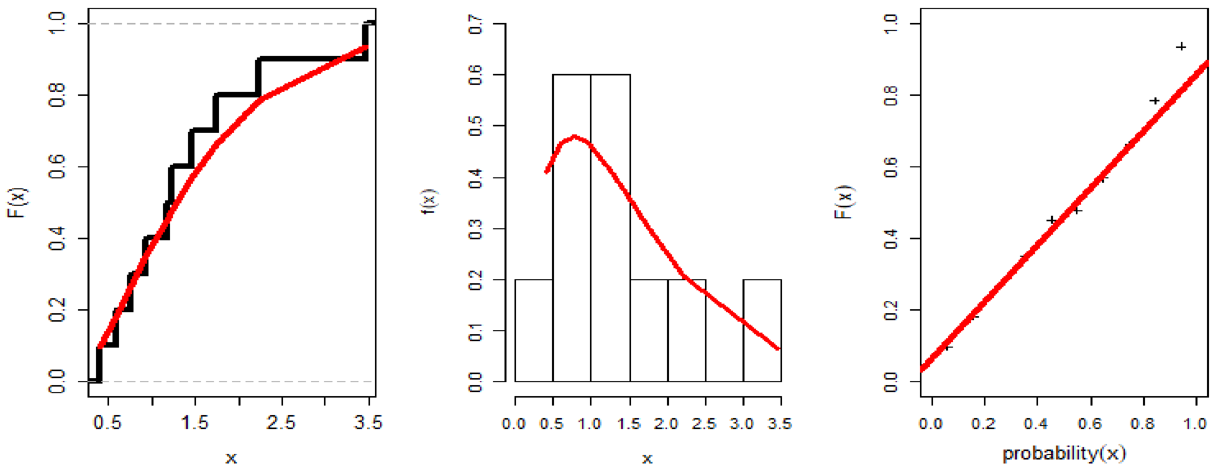


Figure 8. *The estimated PDF, CDF and PP plots for the LBWLo distribution*

Now, let us examine what occurs if the data set is censored. Using the uncensored data set, we produce two artificial PT2C sets in the ways described below (see Table 1):

- Sc.1: $R_1 = n - m, R_2 = R_3 = ... = R_m = 0$.
- Sc.2: $R_1 = R_2 = (n-m)/2, R_3 = R_4 = ... = R_m = 0$.

Table 1 discusses MLEs and BEs along with their standard errors (SEs) based on PT2C samples. We employed a NIFP to calculate the BEs because we have no knowledge of the priors; thus, we chose.

Table 1. *The MLEs and BEs under the PT2C*

Scheme	m		ML		Bayesian	
			Estimate	SE	Estimate	SE
1	10	λ	33.116	73.435	131.651	34.607
		θ	21.801	50.746	87.265	41.757
		$S(t)$	0.753	0.050	0.739	0.047
		$h(t)$	0.266	0.030	0.290	0.027
	20	λ	31.120	27.108	104.367	24.179
		θ	34.400	29.793	107.602	21.628
		$S(t)$	0.578	0.053	0.593	0.051
		$h(t)$	0.569	0.032	0.548	0.029
2	10	λ	27.116	50.650	145.010	48.942

Scheme	m		ML		Bayesian	
			Estimate	SE	Estimate	SE
		θ	30.935	59.721	127.989	45.024
		$S(t)$	0.569	0.048	0.647	0.040
		$h(t)$	0.588	0.019	0.443	0.018
	20	λ	75.000	56.737	112.992	47.187
		θ	80.199	64.818	128.974	54.747
		$S(t)$	0.582	0.050	0.556	0.049
		$h(t)$	0.571	0.029	0.633	0.027

The Bayesian estimation strategy, which has the lowest SE values, is the best choice for estimating the parameters, RF, and HRF for the LBWLo distribution based on PT2C. Additionally, it is noted that Scheme 2 fits the data better than other schemes because it has the lowest value among SE.

7. SUMMARY AND CONCLUSION

This paper investigates parameter estimators, reliability function estimators, and hazard rate function estimators for the LBWLo distribution under PT2C samples using ML and Bayesian methodologies. The SELF is used to derive the BAEs while accounting for gamma and uniform priors. ACIs and BCIs are built using IFP and NIFP as a foundation. Simulation study is carried out to compare the performance of estimates. According to simulation study, BEs perform better than MLEs in the majority of situations. When adopting gamma and uniform priors, respectively, the MSEs of BEs typically take the largest value for Scheme 1 and the smallest value for Scheme 2. The BEs under gamma priors have a higher coverage probability than the ML and BEs with uniform priors. Finally, one set of actual data supports the suggested estimations.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

ACKNOWLEDGMENTS

We are grateful to the editors and the anonymous referees for their hard work, wise counsel, and helpful critiques that greatly improved the paper.

REFERENCES

[1] Lomax, K. S., “Business failures: Another example of the analysis of failure data”, Journal of the American Statistical Association, 49(268): 847–852, (1954).

[2] Harris, C. M., “The Pareto distribution as a queue service discipline”, Operations Research, 16(2): 307–313, (1968).

[3] Atkinson, A.B., and Harrison, A.J., “Distribution of Personal Wealth in Britain”, Cambridge University Cambridge, (1987).

[4] Holland, O., Golaup, A., and Aghvami, A. H., “Traffic characteristics of aggregated module downloads for mobile terminal reconfiguration”, IEE Proceedings- Communications, 153(5): 683–690, (2006). http: //dx.doi.org/10.1049/ip-com:20045155

[5] Hassan, A. S., and Al-Ghamdi, A. S., “Optimum step stress accelerated life testing for Lomax distribution”, Journal of Applied Sciences Research, 5: 2153–2164, (2009).

[6] Hassan, A. S., Assar, S. M., and Shelbaia, A., “Optimum step stress accelerated life test plan for Lomax distribution with an adaptive type-II progressive hybrid censoring”, British Journal of Mathematics and Computer Science, 13(2): 1−19, (2016).

[7] Hassan, A.S., and Mohamed, R.E., “Parameter estimation of inverse exponentiated Lomax with right censored data”, Gazi University Journal of Science, 32(4): 1370–1386,(2019).

[8] Muhammad, IJAZ, “ Bayesian estimation of the shape parameter of Lomax distribution under uniform and Jeffery prior with engineering applications”, Gazi University Journal of Science, 34(2): 562–577, (2021).

[9] Hassan, A.S., and Ismail, D., “Estimation of parameters of Topp-Leone inverse Lomax distribution in presence of right censored samples”, Gazi University Journal of Science, 34(4): 1193–1208, (2021).

[10] Ahmad, A., Ahmad, S.P., and Ahmed, A., Length-biased weighted Lomax distribution: statistical properties and application”, Pakistan Journal of Statistics and Operation Research, 12: 245-255, (2016).

[11] Karimi, H., and Nasiri, P., “Estimation parameter of R = P(Y < X) for length-biased weighted Lomax distributions in the presence of outliers,” Mathematical and Computational Applications, 23(9): 1–9, (2018).

[12] Bantan, R., Hassan, A.S., Almetwally, E., Elgarhy, M. Jamal, F., Chesneau, C., and Elsehetry, M., “Bayesian analysis in partially accelerated life tests for weighted Lomax distribution”, Computers, Materials & Continua, 68(3): 2859–2875, (2021).

[13] Hofmann, G., Cramer, E., Balakrishnan, N., and Kunert, G., “An asymptotic approach to progressive censoring”, Journal of Statistical Planning and Inference, 130: 207–227, (2005).

[14] Krishna, H., and Kumar, K., “Reliability estimation in Lindley distribution with progressively type II right censored sample”, Mathematics and Computers in Simulation, 82: 281–294, (2011).

[15] Kohansal, A.,“ Statistical analysis of two-parameter bathtub-shaped lifetime distribution under progressive censoring with binomial removals”, Gazi University Journal of Science, 29(4): 783-792, (2016).

[16] Cetinkaya, C., “Estimation in step-stress partially accelerated life tests for the power Lindley distribution under progressive censoring”, Gazi University Journal of Science, 34(2): 579–590, (2021).

[17] Shrahili, M., El-Saeed, A.R., Hassan, A.S., Elbatal, I., and Elgarhy, M., “Estimation of entropy for log-Logistic distribution under progressive type II censoring”, Journal of Nanomaterials, (2022). doi.org/10.1155/2022/2739606

[18] Hassan, A. S., Mousa, R. M., and Abu-Moussa, M. H., “Analysis of progressive type-II competing risks data with applications”, Lobachevskii Journal of Mathematics, 43(9): 2479–2492, (2022).

[19] Akdogan, Y., Kus, C., and Wu,S-J.,“ Planning life tests for Burr XII distributed products under progressive group-censoring with cost considerations”, Gazi University Journal of Science, 25(2): 425-434 (2012).

[20] Balakrishnan, N., and Aggarwala, R., “Progressive Censoring Theory, Methods and Applications”, Birkhauser Boston, MA, (2000).

[21] Cohen, A. C., "Maximum likelihood estimation in the Weibull distribution based on complete and censored samples", Technometrics, 7: 579−588, (1965).

[22] Lawless, J.F., "Statistical models and methods for lifetime data", Wiley, New York, (1982).

[23] Greene, W.H., "Econometric analysis", 4th edn. Prentice-Hall, New York, (2000).

[24] Balakrishnan, N., and Sandhu, R.A., "A simple simulation algorithm for generating progressively type II censored samples", American Statistical Association, 49(2): 229–230, (1995).

[25] Dey, S., and Pradhan, B., "Generalized inverted exponential distribution under hybrid censoring", Statistical Methodology, 18: 101−114, (2014).

[26] Dey, S., Singh, S., Tripathi, Y.M., and Asgharzadeh, A., "Estimation and prediction for a progressively censored generalized inverted exponential distribution", Statistical Methodology, 32: 185–202, (2016).

[27] Hassan, A. S., and Assar, S. M., "The exponentiated Weibull-power function distribution", Journal of Data Sciences", 15(4): 589-614, (2017).

[28] Gadde, S.R., and Al-Omari, A.I., "Attribute control charts based on TLT for length-biased weighted Lomax distribution", Journal of Mathematics, (2022).

Appendix

Table A1. *MSE and Bias for different Estimates of parameters, RF, and HRF at* $\lambda = 0.5$ *and* $\theta = 1.5$

		Scheme I						
n	**m**	**Estimate**	**ML**		**IFP**		**NIFP**	
			Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
50	**20**	λ	0.153	0.308	0.152	0.024	0.153	0.024
		θ	0.085	0.078	0.084	0.008	0.085	0.008
		$S(t)$	0.260	0.067	0.259	0.067	0.260	0.068
		$h(t)$	0.125	0.016	0.125	0.016	0.125	0.016
	30	λ	0.109	0.143	0.109	0.012	0.108	0.012
		θ	0.055	0.035	0.055	0.004	0.054	0.003
		$S(t)$	0.273	0.075	0.273	0.075	0.273	0.075
		$h(t)$	0.134	0.018	0.134	0.018	0.134	0.018
100	**20**	λ	0.129	0.167	0.128	0.017	0.128	0.017
		θ	0.068	0.055	0.067	0.005	0.067	0.005
		$S(t)$	0.304	0.093	0.304	0.092	0.304	0.093
		$h(t)$	0.150	0.022	0.150	0.022	0.150	0.022
	50	λ	0.065	0.059	0.062	0.004	0.052	0.003
		θ	0.032	0.014	0.029	0.001	0.027	0.001
		$S(t)$	0.266	0.071	0.302	0.091	0.334	0.112
		$h(t)$	0.135	0.018	0.144	0.021	0.146	0.021
	70	λ	0.038	0.034	0.037	0.002	0.038	0.002
		θ	0.015	0.008	0.015	0.001	0.015	0.001
		$S(t)$	0.247	0.061	0.247	0.061	0.247	0.061

		Scheme I						
n	**m**	**Estimate**	**ML**		**IFP**		**NIFP**	
			Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
		$h(t)$	0.122	0.015	0.122	0.015	0.122	0.015
150	**50**	λ	0.060	0.053	0.064	0.005	0.062	0.004
		θ	0.028	0.014	0.032	0.001	0.030	0.001
		$S(t)$	0.248	0.062	0.308	0.095	0.346	0.119
		$h(t)$	0.128	0.016	0.147	0.022	0.160	0.026
	70	λ	0.031	0.027	0.031	0.001	0.029	0.001
		θ	0.016	0.008	0.015	0.001	0.017	0.001
		$S(t)$	0.287	0.082	0.286	0.082	0.287	0.083
		$h(t)$	0.139	0.019	0.139	0.019	0.138	0.019
	100	λ	0.025	0.021	0.038	0.002	0.012	0.002
		θ	0.009	0.005	0.016	0.001	0.015	0.001
		$S(t)$	0.325	0.105	0.339	0.115	0.357	0.128
		$h(t)$	0.151	0.023	0.149	0.022	0.157	0.025
	130	λ	0.021	0.017	0.021	0.001	0.003	0.001
		θ	0.012	0.004	0.008	0.000	0.011	0.001
		$S(t)$	0.275	0.076	0.264	0.070	0.310	0.096
		$h(t)$	0.131	0.017	0.127	0.016	0.144	0.021

Continued Table A1.

		Scheme 2						
n	**m**	**Estimate**	**ML**		**IFP**		**NIFP**	
			Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
50	**20**	λ	0.162	0.288	0.161	0.026	0.161	0.026
		θ	0.080	0.068	0.079	0.007	0.080	0.007
		$S(t)$	0.431	0.186	0.431	0.185	0.431	0.186
		$h(t)$	0.183	0.033	0.183	0.033	0.183	0.033
	30	λ	0.104	0.114	0.104	0.011	0.103	0.011
		θ	0.050	0.027	0.050	0.003	0.050	0.003
		$S(t)$	0.261	0.068	0.261	0.068	0.261	0.068
		$h(t)$	0.128	0.016	0.128	0.016	0.128	0.016
100	**20**	λ	0.098	0.168	0.097	0.010	0.098	0.010
		θ	0.060	0.058	0.060	0.004	0.059	0.004
		$S(t)$	0.149	0.022	0.149	0.022	0.148	0.022
		$h(t)$	0.109	0.012	0.108	0.012	0.109	0.012
	50	λ	0.057	0.051	0.067	0.005	0.062	0.004
		θ	0.029	0.013	0.027	0.001	0.028	0.001
		$S(t)$	0.273	0.074	0.273	0.075	0.291	0.085
		$h(t)$	0.135	0.018	0.139	0.019	0.137	0.019
	70	λ	0.044	0.034	0.044	0.002	0.043	0.002
		θ	0.018	0.008	0.017	0.001	0.018	0.001
		$S(t)$	0.333	0.111	0.332	0.110	0.333	0.111
		$h(t)$	0.163	0.027	0.163	0.027	0.163	0.027

			Scheme 2					
n	***m***	**Estimate**	**ML**		**IFP**		**NIFP**	
			Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
150	**50**	λ	0.058	0.046	0.053	0.003	0.050	0.003
		θ	0.033	0.015	0.026	0.001	0.024	0.001
		$S(t)$	0.327	0.107	0.306	0.094	0.349	0.122
		$h(t)$	0.151	0.023	0.143	0.021	0.158	0.025
	70	λ	0.041	0.032	0.042	0.002	0.041	0.002
		θ	0.018	0.008	0.018	0.001	0.018	0.001
		$S(t)$	0.283	0.080	0.283	0.080	0.284	0.080
		$h(t)$	0.134	0.018	0.134	0.018	0.134	0.018
	100	λ	0.023	0.019	0.029	0.001	0.029	0.001
		θ	0.012	0.005	0.015	0.001	0.009	0.001
		$S(t)$	0.294	0.086	0.273	0.075	0.255	0.065
		$h(t)$	0.139	0.019	0.127	0.016	0.122	0.015
	130	λ	0.012	0.017	0.025	0.001	0.028	0.001
		θ	0.012	0.004	0.009	0.000	0.010	0.001
		$S(t)$	0.333	0.111	0.307	0.094	0.320	0.102
		$h(t)$	0.162	0.026	0.144	0.021	0.148	0.022

Table A2. *MSE and Bias for different Estimates of parameters, RF, and HRF at* $\lambda = 2$ *and* $\theta = 1.5$

			Scheme 1					
n	***m***	**Estimate**	**ML**		**IFP**		**NIFP**	
			Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
50	**20**	λ	0.376	2.225	0.153	0.024	0.452	0.205
		θ	0.050	0.049	0.085	0.008	0.068	0.005
		$S(t)$	0.328	0.108	0.260	0.068	0.433	0.187
		$h(t)$	0.042	0.002	0.125	0.016	0.057	0.003
	30	λ	0.302	0.897	0.108	0.012	0.322	0.104
		θ	0.045	0.022	0.054	0.003	0.050	0.003
		$S(t)$	0.506	0.256	0.273	0.075	0.459	0.211
		$h(t)$	0.060	0.004	0.134	0.018	0.056	0.003
100	**20**	λ	0.389	1.456	0.128	0.017	0.356	0.127
		θ	0.070	0.049	0.067	0.005	0.056	0.004
		$S(t)$	0.442	0.195	0.304	0.093	0.500	0.250
		$h(t)$	0.053	0.003	0.150	0.022	0.063	0.004
	50	λ	0.179	0.446	0.052	0.003	0.169	0.029
		θ	0.029	0.013	0.027	0.001	0.026	0.001
		$S(t)$	0.425	0.181	0.334	0.112	0.429	0.184
		$h(t)$	0.051	0.003	0.146	0.021	0.051	0.003
	70	λ	0.107	0.268	0.038	0.002	0.128	0.017
		θ	0.017	0.007	0.015	0.001	0.023	0.001
		$S(t)$	0.403	0.163	0.247	0.061	0.481	0.231
		$h(t)$	0.049	0.002	0.122	0.015	0.059	0.003
150	**50**	λ	0.188	0.488	0.062	0.004	0.140	0.020

			Scheme 1					
n	***m***	**Estimate**	**ML**		**IFP**		**NIFP**	
			Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
		θ	0.028	0.013	0.030	0.001	0.022	0.001
		$S(t)$	0.422	0.178	0.346	0.119	0.458	0.210
		$h(t)$	0.055	0.003	0.160	0.026	0.056	0.003
	70	λ	0.130	0.298	0.029	0.001	0.115	0.014
		θ	0.020	0.008	0.017	0.001	0.018	0.001
		$S(t)$	0.484	0.234	0.287	0.083	0.378	0.143
		$h(t)$	0.059	0.003	0.138	0.019	0.048	0.002
	100	λ	0.094	0.193	0.012	0.002	0.098	0.010
		θ	0.015	0.006	0.015	0.001	0.016	0.001
		$S(t)$	0.431	0.186	0.357	0.128	0.434	0.188
		$h(t)$	0.050	0.003	0.157	0.025	0.053	0.003
	130	λ	0.046	0.132	0.003	0.001	0.091	0.009
		θ	0.007	0.004	0.011	0.001	0.014	0.001
		$S(t)$	0.426	0.181	0.310	0.096	0.412	0.170
		$h(t)$	0.050	0.003	0.144	0.021	0.050	0.002

Continued Table A2.

			Scheme 2					
n	***m***	**Estimate**	**ML**		**IFP**		**NIFP**	
			Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
50	**20**	λ	0.369	1.371	0.161	0.026	0.368	0.136
		θ	0.054	0.041	0.080	0.007	0.056	0.004
		$S(t)$	0.500	0.250	0.431	0.186	0.496	0.246
		$h(t)$	0.059	0.004	0.183	0.033	0.063	0.004
	30	λ	0.249	0.718	0.103	0.011	0.285	0.082
		θ	0.038	0.019	0.050	0.003	0.042	0.002
		$S(t)$	0.491	0.241	0.261	0.068	0.433	0.188
		$h(t)$	0.058	0.003	0.128	0.016	0.054	0.003
100	**20**	λ	0.377	1.356	0.098	0.010	0.329	0.109
		θ	0.069	0.054	0.059	0.004	0.065	0.005
		$S(t)$	0.469	0.220	0.148	0.022	0.502	0.252
		$h(t)$	0.059	0.004	0.109	0.012	0.066	0.004
	50	λ	0.155	0.407	0.062	0.004	0.158	0.026
		θ	0.028	0.012	0.028	0.001	0.025	0.001
		$S(t)$	0.453	0.206	0.291	0.085	0.417	0.174
		$h(t)$	0.056	0.003	0.137	0.019	0.052	0.003
	70	λ	0.086	0.244	0.043	0.002	0.162	0.027
		θ	0.012	0.007	0.018	0.001	0.020	0.001
		$S(t)$	0.392	0.154	0.333	0.111	0.451	0.203
		$h(t)$	0.047	0.002	0.163	0.027	0.057	0.003
150	**50**	λ	0.181	0.446	0.050	0.003	0.142	0.021
		θ	0.033	0.014	0.024	0.001	0.025	0.001

			Scheme 2					
n	**m**	**Estimate**	**ML**		**IFP**		**NIFP**	
			Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
		$S(t)$	0.482	0.233	0.349	0.122	0.408	0.166
		$h(t)$	0.060	0.004	0.158	0.025	0.050	0.003
	70	λ	0.105	0.252	0.041	0.002	0.124	0.016
		θ	0.013	0.007	0.018	0.001	0.018	0.001
		$S(t)$	0.422	0.178	0.284	0.080	0.391	0.153
		$h(t)$	0.052	0.003	0.134	0.018	0.046	0.002
	100	λ	0.094	0.197	0.029	0.001	0.099	0.010
		θ	0.014	0.005	0.009	0.001	0.014	0.001
		$S(t)$	0.408	0.166	0.255	0.065	0.436	0.190
		$h(t)$	0.048	0.002	0.122	0.015	0.053	0.003
	130	λ	0.051	0.128	0.028	0.001	0.072	0.006
		θ	0.010	0.004	0.010	0.001	0.010	0.001
		$S(t)$	0.429	0.184	0.320	0.102	0.464	0.215
		$h(t)$	0.051	0.003	0.148	0.022	0.056	0.003

Table A3. *AIL and CP for different estimates for parameters, RF, and HRF at* $\lambda = 0.5$ *and* $\theta = 1.5$

			Scheme 1					
n	**m**	**Estimate**	**ML**		**IFP**		**NIFP**	
			AIL	**CP**	**AIL**	**CP**	**AIL**	**CP**
50	**20**	λ	2.094	97.9	0.082	97.3	0.081	97.4
		θ	1.040	96.0	0.086	97.9	0.086	97.5
		$S(t)$	0.922	95.0	0.977	100.0	0.977	100.0
		$h(t)$	0.269	95.0	0.275	100.0	0.276	100.0
	30	λ	1.417	96.2	0.082	97.2	0.076	97.7
		θ	0.696	96.0	0.086	96.8	0.085	98.3
		$S(t)$	0.855	96.7	0.856	100.0	0.855	100.0
		$h(t)$	0.272	96.7	0.271	100.0	0.272	100.0
100	**20**	λ	1.524	96.3	0.087	97.3	0.079	97.0
		θ	0.877	96.3	0.083	97.3	0.088	97.1
		$S(t)$	0.916	95.0	0.928	100.0	0.928	100.0
		$h(t)$	0.268	95.0	0.277	100.0	0.277	100.0
	50	λ	0.914	95.4	0.080	98.1	0.084	97.4
		θ	0.451	96.0	0.081	97.5	0.084	99.4
		$S(t)$	0.949	96.0	0.903	96.0	0.903	96.0
		$h(t)$	0.286	96.0	0.284	100.0	0.278	96.0
	70	λ	0.703	95.4	0.083	98.2	0.083	97.8
		θ	0.341	95.5	0.076	97.7	0.085	9.8
		$S(t)$	0.977	97.1	0.955	100.0	0.955	100.0
		$h(t)$	0.289	97.1	0.289	95.7	0.289	95.7
150	**50**	λ	0.871	95.8	0.086	96.8	0.086	96.7
		θ	0.456	96.3	0.080	98.0	0.085	97.3

			Scheme 1					
n	***m***	**Estimate**	**ML**		**IFP**		**NIFP**	
			AIL	**CP**	**AIL**	**CP**	**AIL**	**CP**
		$S(t)$	0.951	96.0	0.959	96.0	0.954	96.0
		$h(t)$	0.286	96.0	0.288	98.0	0.281	96.0
	70	λ	0.637	95.8	0.082	97.9	0.086	96.3
		θ	0.347	95.4	0.078	9.8	0.087	97.5
		$S(t)$	0.945	97.1	0.939	98.6	0.939	98.6
		$h(t)$	0.294	97.1	0.287	95.7	0.289	95.7
	100	λ	0.558	95.0	0.084	98.5	0.085	97.9
		θ	0.283	96.4	0.080	97.6	0.080	98.3
		$S(t)$	0.935	97.0	0.921	97.0	0.844	97.0
		$h(t)$	0.293	97.0	0.283	96.0	0.279	96.0
	130	λ	0.507	95.5	0.085	98.1	0.085	97.5
		θ	0.258	96.0	0.078	97.5	0.083	96.4
		$S(t)$	0.933	96.9	0.929	99.2	0.944	100.0
		$h(t)$	0.298	96.9	0.294	99.2	0.292	95.4

Continued Table A3.

			Scheme 2					
n	***m***	**Estimate**	**ML**		**IFP**		**NIFP**	
			AIL	**CP**	**AIL**	**CP**	**AIL**	**CP**
50	**20**	λ	2.008	96.2	0.083	97.7	0.085	98.5
		θ	0.976	95.9	0.084	96.5	0.083	97.9
		$S(t)$	0.923	95.0	0.944	100.0	0.944	100.0
		$h(t)$	0.255	95.0	0.265	100.0	0.265	100.0
	30	λ	1.260	95.9	0.079	97.9	0.083	97.6
		θ	0.617	95.9	0.084	98.5	0.081	98.1
		$S(t)$	0.956	96.7	0.953	100.0	0.953	100.0
		$h(t)$	0.273	96.7	0.271	96.7	0.271	96.7
100	**20**	λ	1.559	96.8	0.084	98.3	0.085	99.3
		θ	0.913	97.0	0.076	97.6	0.085	97.9
		$S(t)$	0.977	95.0	0.983	100.0	0.983	100.0
		$h(t)$	0.281	95.0	0.284	100.0	0.283	100.0
	50	λ	0.860	95.0	0.085	96.4	0.083	97.9
		θ	0.431	96.3	0.082	98.0	0.085	97.8
		$S(t)$	0.949	96.0	0.865	100.0	0.942	100.0
		$h(t)$	0.283	96.0	0.275	100.0	0.284	96.0
	70	λ	0.702	95.5	0.085	97.2	0.083	98.2
		θ	0.344	95.7	0.082	97.7	0.085	97.9
		$S(t)$	0.957	97.1	0.938	100.0	0.938	100.0
		$h(t)$	0.287	97.1	0.285	95.7	0.287	95.7
150	**50**	λ	0.814	95.6	0.087	97.3	0.083	97.0
		θ	0.455	95.7	0.078	98.2	0.091	98.3
		$S(t)$	0.937	96.0	0.951	100.0	0.941	96.0

			Scheme 2					
n	**m**	**Estimate**	**ML**		**IFP**		**NIFP**	
			AIL	**CP**	**AIL**	**CP**	**AIL**	**CP**
		$h(t)$	0.288	96.0	0.288	96.0	0.277	96.0
	70	λ	0.687	95.8	0.080	97.7	0.087	97.2
		θ	0.340	96.2	0.085	97.7	0.084	99.2
		$S(t)$	0.915	97.1	0.912	100.0	0.912	100.0
		$h(t)$	0.285	97.1	0.282	95.7	0.282	95.7
	100	λ	0.527	96.2	0.084	98.8	0.085	97.3
		θ	0.272	96.1	0.083	98.1	0.081	98.6
		$S(t)$	0.931	97.0	0.916	96.0	0.941	96.0
		$h(t)$	0.291	97.0	0.294	97.0	0.288	96.0
	130	λ	0.498	95.8	0.086	97.2	0.088	97.4
		θ	0.253	96.3	0.077	97.0	0.081	98.0
		$S(t)$	0.955	96.9	0.948	95.4	0.908	95.4
		$h(t)$	0.287	96.9	0.288	95.4	0.293	95.4

Table A4. *AIL and CP for different estimates for parameters, RF, and HRF at* $\lambda = 2$ *and* $\theta = 1.5$

			Scheme 1					
N	**m**	**Estimate**	**ML**		**IFP**		**NIFP**	
			AIL	**CP**	**AIL**	**CP**	**AIL**	**CP**
50	**20**	λ	5.664	97.4	0.081	97.4	0.087	97.7
		θ	0.859	96.9	0.086	97.5	0.083	97.9
		$S(t)$	0.807	96.0	0.977	100.0	0.943	100.0
		$h(t)$	0.085	96.0	0.276	100.0	0.088	96.0
	30	λ	3.520	94.8	0.076	97.7	0.081	97.3
		θ	0.560	95.8	0.085	98.3	0.084	96.8
		$S(t)$	0.908	97.1	0.855	100.0	0.915	97.1
		$h(t)$	0.091	97.1	0.272	100.0	0.091	97.1
100	**20**	λ	4.480	95.4	0.079	97.0	0.083	98.3
		θ	0.823	95.5	0.088	97.1	0.083	98.2
		$S(t)$	0.895	96.0	0.928	100.0	0.949	100.0
		$h(t)$	0.092	96.0	0.277	100.0	0.087	96.0
	50	λ	2.524	96.2	0.084	97.4	0.084	96.9
		θ	0.424	95.9	0.084	99.4	0.084	97.5
		$S(t)$	0.954	96.0	0.903	96.0	0.940	96.0
		$h(t)$	0.096	96.0	0.278	96.0	0.094	96.0
	70	λ	1.985	96.2	0.083	97.8	0.087	97.0
		θ	0.329	96.1	0.085	9.8	0.085	97.9
		$S(t)$	0.869	97.3	0.955	100.0	0.937	98.7
		$h(t)$	0.097	97.3	0.289	95.7	0.096	96.0
150	**50**	λ	2.639	95.6	0.086	96.7	0.084	97.5
		θ	0.427	95.8	0.085	97.3	0.081	98.2
		$S(t)$	0.918	96.0	0.954	96.0	0.965	100.0

			Scheme 1					
N	***m***	**Estimate**	**ML**		**IFP**		**NIFP**	
			AIL	**CP**	**AIL**	**CP**	**AIL**	**CP**
		$h(t)$	0.094	96.0	0.281	96.0	0.096	96.0
	70	λ	2.080	95.2	0.086	96.3	0.085	97.2
		θ	0.352	95.6	0.087	97.5	0.081	97.5
		$S(t)$	0.949	97.3	0.939	98.6	0.866	100.0
		$h(t)$	0.096	97.3	0.289	95.7	0.095	100.0
	100	λ	1.684	95.7	0.085	97.9	0.084	97.2
		θ	0.286	96.2	0.080	98.3	0.082	98.6
		$S(t)$	0.907	97.0	0.844	97.0	0.935	100.0
		$h(t)$	0.098	97.0	0.279	96.0	0.097	96.0
	130	λ	1.415	95.6	0.085	97.5	0.082	96.5
		θ	0.235	96.4	0.083	96.4	0.085	97.0
		$S(t)$	0.934	96.9	0.944	100.0	0.941	97.7
		$h(t)$	0.099	96.9	0.292	95.4	0.098	100.0

Continued Table A4.

			Scheme 2					
n	***m***	**Estimate**	**ML**		**IFP**		**NIFP**	
			AIL	**CP**	**AIL**	**CP**	**AIL**	**CP**
50	**20**	λ	4.332	95.5	0.085	98.5	0.084	96.9
		θ	0.750	96.3	0.083	97.9	0.084	97.2
		$S(t)$	0.964	96.0	0.944	100.0	0.929	96.0
		$h(t)$	0.091	96.0	0.265	100.0	0.089	100.0
	30	λ	3.172	94.3	0.083	97.6	0.081	98.0
		θ	0.517	94.9	0.081	98.1	0.090	97.2
		$S(t)$	0.879	97.1	0.953	100.0	0.975	97.1
		$h(t)$	0.093	97.1	0.271	96.7	0.093	100.0
100	**20**	λ	4.321	96.1	0.085	99.3	0.081	97.8
		θ	0.868	96.6	0.085	97.9	0.083	96.7
		$S(t)$	0.889	96.0	0.983	100.0	0.915	100.0
		$h(t)$	0.090	96.0	0.283	100.0	0.089	100.0
	50	λ	2.429	95.6	0.083	97.9	0.085	98.1
		θ	0.420	96.4	0.085	97.8	0.086	98.0
		$S(t)$	0.935	96.0	0.942	100.0	0.946	96.0
		$h(t)$	0.096	96.0	0.284	96.0	0.096	96.0
	70	λ	1.908	95.4	0.083	98.2	0.081	97.3
		θ	0.332	96.7	0.085	97.9	0.083	99.0
		$S(t)$	0.914	97.3	0.938	100.0	0.940	97.3
		$h(t)$	0.097	97.3	0.287	95.7	0.095	97.3
150	**50**	λ	2.520	95.7	0.083	97.0	0.085	98.4
		θ	0.450	95.3	0.091	98.3	0.087	99.2

			Scheme 2					
n	**m**	**Estimate**	**ML**		**IFP**		**NIFP**	
			AIL	**CP**	**AIL**	**CP**	**AIL**	**CP**
		$S(t)$	0.969	96.0	0.941	96.0	0.927	100.0
		$h(t)$	0.097	96.0	0.277	96.0	0.097	100.0
	70	λ	1.924	95.8	0.087	97.2	0.085	97.7
		θ	0.331	96.1	0.084	99.2	0.085	97.4
		$S(t)$	0.914	97.3	0.912	100.0	0.933	100.0
		$h(t)$	0.096	97.3	0.282	95.7	0.097	100.0
	100	λ	1.701	96.0	0.085	97.3	0.087	98.0
		θ	0.283	96.1	0.081	98.6	0.079	97.5
		$S(t)$	0.965	97.0	0.941	96.0	0.946	100.0
		$h(t)$	0.098	97.0	0.288	96.0	0.097	96.0
	130	λ	1.390	96.4	0.088	97.4	0.084	98.2
		θ	0.238	95.8	0.081	98.0	0.084	96.7
		$S(t)$	0.921	96.9	0.908	95.4	0.939	98.5
		$h(t)$	0.099	96.9	0.293	95.4	0.097	95.4