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Solitary Wave Solutions to the General Class of Nonlocal Nonlinear Coupled Wave Equations

Şenay PASİNLİOĞLU [a,*], Gulcin M. MUSLU [a]

[a] *Department of Mathematics, Faculty of Science and Letters, Istanbul Technical University, Maslak, Istanbul, TÜRKİYE*

* *Corresponding author's e-mail address: pasinliogl@itu.edu.tr*

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ABSTRACT

In this paper, we study a general class of nonlocal nonlinear coupled wave equations that includes the convolution operation with kernel functions. For appropriate selections of the kernel functions, the system becomes well-known nonlinear coupled wave equations, for instance Toda lattice system, coupled improved Boussinesq equations. A numerical scheme is proposed for the solitary wave solutions of the system using the Pethiashvili method. Using the different kernels, the validity of the numerical method has been tested.

Keywords: *Coupled Boussinesq equations, Petviashvili's iteration method, solitary wave solutions.*

Yerel ve Lineer Olmayan Kuple Dalga Denklemlerinin Genel Sınıfı için Yalnız Dalga Çözümleri

ÖZET

Bu makalede çekirdek fonksiyonları ile konvolüsyon işlemini içeren, yerel ve doğrusal olmayan kuple dalga denklemlerinin genel bir sınıfını inceliyoruz. Çekirdek fonksiyonlarının uygun seçimleri için sistem, Toda kafes sistemi, kuple Boussinesq denklemleri gibi iyi bilinen doğrusal olmayan kuple dalga denklemleri haline gelir. Petviashvili yöntemi kullanılarak, sistemin yalnız dalga çözümleri için bir sayısal şema önerilmiştir. Farklı çekirdekler kullanılarak, sayısal yöntemin geçerliliği test edilmiştir.

Anahtar Kelimeler: *Kuple Boussinesq denklemler, Petviashvili iterasyon yöntemi, yalnız dalga çözümleri.*

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I. INTRODUCTION

In this paper, we consider the nonlinear nonlocal coupled wave equations

$$u_{1tt} = \left[\beta_1 * \left(u_1 + g_1(u_1,u_2)\right)\right]_{xx}, \quad x \in R,\ t > 0, \tag{1}$$

$$u_{2tt} = \left[\beta_2 * \left(u_2 + g_2(u_1,u_2)\right)\right]_{xx}, \quad x \in R,\ t > 0, \tag{2}$$

where g_1 and g_2 are nonlinear functions of $u_1 = u_1(x,t)$ and $u_2 = u_2(x,t)$, the subscripts indicate partial derivatives. Here the symbol * indicates the convolution

$$\beta_i * v = \int_R \beta_i(x-y)v(y)dy. \tag{3}$$

The functions $g_i(u_1,u_2)\,(i = 1,2)$ satisfy the exactness condition $\frac{\partial g_1}{\partial u_2} = \frac{\partial g_2}{\partial u_1}$ and assume that $g_i(u_1,u_2) \in C^2(R^2)$, $i = 1,2$. The general kernel functions $\beta_i(x)$ are integrable. The Fourier transforms of kernel functions satisfy the following condition

$$0 \le \hat{\beta}_i(k) \le C_i(1+k^2)^{-r_i/2} \quad \text{for all} \quad k \in R\ (i = 1,2) \tag{4}$$

for some constants $C_i > 0$, $r_i \in$ R and $r_i \ge 2$.

The system (1)-(2) turns into well-known coupled systems of nonlinear wave equations for some appropriate selections of the kernel functions $\beta_i(x)$. For the kernel functions $\beta_1(x) = \beta_2(x) = \delta$, the system (1)-(2) reduces to the coupled nonlinear wave equations

$$u_{1tt} - u_{1xx} = [g_1(u_1,u_2)]_{xx}, \tag{5}$$

$$u_{2tt} - u_{2xx} = [g_2(u_1,u_2)]_{xx}, \tag{6}$$

where δ is the Dirac delta function.

In the case of the exponential kernel [1], $\beta_1(x) = \beta_2(x) = \frac{1}{2}e^{-|x|}$, the system (1)-(2) becomes the coupled improved Boussinesq equations

$$u_{1tt} - u_{1xx} - u_{1xxtt} = [g_1(u_1,u_2)]_{xx}, \tag{7}$$

$$u_{2tt} - u_{2xx} - u_{2xxtt} = [g_2(u_1,u_2)]_{xx}. \tag{8}$$

In various contexts, the system (7)-(8) has been obtained to describe bi-directional wave propagation, for example, in a diatomic lattice [2], in a Toda lattice model [3] and in a two layered lattice model [4].

If the kernel functions are chosen as $\beta_1(x) = \beta_2(x) = \frac{1}{2c\kappa}e^{-\frac{|x|}{c}}$, $\kappa = \frac{\rho}{a}$ and $c = \frac{\ell}{\sqrt{12}}$, and taking $g_1(u_1,u_2) = (b-1)u_1 - \frac{b^2}{2}u_1^2 + \frac{b}{2}u_2^2$ and $g_2(u_1,u_2) = bu_1u_2 - u_2$ the system (1)-(2) becomes the Toda lattice system

$$\frac{\rho}{a}u_{1tt} = \mathrm{b}u_{1xx} - \frac{b^2}{2}({u_1}^2)_{xx} + \frac{b}{2}(u_2^2)_{xx} + \frac{\rho}{a}\frac{\ell^2}{12}u_{1xxtt}, \tag{9}$$

$$\frac{\rho}{a}u_{2tt} = \mathrm{b}(u_1u_2)_{xx} + \frac{\rho}{a}\frac{\ell^2}{12}u_{2xxtt}, \tag{10}$$

where ρ is the linear mass density, a is a constant, b is a coupling parameter and ℓ is the characteristic length. These equations describe the propagation of longitudinal and transversal waves on molecules of DNA (deoxyribonucleic acid) [3], [5]. This system has been treated numerically in [3] and theoretically in [5]. The Cauchy problem for (9)-(10) has been studied in [6], [7].

In the case of the double-exponential kernel [8],

$$\beta_1(x) = \beta_2(x) = \frac{1}{2(c_1^2 - c_2^2)}\left(c_1 e^{-|x|/c_1} - c_2 e^{-|x|/c_2}\right), \tag{11}$$

the system (1)-(2) becomes the coupled higher-order Boussinesq equations

$$u_{1tt} - u_{1xx} - \eta_1 u_{1xxtt} + \eta_2 u_{1xxxxtt} = [g_1(u_1, u_2)]_{xx}, \tag{12}$$

$$u_{2tt} - u_{2xx} - \eta_1 u_{2xxtt} + \eta_2 u_{2xxxxtt} = [g_2(u_1, u_2)]_{xx}, \tag{13}$$

where c_1 and c_2 are real, positive constants and $\eta_1 = c_1^2 + c_2^2$ and $\eta_2 = c_1^2 c_2^2$.

The uncoupled form of Eqs. (12)-(13) appears as a model for a dense chain of particles with elastic couplings [9] and for longitudinal waves in a nonlocal nonlinear elastic medium [10]. It can be found in [11] different types of the kernel functions used in the literature.

In this paper, we focus on a general class of kernel functions. The global existence of the system (1)-(2) with initial conditions

$$u_1(x,0) = \phi(x), \qquad u_{1t}(x,0) = \phi_1(x), \tag{14}$$

$$u_2(x,0) = \psi(x), \qquad u_{2t}(x,0) = \psi_1(x) \tag{15}$$

has been proved in [12]. For special cases of kernel functions, the exact solitary wave solutions for the system (1)-(2) can be found in the literature. However, solitary wave solutions for the general cases of kernel functions are unknown. The aim of our study is to generate the solitary wave solutions of the nonlocal nonlinear coupled system by using the Petviashvili method numerically.

We organized this paper as follows. In Section II, we present the Petviashvili's iteration method and we obtain the solitary wave solutions of the system (1)-(2) numerically by using this method. In Section III, we perform some numerical tests for the nonlinear nonlocal coupled wave equations. The conclusion is given in Section IV.

II. THE PETVIASHVILI'S ITERATION METHOD

In this section, we propose the Petviashvili method to obtain the solitary wave solutions of the nonlinear nonlocal coupled wave eqs. (1)-(2). The Petviashvili's method was first introduced in [13] to obtain the solitary wave solutions of nonlinear wave equations numerically. The conditions in which convergence and necessary to obtain the optimal convergence rate were found in [14]. It has been reported in some articles that this method can be applied to nonlinear dispersive wave equations [15]-[21]. In this method, a stabilizing factor is added to the fixed point iteration scheme. The detailed information about this method can be found in [17], [22], [23].

To apply this method to our system, we first use the ansatz

$$u_1(x,t) = \phi(\xi), \qquad u_2(x,t) = \psi(\xi), \qquad \xi = x - ct, \tag{16}$$

where c is the wave propagation speed. Using the asymptotic boundary conditions and substituting these solutions into (1)-(2) and then integrating twice we have

$$c^2\phi = \beta_1 * [\phi + g_1(\phi,\psi)]\,, \tag{17}$$

$$c^2\psi = \beta_2 * [\psi + g_2(\phi,\psi)]\,. \tag{18}$$

Taking the Fourier transform,

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k)e^{ikx}\,dk, \qquad \hat{u}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)e^{ikx}\,dx, \tag{19}$$

of the Eqs. (17)-(18), we find

$$\left[c^2 - \hat{\beta}_1(k)\right]\hat{\phi}(k) = \hat{\beta}_1(k)\hat{g}_1\,, \tag{20}$$

$$\left[c^2 - \hat{\beta}_2(k)\right]\hat{\psi}(k) = \hat{\beta}_2(k)\hat{g}_2\,. \tag{21}$$

The numerical calculation of $\hat{\phi}(k)$ and $\hat{\psi}(k)$ for the Eqs. (20)-(21) can be given in the form

$$\hat{\phi}_{n+1}(k) = \frac{\hat{\beta}_1(k)}{c^2-\hat{\beta}_1(k)}\hat{g}_{1n}\,, \tag{22}$$

$$\hat{\psi}_{n+1}(k) = \frac{\hat{\beta}_2(k)}{c^2-\hat{\beta}_2(k)}\hat{g}_{2n}\,, \tag{23}$$

where $\hat{\phi}_n(k)$ and $\hat{\psi}_n(k)$ are the Fourier transforms of $\phi_n(x)$ and $\psi_n(x)$ which are the nth iterations of the numerical solutions. We add stabilizing factors $M_{1,n}$ and $M_{2,n}$ to ensure the convergence [13]. The new algorithm can be given in the form

$$\hat{\phi}_{n+1}(k) = \left(M_{1,n}\right)^{\gamma_1}\frac{\hat{\beta}_1(k)}{c^2-\hat{\beta}_1(k)}\hat{g}_{1n}\,, \tag{24}$$

$$\hat{\psi}_{n+1}(k) = \left(M_{2,n}\right)^{\gamma_2}\frac{\hat{\beta}_2(k)}{c^2-\hat{\beta}_2(k)}\hat{g}_{2n}\,, \tag{25}$$

where the stabilizing factors are

$$M_{1,n} = \frac{\int_{-\infty}^{\infty}\left(c^2-\hat{\beta}_1(k)\right)[\hat{\phi}_n(k)]^2dk}{\int_{-\infty}^{\infty}\hat{\beta}_1(k)\hat{g}_1(k)\hat{\phi}_n(k)dk}\,, \tag{26}$$

$$M_{2,n} = \frac{\int_{-\infty}^{\infty}\left(c^2-\hat{\beta}_2(k)\right)[\hat{\psi}_n(k)]^2dk}{\int_{-\infty}^{\infty}\hat{\beta}_2(k)\hat{g}_1(k)\hat{\psi}_n(k)dk}\,, \tag{27}$$

and γ_1 and γ_2 are free parameters. Solitary wave solutions for nonlocal nonlinear coupled wave equations can only be constructed under the assumptions

$$c^2 - \widehat{\beta_1}(k) \neq 0 \ \text{ and } \ c^2 - \widehat{\beta_2}(k) \neq 0 \quad \text{for all } k \in R\,. \tag{28}$$

III. NUMERICAL EXAMPLES

In this section, we give some numerical experiments. We first compare the numerical solutions obtained by the Petviashvili's method with the exact solutions available in the literature. Then, we obtain the solitary wave profile for the nonlinear nonlocal coupled wave equations (1)-(2), generated by the Petviashvili method.

In all experiments, we choose the spatial interval $-100 \leq x \leq 100$ and the number of spatial grid points $N = 1024$. The numerical operations are performed by using Matlab. We use three different errors to control the overall iterative process. These are the stabilization factor error

$$\mathrm{M} = \left|1 - M_{i,n}\right|, \; M_{i,n} \text{ are given by (26)-(27)}, \; i = 1,2 \text{ and } n = 0,1,2 \ldots, \tag{29}$$

the residual error

$$\mathrm{RES(n)} = ||\mathcal{R}_1\phi_n||, \quad \mathrm{RES(m)} = ||\mathcal{R}_2\psi_m||, \quad n,m = 0,1,2 \ldots, \tag{30}$$

where

$$\mathcal{R}_1\phi = c^2\phi - \left[\beta_1 * \left(\phi + g_1(\phi,\psi)\right)\right], \tag{31}$$

$$\mathcal{R}_2\psi = c^2\psi - \left[\beta_2 * \left(\psi + g_2(\phi,\psi)\right)\right], \tag{32}$$

and error between two consecutive iterations

$$\mathrm{Error}_1(n) = ||\phi_n - \phi_{n-1}||, \qquad \mathrm{Error}_2(\mathrm{m}) = ||\psi_m - \psi_{m-1}||, \qquad n,m = 0,1,2 \ldots. \tag{33}$$

In Example 1 and Example 2, we use exponential kernels $\beta_1(x) = \beta_2(x) = \frac{1}{2}e^{-|x|}$ with Fourier transforms are $\hat{\beta}_1(k) = \hat{\beta}_2(k) = \frac{1}{1+k^2}$ to compare the solitary wave solutions obtained by the Petviashvili's method with the exact solutions.

Example 1. In this experiment, we compare our numerical solution with the exact solution of the uncoupled improved Boussinesq equations to test our scheme. For the kernel functions $\beta_1(x) = \beta_2(x) = \frac{1}{2}e^{-|x|}$, if we take nonlinear functions g_1 and g_2 as $g_1(u_1,u_2) = u_1^2$ and $g_2(u_1,u_2) = u_2^2$, we obtain uncoupled improved Boussinesq equations

$$u_{1tt} = u_{1xx} + u_{1xxtt} + (u_1^2)_{xx}, \tag{34}$$
$$u_{2tt} = u_{2xx} + u_{2xxtt} + (u_2^2)_{xx}. \tag{35}$$

Since equations (34) and (35) are the same, we only use the first equation. The solitary wave solution of (34)-(35) is given by

$$u_1(x,t) = u_2(x,t) = \alpha\,\mathrm{sech}^2\left(\frac{1}{A}\sqrt{\frac{\alpha}{6}}(x - At - x_0)\right), \tag{36}$$

where $\alpha = 0.25$ is the initial amplitude, and $A = \sqrt{1 + \frac{2}{3}\alpha}$ is the velocity of the pulse with $A^2 > 1$ [24].

In the left side of Figure 1, we compare the numerically obtained solitary wave solution of (34)-(35) with exact solitary wave solution. In the right side of Figure 1, we present the variation of the stabilization factor error M, the residual error RES and Error given by (29)-(33). We choose $\gamma_i = 1.2$ and initial guess $u_1 = e^{-x^2}$. It can be seen from the figure, the solitary wave profile obtained by proposed method is compatible with the exact solitary wave solution of equations (34)-(35).

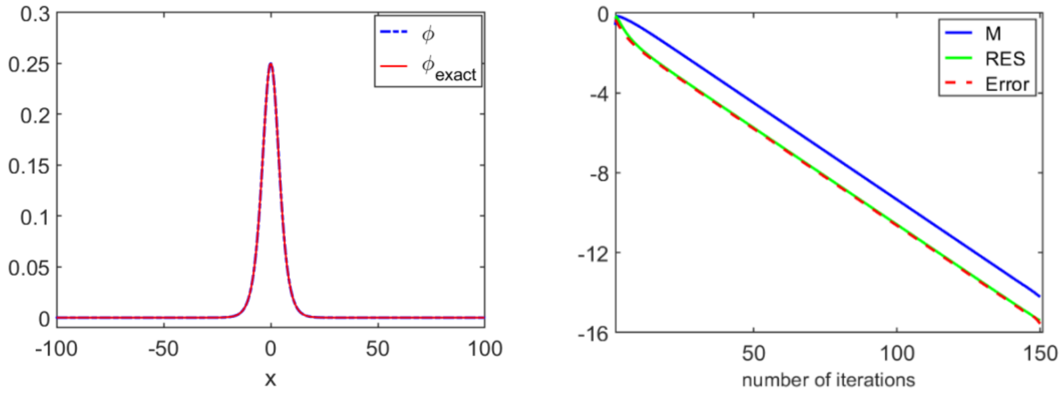


Figure 1. *The numerical and exact solitary wave profiles of the uncoupled improved Boussinesq equation and the variation of errors.*

Example 2. For the kernel functions $\beta_1(x) = \beta_2(x) = \frac{1}{2}e^{-|x|}$, if we choose $g_1(u_1, u_2) = -bu_1^2 + u_2^2$ and $g_2(u_1, u_2) = 2u_1u_2$ we obtain the following system

$$u_{1tt} = u_{1xx} - b(u_1^2)_{xx} + (u_2^2)_{xx} + u_{1xxtt}\,, \tag{37}$$

$$u_{2tt} = u_{2xx} + 2(u_2u_1)_{xx} + u_{2xxtt}\,, \tag{38}$$

where $u_1(x,t)$ and $u_2(x,t)$ describe the longitudinal strain and the transverse strain respectively. This system emerges from a weakly nonlinear model of wave propagation in a simple cubic lattice [25]. The solitary wave solutions of the system (37)-(38) are given by

$$u_1(x,t) = u_{1c}(x - ct)\,, \quad u_2(x,t) = \alpha u_{1c}(x - ct)\,, \tag{39}$$

where

$$u_{1c}(x) = \frac{3}{4}(c^2 - 1)sech^2(\gamma x)\,, \quad \alpha = \sqrt{2 + b}\,, \quad \gamma = \frac{1}{2}\sqrt{\frac{c^2 - 1}{c^2}}\,. \tag{40}$$

The stability of the solitary wave solutions for this system has been studied numerically by [3] and [25].

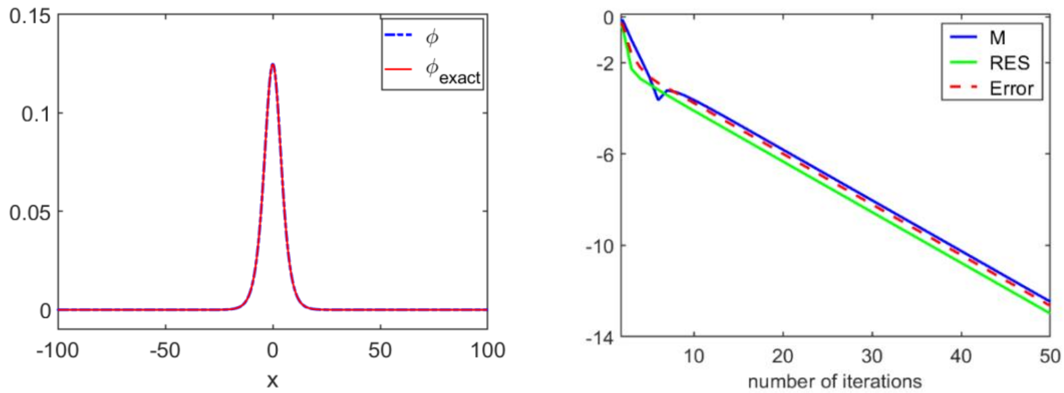


Figure 2. *The numerical and exact solitary wave profiles of the system of (37)-(38) and the variation of errors.*

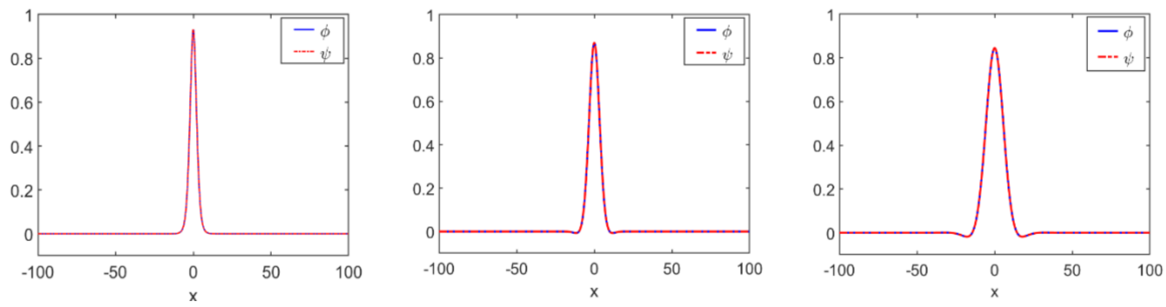
In Figure 2, we show the only one solution of the system (37)-(38) because the solutions are linearly dependent. Figures for the solution of the second equation of the system (37)-(38) are the same. Here we choose $b = -1$, $c = 1.08$ and initial guess $u_1 = e^{-x^2}$. It can be seen from the Figure 2 the solitary wave profile obtained by the proposed method is compatible with the exact solitary wave solution of equations (37)-(38).

In the following examples, we use kernel functions whose Fourier transforms are known. To our knowledge, the exact solitary wave solutions of the system (1)-(2) with the following kernels are unknown. So, we construct the solitary wave solutions of the system (1)-(2) with the following kernels by using the Petviashvili iteration method given in Section II.

Example 3. In this example, we take the general kernel function with the Fourier transform is

$$\hat{\beta}_i(k) = \frac{1}{1+k^2+\eta k^2 \sin^2(k^2)}\ , \quad i = 1,2\ , \tag{41}$$

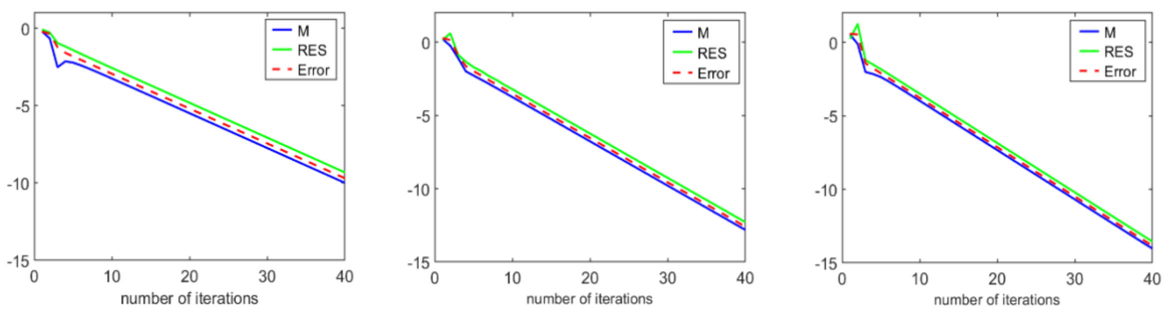
where η is a positive parameter. Choosing $g_1(u_1,u_2) = u_1^2 + u_2^2$ and $g_2(u_1,u_2) = 2u_1u_2$, we carry out some numerical experiments for different values of η . In Figure 3, we show the numerical solutions obtained by the proposed method for the system (1)-(2) with (41).



(a) **(b)** **(c)**

Figure 3. *The numerical solitary wave profiles obtained by proposed method for the the system (1)-(2) with (41).*

In this case, we take **(a)** $\eta = 0.1$, **(b)** $\eta = 5$ and **(c)** $\eta = 50$ for $c = 1.5$. We see that the amplitude of the solitary wave solution decreases as we increase values of η. Since the exact solution of the system (1)-(2) with (41) is not known, we cannot compare our numerical solution with the exact solution. Therefore we show the stabilization factor error $|1 - M_n|$, the residual error RES, and Error with the number of iterations in semi-log scale in Figure 4, respectively, for $\eta = 0.1$, $\eta = 5$ and $\eta = 50$. From these results we observe that the solitary wave solutions for the system (1)-(2) with (41) by the proposed method converges rapidly to the exact solutions of the given system.



(a) **(b)** **(c)**

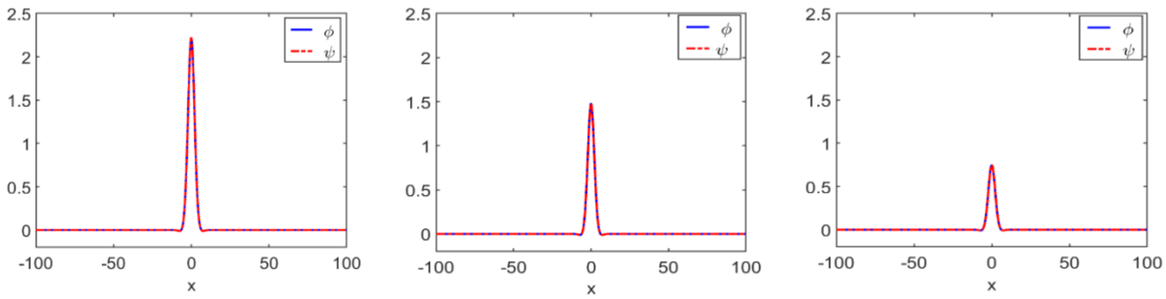
Figure 4. *The variation of errors for* $\eta = 0.1, \eta = 5$ *and* $\eta = 50$ *in semi-log scale.*

Example 4. In this example, we take the general kernel function with the Fourier transform is

$$\hat{\beta}_i(k) = \frac{1}{1+k^2+k^4} + \frac{\mu}{1+k^4}\,, \quad i = 1,2\,, \tag{42}$$

where μ is a positive parameter. We choose the wave speed c satisfying the condition (28). In this case, we choose $g_1(u_1, u_2) = u_1^2 + u_2^2$ and $g_2(u_1, u_2) = 2u_1u_2$. We carry out some numerical tests for **(a)** $\mu = 0.5$, **(b)** $\mu = 1$ and **(c)** $\mu = 2$ with $c = 2.5$.

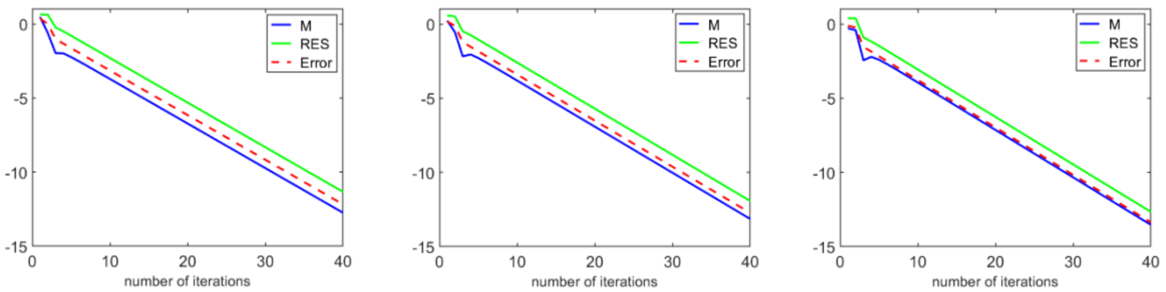
In Figure 5, we present the numerical solitary wave profiles of the system (1)-(2) with (42). We see that the amplitude of the solitary wave solution decreases as we increase values of μ .



(a) **(b)** **(c)**

Figure 5. *The solitary wave profiles obtained by proposed method for the system (1)-(2) with (42).*

In the Figure 6, we show the variation of three errors $|1 - M_n|$, RES and Error. The presented figures show that the solitary wave solutions for the system (1)-(2) with (42) by the proposed method converges rapidly to the exact solutions of the given system.



(a) **(b)** **(c)**

Figure 6. *The variation of errors for* $\mu = 0.5$, $\eta = 1$ *and* $\eta = 2$ *in semi-log scale.*

IV. CONCLUSION

In this work, we study a general class of nonlinear nonlocal coupled wave equations (1)-(2). Since the solitary wave solution for the nonlocal nonlinear coupled system is not known for general kernels, we propose a method for numerically constructing the solitary wave profile by using the Pethviashvili's method. The efficiency of the numerical methods is tested for different kernels. As it can be seen from the presented figures, our proposed numerical scheme converges considerably well with the solution.

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