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Research Article

Construction of rational interpolations using Mamquist-Takenaka systems

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ABSTRACT. Rational functions have deep system-theoretic significance. They represent the natural way of modeling linear dynamical systems in the frequency (Laplace) domain. Using rational functions, the goal of this paper to compute models that match (interpolate) given data sets of measurements. In this paper, the authors show that using special rational orthonormal systems, the Malmquist-Takenaka systems, it is possible to write the rational interpolant $r_{(n,m)}$, for $n = N-1, m = N$ using only N sampling nodes (instead of $2N$ nodes) if the interpolating nodes are in the complex unit circle or on the upper half-plane. Moreover, the authors prove convergence results related to the rational interpolant. They give an efficient algorithm for the determination of the rational interpolant.

Keywords: Rational interpolation, Hardy spaces, Malmquist-Takenaka systems, discrete biorthogonality.

2020 Mathematics Subject Classification: 42C05, 30H10, 30J10, 31C20, 65D05, 65D25.

1. INTRODUCTION

Rational functions have deep system-theoretic significance. They represent the natural way of modeling linear dynamical systems in the frequency (Laplace) domain, because the Laplace transform of a sum of complex exponentials is a rational function; more precisely, the transfer functions (or frequency responses) of such systems are rational functions. Using rational functions, our goal is to compute models that match (interpolate) given data sets of measurements.

We give first a short summary related to the general solution of the rational interpolation problem. Let us consider a function $f : H \to \mathbb{C}$, $H \subset \mathbb{C}$, and a general rational function of the form:

$$r_{(n,m)}(x) = \frac{\sum_{i=0}^{n} \alpha_i x^i}{\sum_{j=0}^{m} \beta_j x^j},$$

where $\alpha_i, \beta_j, x \in \mathbb{C}$, and m and n are not necessarily equal natural numbers. To find a rational interpolant $r_{(n,m)}$ of type (n,m) requires $n+m+1$ sample points (or in other word nodes), because we have to determine the α_i and β_j coefficients (one coefficient can be set to 1). Knowing $(x_k, f(x_k))$, $k = 1, \dots, n+m+1$, we search the solution of the interpolation problem satisfying the following conditions

$$r_{(n,m)}(x_k) = f(x_k), \ k = 1, \dots, n+m+1.$$

In this paper, we show that using special rational orthonormal systems, the Malmquist-Takenaka systems, it is possible to write the rational interpolant $r_{(n,m)}$, for $m = N, n = N-1$ using only N sampling nodes (instead of $2N$ nodes) if the interpolating nodes are in the unit circle or on

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the upper half-plane, moreover we can prove convergence results related to the rational interpolant. We give an efficient algorithm for the determination of the rational interpolant. We will introduce new rational interpolation operators of type $(N-1,N)$ using N special nodes in the closed unit disc. These nodes are solutions of certain equation related to the Malmquist-Takenaka systems and its dual systems, and we will study the properties of the new interpolation operators. We will study the analogue of the problem also for the closed upper half-plane. Before we present our results, let us revise the classical method to find $r_{(n,m)}$ (see for example Berrut, Trefethen or Ionita [1, 11] and the reference list therein). We write our interpolation conditions in the following form:

$$\left(\sum_{i=0}^{n}\alpha_i x_k^i\right) - f(x_k)\left(\sum_{j=0}^{m}\beta_j x_k^j\right) = 0.$$

In matrix form, this is equivalent to

$$\mathbf{A}\mathbf{b} = \mathbf{0},$$

where

$$\mathbf{A} := \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n & -f(x_0) & -f(x_0)x_0 & -f(x_0)x_0^2 & \dots & -f(x_0)x^m \\ 1 & x_1 & x_1^2 & \dots & x_1^n & -f(x_1) & -f(x_1)x_1 & -f(x_1)x_1^2 & \dots & -f(x_1)x^m \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_M & x_M^2 & \dots & x_M^n & -f(x_M) & -f(x_M)x_M & -f(x_M)x_M^2 & \dots & -f(x_M)x_M^m \end{bmatrix}$$

and

$$\mathbf{b} := [\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n, \beta_0, \beta_1, \beta_2, \dots \beta_m]^T.$$

However, there is no any guarantee that the solution exists, and it is unique. It is possible that there are more **b** vectors satisfying the equation -if it exists at all. When $\beta_0 = 1, \beta_1 = \beta_2 = \dots = \beta_m = 0$, then the problem reduces to the construction of a polynomial interpolant. In this case, if the nodes x_k are different from each other and we have $M = n+1$ samples, the problem has unique solution. If we want to express the interpolation polynomial $r_{(n,0)}(x) = P_n(x) = \sum_{i=0}^{n} c_i x^i$ in the basis $\Phi_k(x) = x^k$ satisfying the condition $P_n(x) = f(x_k),\ k = 1,\dots,M = n+1$, then the solution $c = (c_0, c_1, \dots, c_n)$ of the system is $\mathbf{c} = \Phi^{-1}\mathbf{f}$, where $\mathbf{f} = (f(x_1),\dots,f(x_{n+1}))^T$ and

$$\Phi = V(x_0, x_1, \dots, x_n) = \begin{bmatrix} 1 & x_0^1 & \dots & x_0^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n^1 & \dots & x_n^n \end{bmatrix}.$$

We don't have to solve the linear equation system if we write the interpolation polynomial in Lagrange form. In this way, we reduce the number of operations. Let us consider the Lagrange interpolation polynomials corresponding to the $n+1$ sample points defined by

$$l_i(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)} = \frac{\prod_{j=0,j\neq i}^{n}(x-x_j)}{\prod_{j=0,j\neq i}^{n}(x_i-x_j)}.$$

Because

$$l_i(x_k) = \delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k, \end{cases} \tag{1.1}$$

the solution of the interpolation problem has the following form:

$$P_n(x) = L_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i).$$

The set $\{l_i(x),\, i = 0,\ldots,n\}$ is the so-called Lagrange basis, thus the resulted interpolation polynomial is the linear combination of the Lagrange basis. There is only one unique Lagrange polynomial basis perfectly fitting to the set of different sample points $\{(x_i, f(x_i)),\, i = 0,\ldots,n\}$. Unfortunately, using the Lagrange method, the basis have to be recalculated when we add a new sample point, requiring $\mathcal{O}(n^2)$ operations. A solution for this problem, to diminish the number (cost) of the operations, is the Barycentric Lagrange polynomial interpolation. Using the divided differences method, we get a much faster algorithm than the Lagrange interpolation, mainly when we have a new, additional sample point. First let us consider the Lagrange polynomial of constant function 1:

$$e_n(x) = \sum_{i=0}^{n} \left\{ \frac{\prod_{j=0,j\neq i}^{n}(x - x_j)}{\prod_{j=0,j\neq i}^{n}(x_i - x_j)} \right\} = 1.$$

Using this, we can write for any function f the interpolant $L_n(x)$ in the following form:

$$L_n(x) = \frac{L_n(x)}{e_n(x)} = \frac{\sum_{i=0}^{n} \left\{ \frac{\prod_{j=0,j\neq i}^{n}(x-x_j)}{\prod_{j=0,j\neq i}^{n}(x_i-x_j)} f(x_i) \right\}}{\sum_{i=0}^{n} \left\{ \frac{\prod_{j=0,j\neq i}^{n}(x-x_j)}{\prod_{j=0,j\neq i}^{n}(x_i-x_j)} \right\}}.$$

Simplifying by $\prod_{j=0}^{n}(x - x_j)$, if we consider that $\prod_{j=0,j\neq i}^{n}(x - x_j) = \prod_{j=0}^{n}(x - x_j)\frac{1}{x-x_i}$, we arrive to the following:

$$L_n(x) = \frac{\sum_{i=0}^{n} \left\{ \frac{1}{\prod_{j=0,j\neq i}^{n}(x_i-x_j)} \frac{f(x_i)}{x-x_i} \right\}}{\sum_{i=0}^{n} \left\{ \frac{1}{\prod_{j=0,j\neq i}^{n}(x_i-x_j)} \frac{1}{x-x_i} \right\}}.$$

Let be $\frac{1}{\prod_{j=0,j\neq i}^{n}(x_i-x_j)} = \lambda_i$, then

$$L_n(x) = \frac{\sum_{i=0}^{n} \left\{ \lambda_i \frac{f(x_i)}{x-x_i} \right\}}{\sum_{i=0}^{n} \left\{ \lambda_i \frac{1}{x-x_i} \right\}}$$

is called polynomial Barycentic formula. After the determination of each λ_i, it is relatively fast to calculate the polynomial in this form, it requires $\mathcal{O}(n)$ operations. Another advantage of the Barycentric formula is that it is numerically stable. In case if we choose λ_i freely, we get a rational interpolant $r_{(n,n)}$ fitting to the sample points (when $\lambda_i = \frac{1}{\prod_{j=0,j\neq i}^{n}(x_i-x_j)}$, we get the Lagrange polynomial $L_n(x)$). These rational functions satisfy the interpolation condition $r_{(n,n)}(x_k) = f(x_k), k = 0,\ldots,n$. These rational interpolants are called Lagrange rational interpolants. Freely choosing the λ_i-s, there are more rational functions fitting to the sample points. In order to determine $r_{(n,n)}$ uniquely, we need to fix $n+1$ more λ_i-s. We can get the Lagrange rational interpolation which satisfies

$$r_{(n,n)}(x_k) = f(x_k), \quad k = 1,\ldots,2n+1$$

without solving the system of equation in the following way:

1. We have to divide the $2n+1$ sample points $\{x_j\}$ into two subgroups, $n+1$ Lagrange nodes denoted by η_i and the remaining n sample points denoting by (μ_j). Similarly, also the set of the corresponding function values ($\{f(x_j)\}$) has to be partitioned.

2. We express $r_{(n,n)}$ with the Lagrange basis $\prod_{j=0,j\neq i}^{n}(x-\eta_j)$ associated to the Lagrange nodes η_i in the following form:

$$r_{(n,n)}(x) = \frac{\sum_{i=0}^{n}\left\{a_i \prod_{j=0,j\neq i}^{n}(x-\eta_j)\right\}}{\sum_{i=0}^{n}\left\{b_i \prod_{j=0,j\neq i}^{n}(x-\eta_j)\right\}}.$$

This rational function can be written in barycentric form:

$$r_{(n,n)}(x) = \frac{\sum_{i=0}^{n}\left\{a_i \frac{1}{x-\eta_i}\right\}}{\sum_{i=0}^{n}\left\{b_i \frac{1}{x-\eta_i}\right\}}.$$

Similarly to the polynomial barycentric formula, also this rational barycentric formula requires only $\mathcal{O}(n)$ operations. When $x=\eta_i$, then $r_{(n,n)}(\eta_i)=\frac{a_i}{b_i}=f(\eta_i)$, so if we set $a_i=b_i f(\eta_i)$, then $r_{(n,n)}$ will exactly interpolate f at the η_i nodes and we get back our earlier formula when the coefficients were the same in the numerator and denominator.

3. Now, what we have to do is only to determine the unknown coefficients b_i using the remaining n sample points (μ_j) and the corresponding $f(\mu_j)$ values (which were still not used). Using that $a_i=b_i f(\eta_i)$, for the μ_j points $j=1,\dots,n$ the rational function satisfies the following:

$$r_{(n,n)}(\mu_j) = f(\mu_j) = \frac{\sum_{i=0}^{n}\left\{\frac{b_i f(\eta_i)}{\mu_j-\eta_i}\right\}}{\sum_{i=0}^{n}\left\{\frac{b_i}{\mu_j-\eta_i}\right\}}.$$

Rearranging these equations, we get the equivalent forms for $j=1,\dots,n$:

$$\sum_{i=0}^{n}\left\{\frac{b_i f(\mu_j)}{\mu_j-\eta_i}\right\} = \sum_{i=0}^{n}\left\{\frac{b_i f(\eta_i)}{\mu_j-\eta_i}\right\},$$
$$\sum_{i=0}^{n}\left\{\frac{b_i(f(\mu_j)-f(\eta_i))}{\mu_j-\eta_i}\right\} = 0.$$

These conditions can be written in a matrix form:

$$\begin{bmatrix} \frac{(f(\mu_0)-f(\eta_0))}{\mu_0-\eta_0} & \cdots & \frac{(f(\mu_0)-f(\eta_n))}{\mu_0-\eta_n} \\ \vdots & \ddots & \vdots \\ \frac{(f(\mu_{n-1})-f(\eta_0))}{\mu_{n-1}-\eta_0} & \cdots & \frac{(f(\mu_{n-1})-f(\eta_n))}{\mu_{n-1}-\eta_n} \end{bmatrix} \begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix} = 0.$$

The matrix is called Loewner matrix ($\mathbb{L}$) and it is an $n\cdot(n+1)$ matrix.

4. The Loewner matrix and its null space has to be computed using the partitioned nodes and the corresponding sample values solving the $\mathbb{L}\bar{b}=0$ equation. In this way, we get the $\bar{b}$ vector of the coefficients.

5. After the above-mentioned steps, we can form the Lagrange rational polynomial using the barycentric formula:

$$r_{(n,n)}(x) = \frac{\sum_{i=0}^{n} \left\{ \frac{b_i f(\eta_i)}{x-\eta_i} \right\}}{\sum_{i=0}^{n} \left\{ \frac{b_i}{x-\eta_i} \right\}}.$$

If the number of sample points is large, then the number of operations to determine the rational interpolant fitting the data is still high. Our goal is to find new methods to write the rational interpolant using less initial data and to reduce the number of operations if it is possible.

The paper is organized as follows. In Section 2, we present rational interpolation using Malmquist-Takenaka systems for the unit disc and also for the upper half plane. In both cases, we give the algorithms how the rational interpolant can be described. We study also the convergence properties of the interpolants. In Section 3, we introduce new rational interpolation operators with special nodes related to discrete biorthogonality of Malmquist-Takenaka systems and we study their properties.

2. Rational interpolation using Malmquist-Takenaka systems

In what follows, we focus on the determination of a rational interpolant of type $(N-1, N)$. According to the algorithms presented in the previous section, to write a rational interpolant of type $(N-1, N)$ in general, we would need $2N$ nodes and the values of the function in these nodes. In this section, we show that choosing a good basis of rational functions, the Mamquist-Takenaka system, we can reduce the number of the data and we can avoid to solve the system of equations associated to the interpolation problem. We will work with some assumptions regarding the nodes and the function f. We assume that the nodes are in the unit disc or in the upper half-plane and the function f belongs to the Hardy space of the unit disc or the Hardy space of the upper half-plane, respectively. Using the corresponding Malmquist-Takenaka systems, we show that it is possible to write a rational interpolant of type $(N-1, N)$ using only N nodes and the values of the function in these nodes. Moreover, we give an algorithm for the determination of the rational interpolant, and we study the convergence properties of the rational interpolant.

2.1. **Rational interpolation with nodes in the unit disc related to Malmquist-Takenaka system of the unit disc.** Let $\mathbb{D}$ denote the open and $\overline{\mathbb{D}}$ denote the closed unit disc, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, and let us denote the unit circle with $\mathbb{T}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let us denote the set of analytic functions over $\mathbb{D}$ with $A(\mathbb{D})$, the Hardy space of the unit disc with

$$H^2(\mathbb{D}) = \left\{ f \in A(\mathbb{D}) : \|f\|_{H^2(\mathbb{D})} = \sup_{r<1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^2 dr \right)^{1/2} < \infty \right\}.$$

For every function $f \in H^2(\mathbb{D})$ and for a.e. $t \in [-\pi, \pi)$, there exists the finite limit $f(e^{it}) := \lim_{r\to 1} f(re^{it})$. Moreover for the limit function holds that $f \in L^2(\mathbb{T})$, and $\|f\|_{H^2(\mathbb{D})} = \|f\|_{L^2(\mathbb{T})}$. The set of the limit functions of $H^2(\mathbb{D})$ is the Hardy space of the unit circle denoted by $H^2(\mathbb{T})$. The Malmquist–Takenaka system ([13, 20]) is an orthonormal system of rational functions, products of Blaschke factors, in the Hardy space of unit disc, which contains as special case the classical ”trigonometric” system. In system identification, it is frequently applied in order to approximate the transfer functions of the systems. Let us consider a sequence $a = (a_1, a_2, \ldots)$

of complex numbers, $a_n \in \mathbb{D}$ of the unit disc $\mathbb{D}$, and denote the Blaschke functions by

$$b_a(z) := \frac{z-a}{1-\overline{a}z} \qquad (a \in \mathbb{D},\ z \in \mathbb{C},\ 1-\overline{b}z \neq 0).$$

The Malmquist-Takenaka (MT) system $\Phi_n = \Phi_n^a$ $(n \in \mathbb{N}^*)$ is defined by

$$\Phi_1(z) = \frac{\sqrt{1-|a_1|^2}}{1-\overline{a_1}z}, \quad \Phi_n(z) = \frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}z}\prod_{k=1}^{n-1} b_{a_k}(z),\ n \geq 2. \tag{2.2}$$

When all parameters are equal, i.e., $a_n = a$, $n \in \mathbb{N}^*$, we obtain the so called discrete Laguerre system and particularly, when $a_n = 0$, $n \in \mathbb{N}^*$, we obtain the trigonometric system. Consequently, these systems can be viewed as extensions of the trigonometric system on the unit circle. These functions form an orthonormal system on the unit circle, i.e.,

$$\langle \Phi_n, \Phi_m \rangle = \frac{1}{2\pi}\int_0^{2\pi} \Phi_n(e^{it})\overline{\Phi_m(e^{it})}dt = \delta_{mn} \quad (m, n \in \mathbb{N}^*).$$

If the sequence $a = (a_1, a_2, \ldots)$ satisfies the non-Blaschke condition

$$\sum_{n\geq 1}(1-|a_n|) = +\infty, \tag{2.3}$$

then the corresponding MT system is complete in the Hardy space of the unit disc. Let us consider the orthogonal projection operator of order N of an arbitrary function $f \in H^2(\mathbb{T})$ with respect to the MT system:

$$P_N f(z) = \sum_{k=1}^{N} \langle f, \Phi_k \rangle \Phi_k(z). \tag{2.4}$$

For a special sequence $a = (a_1, a_2, \ldots)$, Pap proved in [15] that the analytic continuation in the unit disc of the projection $P_N f$ is at the same time a rational interpolation operator in the unit disc for the analytic continuation of f in the unit disc. In this paper, we show that this interpolation property is true in general for any sequence $a = (a_1, a_2, \ldots)$, with elements from $\mathbb{D}$, different from each other.

Theorem 2.1. *Let us consider a sequence $a = (a_1, a_2, \ldots)$, with elements from $\mathbb{D}$, different from each other ($a_k \neq a_j$, $k \neq j$). For every $f \in H^2(\mathbb{T})$, the projection operator $P_N f$ is a rational interpolation operator of type $(N-1, N)$ at the points $a_1, a_2, \ldots, a_N$ for the analytic continuation of f in the unit disc.*

Proof. In order to prove the interpolation property of $P_N f$, let us consider the kernel function of this projection operator:

$$K_N(z, \xi) = \sum_{k=1}^{N} \overline{\Phi_k(\xi)}\Phi_k(z). \tag{2.5}$$

According to the Christoffel-Darboux formula (see [12, 16, 2]), the kernel function can be written in closed form

$$K_N(z, \xi) = (1-z\overline{\xi})^{-1}\left(1 - \overline{\prod_{k=1}^{N}\frac{\xi-a_k}{1-\overline{a_k}\xi}}\prod_{k=1}^{N}\frac{z-a_k}{1-\overline{a_k}z}\right). \tag{2.6}$$

From this relation, it follows that the values of the kernel-function at the points $(a_m,\ m = 1,\dots,N)$ are equal to localized Cauchy kernels

$$K(a_m,\xi) = \frac{1}{1-a_m\overline{\xi}}.$$

From this property and the Cauchy integral formula, we get that the interpolation property holds, i.e.,

$$P_N f(a_m) = \langle f, K_N(.,a_m)\rangle = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{it})}{1-a_m e^{-it}}dt = f(a_m) \qquad (m = 1,\dots,N).$$

$\square$

For special choice of $a = (a_1, a_2, \dots)$, $a_i \in \mathbb{D}, i \in \{1,\dots,N\}$ (in Pap [15]), it has been shown that the coefficients of the projection operator $P_N f$ can be computed exactly if we know f at $a_1, a_2, \dots, a_N$. We show that this algorithm can be extended in general, when we can measure f at $a_1, a_2, \dots, a_N \in \mathbb{D}$ with $a_i \neq a_j,\ i \neq j,\ i, j, \in \{1,\dots,N\}$. Consequently, $P_N f$ can be written exactly if we know the values of $f(a_i)$. We present here the steps of the algorithm.

1. Step: For $k = 1,\dots,N$, we write the partial fraction decomposition of Φ_k:

$$\Phi_k(\xi) = \sum_{k'=1}^{k} c_{kk'}\frac{1}{1-\overline{a_{k'}}\xi}.$$

Using the orthonormality of the functions $\{\Phi_{k'},\ k' = 1,\dots,k\}$ and the Cauchy formula, we get that

$$\delta_{kn} = \langle \Phi_n, \Phi_k\rangle = \sum_{k'=1}^{k} \overline{c_{kk'}}\Phi_n(a_{k'}), \quad (n = 1,\dots,k).$$

If we order these equality's so that we write first the relations for $n = k$ then for $n = k-1$ etc., this is equivalent to

$$\begin{pmatrix} 1\\0\\0\\ \cdot\\ \cdot\\ \cdot\\0\end{pmatrix} = \begin{pmatrix} \Phi_k(a_k) & 0 & 0 & 0 & \dots & 0\\ \Phi_{k-1}(a_k) & \Phi_{k-1}(a_{k-1}) & 0 & 0 & \dots & 0\\ \Phi_{k-2}(a_k) & \Phi_{k-2}(a_{k-1}) & \Phi_{k-2}(a_{k-2}) & 0 & \dots & 0\\ \vdots & & & \vdots & & \\ \Phi_1(a_k) & \Phi_1(a_{k-1}) & \Phi_1(a_{k-2}) & \dots & & \Phi_1(a_1)\end{pmatrix} \begin{pmatrix} \overline{c_{kk}}\\ \overline{c_{kk-1}}\\ \overline{c_{kk-2}}\\ \vdots\\ \overline{c_{k1}}\end{pmatrix}.$$

2. Step: We solve the previous system of equations. Because of the elements from the main diagonal are different from zero, this system has a unique solution

$$(\overline{c_{kk}}, \overline{c_{kk-1}}, \overline{c_{kk-2}}, \dots, \overline{c_{k1}})^T.$$

3. Step: For $k = 1,\dots,N$, we determine the vectors $(\overline{c_{kk}}, \overline{c_{kk-1}}, \overline{c_{kk-2}}, \dots, \overline{c_{k1}})^T$, then based on Cauchy formula, we can compute the exact value of $\langle f, \Phi_k\rangle$ knowing the values of f on the set $a_1,\dots,a_N$. Indeed, using again the partial fraction decomposition of ψ_k and the Cauchy integral formula, we get that

$$\begin{aligned}\langle f, \Phi_k\rangle &= \sum_{k'=1}^{k} \overline{c_{kk'}}\left\langle f(\xi), \frac{1}{1-\overline{a_{k'}}\xi}\right\rangle\\ &= \sum_{k'=1}^{k} \overline{c_{kk'}} f(a_{k'}).\end{aligned}$$

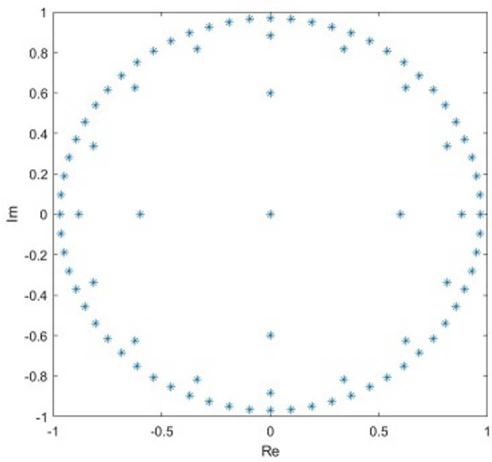

FIGURE 1. The first 85 elements of the sequence a

4. Step: We write

$$P_N f(z) = \sum_{k=1}^{N} \langle f, \Phi_k \rangle \Phi_k(z),$$

which is in the same time a projection operator and a rational interpolation operator of type $(N-1, N)$ at the points $a_1, a_2, \ldots, a_N$. A Matlab code was developed for the interpolation process on the unit disc (see **code**). In the code, we defined the sequence $a = (a_1, a_2, \ldots)$ as it is given in [15], in equations (2.4), (2.6) and (2.7), where the points of the sequence form concentric circles. For $k = 3$, we get the first 85 elements of the sequence (see on Figure 1). We apply the Steps 1–4 mentioned above to create $P_n f$ for the function

$$f(z) = \frac{1}{2 - z^2}.$$

We plot the function f and the projection operator $P_N f$ at the points $z_i = a_i$. As one can see on Figure 2, the values of the function and the projection operator are equal at these points, as it was stated in Theorem 2.1. In general, it is a hard task to study the convergence properties of an interpolation operator. In this case using that $P_N f$ is at the same time projection operator, we can derive more easily convergence results. The properties of orthogonal projection $P_N f$ on the unit circle were studied by Malmquist and Takenaka [13, 20]. If the sequence a is non-Blaschke sequence, i.e., $\sum_{n=0}^{\infty}(1 - |a_n|) = \infty$, then the Malmquist-Takenaka system is complete in the $H^p(\mathbb{T})$ for $0 < p < \infty$ (it follows from K. Hoffman, (1962, pp. 64) [10], J. B. Garnett, (1981, pp. 53) [9] and Z. Szabó [19, 18]), and $P_N f$ converge to f in norm on the circle and the convergence is compactly uniform on the disc for every $f \in H^2(\mathbb{D})$.

2.2. **Rational interpolation with nodes on the upper half-plane related to the Malmquist-Takenaka systems on the upper half-plane.** Let us denote the upper half-plane with $\mathbb{C}_+$, $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$. Let us denote the set of analytic functions over $\mathbb{C}_+$ with $A(\mathbb{C}_+)$, respectively,

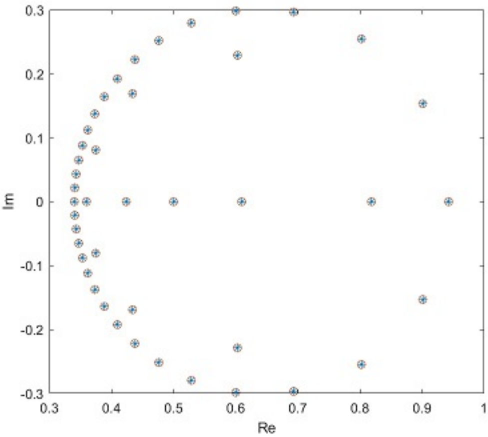

FIGURE 2. The interpolated function f (star) and the interpolation operator $P_N f$ (circle) at $z_i = a_i$

and consider the Hardy space of the upper half-plane

$$H^2(\mathbb{C}_+) = \left\{ f \in A(\mathbb{C}_+) : \|f\|_{H^2(\mathbb{C}_+)} = \sup_{0<y} \left(\int |f(x+iy)|^2 dx \right)^{1/2} < \infty \right\}.$$

If $f \in H^2(\mathbb{C}_+)$, for a.e. $x \in \mathbb{R}$ there exist the finite limit $f(x) := \lim_{y \to 0_+} f(x+iy)$, the limit function of f satisfies the following conditions $f \in L^2(\mathbb{R})$ and $\|f\|_{L^2(\mathbb{R})} = \|f\|_{H^2(\mathbb{C}_+)}$. The set of limit functions is the Hardy space of the real line denoted by $H^2(\mathbb{R})$. The Hardy space of the upper half-plane and the Hardy space of the unit disc $\mathrm{H}^2(\mathbb{D})$ may be connected through the Cayley transform. The conformal mapping from $\mathbb{C}_+$ to $\mathbb{D}$ defined by

$$C(\omega) = \frac{i-\omega}{i+\omega} \qquad (\omega \in \mathbb{C}_+) \tag{2.7}$$

is called Cayley transform and it extends continuously as a bijective mapping from the extended real line to $\mathbb{T}$. With the Cayley transform, the linear transformation from $\mathrm{H}^2(\mathbb{D})$ to $\mathrm{H}^2(\mathbb{C}_+)$ defined for $f \in \mathrm{H}^2(\mathbb{D})$ by

$$Tf(z) := \frac{1}{\sqrt{\pi}} \frac{1}{i+z} (f \circ C)(z) \tag{2.8}$$

is an isomorphism. Consequently, the theory of the real line is a close analogy with what we have for the circle. Using the Caley transform given by (2.7) and (2.8), we can make the transition of MT system to the upper half-plane. The system

$$\Psi_n(z) := c_n(T\Phi_n)(z) = (Tf)(z) := c_n \frac{1}{\sqrt{\pi}} \frac{1}{i+z} \Phi_n(C(z)) \qquad (\Im z \geq 0,\ n \in \mathbb{N}^*)$$

is the analogue of the Malmquist-Takeneka system for the upper half-plane. It is easy to check that for $a \in \mathbb{D}$ with $a^* := 1/\overline{a}$,

$$\lambda_a := C^{-1}(a) = i\frac{1-a}{1+a} \in \mathbb{C}_+, \quad \lambda_{a^*} = \overline{\lambda}_a, \quad \frac{\sqrt{1-|a|^2}}{|1+\overline{a}|} = \sqrt{\Im \lambda_a}, \tag{2.9}$$

and

$$\tilde{b}_a(z) = b_a(-1)\frac{z-\lambda_a}{z-\overline{\lambda}_a}, \quad \tilde{r}_a(z) = r_a(-1)\frac{z+i}{z-\overline{\lambda}_a} \quad (z \in \overline{\mathbb{C}}_+). \tag{2.10}$$

This implies that the functions $\Psi_n = c_n T\Phi_n$ $(n \in \mathbb{N}^*)$, $c_n = \frac{\sqrt{\Im\lambda_{a_n}}}{\Phi_n(-1)}$ are of the form

$$\Psi_1(z) = \frac{1}{\sqrt{\pi}}\frac{\sqrt{\Im\lambda_{a_1}}}{z-\overline{\lambda}_{a_1}}, \quad \Psi_n(z) = \frac{1}{\sqrt{\pi}}\frac{\sqrt{\Im\lambda_{a_n}}}{z-\overline{\lambda}_{a_n}}\prod_{k=1}^{n-1}\frac{z-\lambda_{a_k}}{z-\overline{\lambda}_{a_k}}. \tag{2.11}$$

The system of functions $\{\Psi_n\}_{n=1}^{\infty}$ is orthonormal on the entire axis in the following sense

$$\int_{-\infty}^{+\infty}\Psi_n(t)\overline{\Psi_m(t)}dt = \delta_{mn}. \tag{2.12}$$

Moreover, if the following non-Blaschke condition for the upper half-plane is satisfied

$$\sum_{k=1}^{\infty}\frac{\Im\lambda_{a_k}}{1+|\lambda_{a_k}|^2} = \infty,$$

then $(\Psi_n,\, n \in \mathbb{N}^*)$ is a complete orthonormal system for $H^2(\mathbb{C}_+)$. Let us consider the orthogonal projection operator of order N of an arbitrary function $f \in \mathrm{H}^2(\mathbb{C}_+)$ with respect to $\mathbf{\Psi}_N = \{\Psi_n, n = 1, 2, \cdots, N\}$ given by

$$Q_N f(z) = \sum_{k=1}^{N}\langle f, \Psi_k\rangle\Psi_k(z). \tag{2.13}$$

Let us consider the kernel function of this projection operator

$$\widetilde{K}_N(\omega, w) = \sum_{k=1}^{N}\overline{\Psi_k(w)}\Psi_k(\omega).$$

Then the projection operator can be expressed as a scalar product:

$$Q_N f(z) = \int\limits_{-\infty}^{\infty} f(t)\widetilde{K}_N(z,t)dt = \langle f(.), \widetilde{K}_N(., z)\rangle. \tag{2.14}$$

According to [3], the kernel function can be written in the following form:

$$\widetilde{K}(\omega, w)_N = \sum_{k=1}^{N}\overline{\Psi_k(w)}\Psi_k(\omega) = \frac{1-\overline{\widetilde{B}_N(w)}\widetilde{B}_N(\omega)}{2i\pi(\overline{w}-\omega)}, \quad \omega \neq \overline{w},$$

where

$$\widetilde{B}_N(\omega) = \prod_{k=1}^{N}\frac{\omega-\lambda_{a_k}}{\omega-\overline{\lambda_{a_k}}}\tau_k, \quad \tau_k = \frac{|1+\lambda_{a_k}^2|}{1+\lambda_{a_k}^2}$$

is the Blaschke product on the upper half-plane. Eisner and Pap [4] proved the following interpolation property of the projection operator:

Theorem 2.2 (Eisner, Pap [4]). *For any $f \in \mathrm{H}^2(\mathbb{C}_+)$, the projection operator $Q_N f$ is an interpolation operator of type $(N-1, N)$ on the set $\{\lambda_{a_k},\ j, k = 1, \ldots, N\}$, $\lambda_{a_k} \neq \lambda_{a_j}, k \neq j$, i.e.*

$$Q_N f(\lambda_{a_k}) = f(\lambda_{a_k}) \qquad (k = 1, \ldots, N).$$

If condition (2.2) is satisfied, $\{\Psi_k,\ k=1,\dots\infty\}$ is a complete orthonormal set in the Hilbert space $\mathrm{H}^2(\mathbb{C}_+)$ and we have $\|f-Q_Nf\|_{\mathrm{H}^2(\mathbb{C}_+)}\to 0$ as $N\to\infty$. Since convergence in $\mathrm{H}^2(\mathbb{C}_+)$ implies uniform convergence to the analytic continuation of f on the upper half-plane on every compact subset, we conclude that $Q_Nf\to f$ uniformly on every compact subset of the upper half-plane. For $\lambda_a=(\lambda_{a_1},\lambda_{a_2},\dots)$, $\lambda_{a_n}\in\mathbb{C}_+$ and $\lambda_{a_n}\neq\lambda_{a_k}$, $n\neq k$, we show that the coefficients of the projection operator Q_Nf can be computed exactly if we know f in $\lambda_{a_1},\lambda_{a_2},\dots$. Consequently, Q_Nf can be written exactly if we know the values of $f(\lambda_{a_i})$. We present here the steps of the algorithm.

1. Step: For $k=1,\dots,N$, we write the partial fraction decomposition of Φ_k:

$$\Psi_k(\xi)=\sum_{k'=1}^{k} b_{kk'}\frac{1}{\xi-\overline{\lambda_{a_{k'}}}}.$$

Using the orthonormality of the functions $\{\Psi_{k'},\ k'=1,\dots,k\}$ and the Cauchy formula, we get that

$$\delta_{kn}=\langle\Psi_n,\Psi_k\rangle=\sum_{k'=1}^{k}\overline{b_{kk'}}\Psi_n(\lambda_{a_{k'}})\qquad(n=1,\dots,k).$$

If we order these equality's so that we write first the relations for $n=k$ then for $n=k-1$ etc., this is equivalent to

$$\begin{pmatrix}1\\0\\0\\.\\.\\.\\0\end{pmatrix}=\begin{pmatrix}\Psi_k(\lambda_{a_k}) & 0 & 0 & 0 & \dots & 0\\ \Psi_{k-1}(\lambda_{a_k}) & \Psi_{k-1}(\lambda_{a_{k-1}}) & 0 & 0 & \dots & 0\\ \Psi_{k-2}(\lambda_{a_k}) & \Psi_{k-2}(\lambda_{a_{k-1}}) & \Psi_{k-2}(\lambda_{a_{k-2}}) & 0 & \dots & 0\\ \vdots & & & \vdots & & \\ \Psi_1(\lambda_{a_k}) & \Psi_1(\lambda_{a_{k-1}}) & \Psi_1(\lambda_{a_{k-2}}) & \dots & & \Psi_1(\lambda_{a_1})\end{pmatrix}\begin{pmatrix}\overline{b_{kk}}\\ \overline{b_{kk-1}}\\ \overline{b_{kk-2}}\\ \vdots\\ \overline{b_{k1}}\end{pmatrix}.$$

2. Step: We solve the previous system of equations. Because of the elements from the main diagonal are different from zero, this system has a unique solution

$$(\overline{b_{kk}},\overline{b_{kk-1}},\overline{b_{kk-2}},\dots,\overline{b_{k1}})^T.$$

3. Step: If we determine the vector $(\overline{b_{kk}},\overline{b_{kk-1}},\overline{b_{kk-2}},\dots,\overline{b_{k1}})^T$, then based on Cauchy formula, we can compute the exact value of $\langle f,\Psi_k\rangle$ knowing the values of f on the set $\lambda_{a_1},\dots,\lambda_{a_n}$. Indeed, using again the partial fraction decomposition of Ψ_k and the Cauchy integral formula for upper half-plane, we get that

$$\begin{aligned}\langle f,\Psi_k\rangle&=\sum_{k'=1}^{k}\overline{b_{kk'}}\left\langle f(\omega),\frac{1}{\omega-\overline{\lambda_{a_{k'}}}}\right\rangle\\&=\sum_{k'=1}^{k}\overline{b_{kk'}}f(\lambda_{a_{k'}}).\end{aligned}$$

4. Step: We write

$$Q_Nf(z)=\sum_{k=1}^{N}\langle f,\Psi_k\rangle\Psi_k(z),$$

which is in the same time a projection operator and a rational interpolation operator of type $(N-1,N)$ at the points $\lambda_{a_1},\lambda_{a_2},\dots,\lambda_{a_n}$. We also developed a Matlab code for the interpolation process on the upper half-plane (see **code**). In the code, we use the sequence $a=(a_1,a_2,\dots)$ as it is given in [15], in equations (2.4), (2.6) and (2.7), and we defined the $\lambda_a=(\lambda_{a_1},\lambda_{a_2},\dots)$

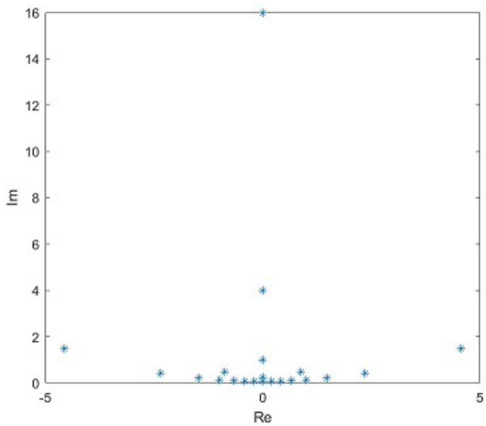

FIGURE 3. The first 21 elements of the sequence λ_a

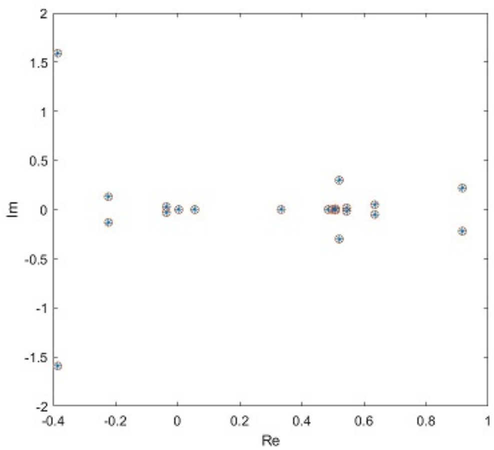

FIGURE 4. The interpolated function f (star) and the interpolation operator $Q_N f$ (circle) at $z_i = \lambda_{a_i}$

sequence with Cayley transformation, see in (2.9). On Figure 3, the first 21 elements of the λ_a sequence can be seen. Following Steps 1–4 mentioned above, we create $Q_N f$ for the function

$$f(z) = \frac{1}{2 - z^2}.$$

Representing the function f and the projection operator $Q_N f$ at the points $z_i = \lambda_{a_i}$, we can see that the values are equal at these points as it was stated in Theorem 2.2 (see Figure 4).

3. RATIONAL INTERPOLATION WITH SPECIAL NODES RELATED TO DISCRETE BIORTHOGONALITY OF MAMQUIST-TAKENAKA SYSTEMS

Discretization results connected to MT systems for unit disc and the upper half-plane were published in [16, 17, 4, 8]. Based on these results, an analogue of discrete Fourier transform

(DFT) was developed and the discrete versions was applied successfully for compression and representation of human ECG signals [5, 6]. In paper [7], Fridli and Schipp introduced the dual of the Malmquist-Takenaka system on the unit disc and proved discrete biorthogonal property on a set of points of the unit disc. Nagy-Csiha and Pap recently introduced the dual system of the Malmquist-Takenaka system on the upper half-plane and proved discrete biorthogonality result on a set of discretization points on upper half-plane [14].

In this section, using the discretization points as nodes on closed disc and on closed upper half-plane respectively, we introduce new rational interpolation operators and we study their properties.

3.1. **Rational interpolation based on the dual of the Mamquist-Takenaka system in the unit disc and discrete biorthogonality.** Let us denote by $z^* = 1/\overline{z}$. Let $\mathcal{Q}$ denote the set of rational functions. For any $f \in \mathcal{Q}$, the domain will be extended to $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by $f(a) = \infty$ if a is a pole of f and $f(\infty) := \lim_{z\to\infty} f(z)$. Let us consider the following two types of inversions:

$$f^*(z) := (f(z))^*, \quad f^\star(z) := f(z^*) \quad (z \in \overline{\mathbb{C}}, f \in \mathcal{Q}).$$

It is obvious that for any $z \in \mathbb{T}$, we have

$$z = z^*. \quad f^*(z) = f^\star(z) = f(z) \quad (f \in \mathcal{Q}).$$

Moreover, in case of Blaschke-products $B_N(z) = \prod_{k=1}^N b_{a_k}(z)$, the operations coincide:

$$B_N^*(z) = B_N^\star(z) = B_N(z^*) \quad (z \in \overline{\mathbb{C}}).$$

Let us consider the following functions:

$$\begin{aligned} &\Phi_1^\star = \overline{z}\frac{\sqrt{1-|a_1|^2}}{\overline{z}-\overline{a}_1} = r_{a_1}^\star(z), \\ (3.15) \qquad &\Phi_n^\star = \Phi_n(z^*) = \overline{z}\frac{\sqrt{1-|a_n|^2}}{\overline{z}-\overline{a}_n}\prod_{k=1}^{n-1}\frac{1-a_k\overline{z}}{\overline{z}-\overline{a}_k} = r_{a_n}^\star(z)\prod_{k=1}^{n-1} b_{a_k}^\star(z) \quad (n\in\mathbb{N}^*), z\in\mathbb{C}\setminus\mathbb{D}. \end{aligned}$$

The system $\mathbf{\Phi}^\star := ((\Phi_n)^\star, n \in \mathbb{N}^*)$ is called the dual of the MT system $\mathbf{\Phi} = (\Phi_n, n \in \mathbb{N}^*)$. If $z \in \mathbb{T}$, then $\Phi_n^\star = \Phi_n, n \in \mathbb{N}^*$. If $|u| \le 1$, it is easy to see that the equation $B_N(z) = u$ has exactly N solutions in the closed unit disc counting with multiplicities. In particular, if $u \in \mathbb{T}$, then all of the roots are of multiplicity one and they are on the unit circle. If $|u| \ge 1$, then $|u^*| \le 1$. In that case $B_N(z) = u$ if and only if $B_N^*(z) = u^*$. But $B_N^*(z) = B_N(z^*)$, which implies that the equation $B_N(z) = u$ has N solutions outside of the open unit disc. In the following, we will consider an $u \in \overline{\mathbb{D}}$ for which the equation has N distinct roots. Let us introduce the set:

$$\mathcal{Z}_{N,u}^{\mathbf{a}} := \{z \in \mathbb{C} : B_N(z) = u,\ (B_N)'(z) \neq 0\} \quad (0 < |u| \le 1).$$

If it has N different elements, denote the elements by z_k and $\mathcal{Z}_{N,u}^{\mathbf{a}} = \{z_k,\ k = 1, \ldots, N\}$. We recall Theorem 2.1. of Fridli and Schipp in [7]. It is easy to verify (see the proof in [7]) that the following theorem holds not just for $0 < |u| \le 1$ as it is mentioned in [7], but for $u \in \mathbb{C} \setminus \{0\}$.

Theorem 3.3 (Fridli, Schipp [7]). *Let $0 < |u| \le 1$ be a parameter for which the set $\mathcal{Z}_{N,u}^{\mathbf{a}}$ has N different elements. Then the $\Phi_n, \Phi_n^\star$ $(1 \le n \le N)$ systems are biorthogonal with respect to the following discrete scalar product*

$$[\Phi_n, \Phi_m^\star]_{\mathbf{a},u} := \sum_{z\in\mathcal{Z}_{N,u}^{\mathbf{a}}} \Phi_n(z)\overline{\Phi_m^\star(z)}/K_N(z, z^*) = \delta_{mn} \quad (1 \le m, n \le N),$$

where $K_N(z,z^*)$ is the Dirichlet kernel,

$$K_N(z,z^*) = \sum_{k=1}^{N} \Phi_k(z)\overline{\Phi_k(z^*)} = \sum_{k=1}^{N} \frac{(1-|a_k|^2)z}{(1-\overline{a_k}z)(z-a_k)}.$$

On the unit circle, the MT system is orthonormal with respect to the continuous measure, i.e.

$$\int_{\mathbb{T}} \Phi_n(z)\overline{\Phi_m(z)}dz = \frac{1}{2\pi}\int_0^{2\pi} \Phi_n(e^{it})\overline{\Phi_m(e^{it})}dt = \delta_{mn}.$$

From the definition of the dual system, it follows that the original system and the dual system are equal on the unite circle $\mathbb{T}$, i.e., if $z \in \mathbb{T}$, then $\Phi_n^\star = \Phi_n, n \in \mathbb{N}$. As a consequence, on the unit circle the original system and the dual system are biorthogonal with respect to the scalar product generated by the continuous measure:

$$\int_{\mathbb{T}} \Phi_n(z)\overline{\Phi_m^\star(z)}dz = \frac{1}{2\pi}\int_0^{2\pi} \Phi_n(e^{it})\overline{\Phi_m^\star(e^{it})}dt = \delta_{mn}.$$

The continuous projection operators connected to the MT and dual MT system for $f \in H^2(\mathbb{D})$ are the following:

$$P_N f(z) = \sum_{k=1}^{N} \langle f, \Phi_k^\star\rangle \Phi_k(z),\ z \in \overline{\mathbb{D}},\ f \in H^2(\mathbb{D}),$$

$$P_N^\bullet f(z) = \sum_{k=1}^{N} \langle f, \Phi_k\rangle \Phi_k^\star(z),\ z \in \mathbb{C}\setminus\mathbb{D},\ f \in H^2(\mathbb{C}\setminus\mathbb{D}),$$

where

$$\langle f, \Phi_k\rangle = \int_{\mathbb{T}} f(z)\overline{\Phi_k(z)}dz = \int_{\mathbb{T}} f(z)\overline{\Phi_k^\star(z)}dz = \langle f, \Phi_k^\star\rangle.$$

Taking into account, that on the circle $\Phi_k = \Phi_k^\star$, the projection $P_N f$ is the same projection which was studied in the previous section and the projections are related to each other in the following way:

$$P_N^\bullet f(z) = \sum_{k=1}^{N} \langle f, \Phi_k^\star\rangle \Phi_k^\star(z) = \sum_{k=1}^{N} \langle f, \Phi_k^\star\rangle \Phi_k(z^*) = P_N f(z^*),\ z \in \mathbb{C}\setminus\mathbb{D}.$$

For $z \in \mathbb{T}$, the two projection operator are the same: $P_N^\bullet f(z) = P_N f(z)$. Consequently, it is enough to study the properties of $P_N f(z), z \in \overline{\mathbb{D}}$. In the previous section, we proved that $P_N f(z)$ is a rational interpolant of type $(N-1,N)$ of f in a_k, $k = 1,\dots,N$. Then $P_N^\bullet f(z)$ will interpolate the analytic continuation of f (if this exists) outside of the disc in a_k^*, $k = 1,\dots,N$. In analog way, we can consider the discrete projection operator associated to the discrete scalar product denoted by $P_N^\circ f$, expressed as follows:

$$P_N^\circ f(z) = \sum_{k=1}^{N} [f, \Phi_k^\star]_{\mathbf{a},u} \Phi_k(z),$$

where the coefficients are expressed by the discrete scalar product as follows

$$[f, \Phi_k^\star]_{\mathbf{a},u} = \sum_{z_j \in \mathcal{Z}_{N,u}^{\mathbf{a}}} \frac{f(z_j)\overline{\Phi_k^\star(z_j)}}{K_N(z_j, z_j^*)}.$$

The question naturally arises, weather $P_N^\circ f$ is an interpolation operator or not. In what follows, we will study the properties of this discrete projection operator. In [19], Szabó studied the properties of $P_N^\circ f$ for $u = 1$ and $z \in \mathbb{T}$. For this special case, he proved that this projection operator is also an interpolation operator on the set of discretazation points

$$\mathcal{Z}_{N,1}^{\mathbf{a}} := \{z \in \mathbb{C} : B_N(z) = 1\},$$

which are on the the unit circle. In this paper, we will extend the results obtained by Szabó for $P_N^\circ f(z)$ when $u \in \overline{\mathbb{D}} \setminus \{0\},\ z \in \overline{\mathbb{D}}$ and the discretization point are in the closed unit disc.

Theorem 3.4. *Let $0 < |u| \leq 1$ be a parameter for which the set $\mathcal{Z}_{N,u}^{\mathbf{a}}$ has N different elements and $P_N^\circ f$ defined as before. For every $f \in C(\overline{\mathbb{D}})$, the projection operator $P_N^\circ f$ is a Lagrange type rational interpolation operator of type $(N-1, N)$ at $z_k \in \mathcal{Z}_{N,u}^{\mathbf{a}}$, i.e.,*

$$P_N^\circ f(z_k) = f(z_k), \ \ z_k \in \mathcal{Z}_{N,u}^{\mathbf{a}}.$$

Proof. Let us consider the kernel function of the discrete projection operator:

$$K_N^\circ(z,\xi) := \overline{\sum_{k=1}^{N} \overline{\Phi_k^\star(\xi)}\Phi_k(z)} = \overline{\sum_{k=1}^{N} \overline{\Phi_k(\xi^*)}\Phi_k(z)} = \overline{K_N(z,\xi^*)}. \tag{3.16}$$

The discrete projection can be expressed using $K_N^\circ(z,\xi)$ and the discrete scalar product more explicitly:

$$P_N^\circ f(z) = [f(.), K_N^\circ(z,\cdot)]_{\mathbf{a},u} = \sum_{z_j \in \mathcal{Z}_{N,u}^{\mathbf{a}}} \frac{K_N(z,z_j^*)}{K_N(z_j,z_j^*)} f(z_j).$$

Let us consider

$$\ell_{N,\xi}(z) = \frac{K_N(z,\xi^*)}{K_N(\xi,\xi^*)}.$$

From the definition, it follows that $\ell_{N,\xi}(\xi) = 1$. According to the Christoffel-Darboux formula for $z \neq \xi$, the kernel function can be written in closed form

$$K_N(z,\xi^*) = (1 - z\overline{\xi^*})^{-1}\left(1 - \overline{B_N(\xi^*)}B_N(z)\right). \tag{3.17}$$

$\ell_{N,\xi}(z)$ is a rational function in z of type $(N-1, N)$. Because of $B_N(z^*) = B_N^*(z) = 1/\overline{B_N(z)}$,

$$\ell_{N,\xi}(z) = \frac{1 - \frac{B_N^a(z)}{B_N^a(\xi)}}{K_N(\xi,\xi^*)(1 - z\overline{\xi^*})}.$$

From here and the definition of $\ell_{N,\xi}(z)$, we get that for $z_j, z_k \in \mathcal{Z}_{N,u}^{\mathbf{a}},\ z_j \neq z_k$ we have $\ell_{N,z_j}(z_k) = \delta_{jk}$, so these functions, behave like the Lagrange interpolation polynomials. Consequently, $P_N^\circ f$ has the following interpolation property

$$P_N^\circ f(z_k) = \sum_{z_j \in \mathcal{Z}_{N,u}^{\mathbf{a}}} \frac{K_N(z_k,z_j^*)}{K_N(z_j,z_j^*)} f(z_j) = f(z_k), \ \ z_k \in \mathcal{Z}_{N,u}^{\mathbf{a}}.$$

□

We consider ℓ_{N,z^*}, the dual of $\ell_{N,z}$. For these functions, we can prove the following orthogonality properties.

Theorem 3.5. *Let $0 < |u| \leq 1$ be a parameter for which the set $\mathcal{Z}_{N,u}^{\mathbf{a}}$ has N different elements. For $z_j, z_m \in \mathcal{Z}_{N,u}^{\mathbf{a}}$, the functions ℓ_{N,z^*}, and $\ell_{N,z}$ satisfy the following biorthogonality relation*

$$\frac{1}{2\pi}\int_0^{2\pi} \ell_{N,z_j}(e^{it})\overline{\ell_{N,z_m^*}(e^{it})}dt = \frac{1}{K_N(z_m,z_m^*)}\delta_{mj}.$$

In addition, $\ell_{N,z}$ satisfies the following discrete orthogonality relation:

$$[\ell_{N,z_n}, \ell_{N,z_m}]_{\mathbf{a},u} = \delta_{nm} \frac{1}{K_N^{\mathbf{a}}(z_n, z_n^*)}.$$

Proof.

$$\begin{aligned}
\langle \ell_{N,z_j}, \ell_{N,z_m^*} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} \ell_{N,z_j}(e^{it}) \overline{\ell_{N,z_m^*}(e^{it})} dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{K_N(e^{it}, z_j^*)}{K_N(z_j, z_j^*)} \overline{\left(\frac{K_N(e^{it}, z_m)}{K_N(z_m^*, z_m)} \right)} dt \\
&= \frac{1}{2\pi} \frac{1}{K_N(z_j, z_j^*) K_N(z_m, z_m^*)} \int_0^{2\pi} \sum_{k=1}^N \Phi_k(e^{it}) \overline{\Phi_k(z_j^*)} \overline{\sum_{k'=1}^N \Phi_{k'}(e^{it}) \overline{\Phi_{k'}(z_m)}} dt \\
&= \frac{1}{K_N(z_j, z_j^*) K_N(z_m, z_m^*)} \sum_{k=1}^N \sum_{k'=1}^N \frac{1}{2\pi} \int_0^{2\pi} \Phi_k(e^{it}) \overline{\Phi_k(z_j^*)} \overline{\Phi_{k'}(e^{it}) \overline{\Phi_{k'}(z_m)}} dt \\
&= \frac{1}{K_N(z_j, z_j^*) K_N(z_m, z_m^*)} \sum_{k=1}^N \sum_{k'=1}^N \overline{\Phi_k(z_j^*)} \Phi_{k'}(z_m) \frac{1}{2\pi} \int_0^{2\pi} \Phi_k(e^{it}) \overline{\Phi_{k'}(e^{it})} dt \\
&= \frac{1}{K_N(z_j, z_j^*) K_N(z_m, z_m^*)} \sum_{k=1}^N \sum_{k'=1}^N \overline{\Phi_k(z_j^*)} \Phi_{k'}(z_m) \delta_{kk'} \\
&= \frac{1}{K_N(z_j, z_j^*) K_N(z_m, z_m^*)} \sum_{k=1}^N \overline{\Phi_k(z_j^*)} \Phi_k(z_m) \\
&= \frac{1}{K_N(z_j, z_j^*) K_N(z_m, z_m^*)} K_N(z_m, z_j^*).
\end{aligned}$$

From here if $z_m = z_j$, then $\langle \ell_{N,z_m}, \ell_{N,z_m^*} \rangle = \frac{1}{K_N(z_m, z_m^*)}$. If $z_m \neq z_j$, then

$$K_N(z_m, z_j^*) = \frac{1 - B_N(z_m)\overline{B_N(z_j^*)}}{1 - z_m \overline{z_j^*}} = \frac{1 - B_N(z_m)\overline{B_N^*(z_j)}}{1 - z_m \overline{z_j^*}} = \frac{1 - u\overline{u^*}}{1 - z_m \overline{z_j^*}} = 0.$$

We get that for every $u \in \overline{\mathbb{D}} \setminus \{0\}$

$$\frac{1}{2\pi} \int_0^{2\pi} \ell_{N,z_j}(e^{it}) \overline{\ell_{N,z_m^*}(e^{it})} dt = \frac{1}{K_N(z_m, z_m^*)} \delta_{mj}.$$

If $u \in \mathbb{T}$, then $z_j \in \mathbb{T}$ for every $j = 1, \ldots, N$, consequently $\ell_{N,z_j} = \ell_{N,z_j^*}$. In this case, we have that the system $\{\ell_{N,z_j},\, j=1,\ldots,N\}$ is orthogonal. If specially $u = 1$, then we get the result of Szabó [19]. The discrete orhtogonality of the system $\{\ell_{N,z_j},\ j = 1, \ldots, N\}$ is true for every $u \in \overline{\mathbb{D}} \setminus \{0\}$. Indeed

$$\begin{aligned}
[\ell_{N,z_n}, \ell_{N,z_m}]_{\mathbf{a},u} &= \sum_{z_j \in \mathcal{Z}_{N,u}^{\mathbf{a}}} \ell_{N,z_n}(z_j) \overline{\ell_{N,z_m}(z_j)} \frac{1}{K_N^{\mathbf{a}}(z_j, z_j^*)} \\
&= \sum_{z_j \in \mathcal{Z}_{N,u}^{\mathbf{a}}} \delta_{nj} \delta_{mj} \frac{1}{K_N^{\mathbf{a}}(z_j, z_j^*)} \\
&= \delta_{nm} \frac{1}{K_N^{\mathbf{a}}(z_n, z_n^*)}.
\end{aligned}$$

□

Beside the interpolation property, $P_N f$ and $P_N^\circ f$ reconstruct f exactly in some cases if we measure the function in the N interpolation points. Let us denote by $\mathcal{P}_k$ the space of polynomials of degree at most k. Let us consider the polynomials of the following form: $\eta(z) = \prod_{n=1}^{N}(1 - z\overline{a}_n)$ and the set

$$\mathcal{R}_N := \left\{ \frac{p}{\eta} : \ p \in \mathcal{P}_{N-1} \right\}.$$

The system

$$\Phi_N^a = \{\Phi_n, n = 1, \ldots, N\}$$

forms an orthonormal basis in $\mathcal{R}_N$:

$$\mathcal{R}_N = \text{span}\{\Phi_l, l = 1, \ldots, N\}.$$

For every $f \in \mathcal{R}_N$, we have $f = P_N f = P_N^\circ f$. Indeed, if $f(z) = \sum\limits_{k=1}^{N} c_k \Phi_k(z)$, the continuous biorthogonality implies that

$$\langle f, \Phi_j^\star \rangle = \sum_{k=1}^{N} c_k \langle \Phi_k, \Phi_j^\star \rangle = \sum_{k=1}^{N} c_k \delta_{kj} = c_j,$$

from which we get that $f = P_N f$. Similarly, from discrete biorthogonality, we get

$$[f, \Phi_j^\star]_{\mathbf{a},u} = \sum_{k=1}^{N} c_k [\Phi_k, \Phi_j^\star]_{\mathbf{a},u} = \sum_{k=1}^{N} c_k \delta_{kj} = c_j,$$

which implies that $f = P_N^\circ f$.

3.2. **Rational interpolation based on the dual of the Mamquist-Takenaka system on the upper half-plane and discrete biorthogonality.** Recently Nagy-Csiha and Pap introduced the dual system for the Malmquist-Takenaka system on the upper half-plane. It was proved that these systems are also discrete biorthogonal with respect to the discrete inner product over a set of discratization points in closed upper half-plane (see [14]). In this subsection, we prove that on the discretisation nodes belonging to the closed upper half-plane, we can construct an interpolation operator of type $(N-1, N)$.

First, we introduce the notations and we present a short summary of the discrete biorthogonality of Malmquist-Takenaka and it's dual on the upper half-plane. We consider the isometric transform of the Malmquist-Takenaka and it's dual to the upper half-plane. With straightforward computation, it is easy to see that for

$$a_k = K(\lambda_k) = \frac{i - \lambda_k}{i + \lambda_k}, \quad \lambda_k \in \mathbb{C}_+, \quad k = 1, \ldots, \infty$$

the dual system of (2.11) is equal to

$$\widetilde{\Psi}_1^\lambda(z) := \frac{i + \overline{z}}{i + z} \frac{\frac{\sqrt{\Im \lambda_1}}{\sqrt{\pi}}}{\overline{z} - \overline{\lambda_1}} = \frac{i + \overline{z}}{i + z} \Psi_1^\lambda(\overline{z}),$$

$$\widetilde{\Psi}_n^\lambda(z) = \frac{i + \overline{z}}{i + z} \frac{\frac{\sqrt{\Im \lambda_n}}{\sqrt{\pi}}}{\overline{z} - \overline{\lambda_n}} \prod_{k=1}^{n-1} \frac{\overline{z} - \lambda_k}{\overline{z} - \overline{\lambda_k}} = \frac{i + \overline{z}}{i + z} \Psi_n^\lambda(\overline{z}).$$

For arbitrary values of the variables ω and w from $\mathbb{C}_+$ and for any N, $1 \leq N < \infty$, the kernel function corresponding to the system (2.11) and its dual can be written also in closed form as follows [3]:

$$\widetilde{K}_N(\omega, w) = \sum_{k=1}^{N} \Psi_k^\lambda(\omega)\overline{\widetilde{\Psi}_k^\lambda(w)} = \overline{\left(\frac{i+\overline{w}}{i+w}\right)} \sum_{k=1}^{N} \Psi_k^\lambda(\omega)\overline{\Psi_k^\lambda(\overline{w})} = \frac{w-i}{\overline{w}-i}\frac{1-\overline{\widetilde{B}_N(\overline{w})}\widetilde{B}_N(\omega)}{2i\pi(w-\omega)},\ \omega \neq w,$$

$$\widetilde{K}_N(\omega, \omega) = \sum_{k=1}^{N} \Psi_k^\lambda(\omega)\overline{\widetilde{\Psi}_k^\lambda(\omega)} =: \frac{1}{\widetilde{\rho}_N(\omega)} = \frac{w-i}{\overline{w}-i}\sum_{k=1}^{N}\frac{\Im\lambda_k}{\pi(\omega-\lambda_k)(\omega-\overline{\lambda}_k)}.$$

For $a_k = K(\lambda_k) = \frac{i-\lambda_k}{i+\lambda_k}$, we assume that the following equation has N different solutions denoted by z_k:

$$\frac{z-a_1}{1-\overline{a}_1 z}\frac{z-a_2}{1-\overline{a}_2 z}\cdots\frac{z-a_N}{1-\overline{a}_N z} = u,\ u \in \mathbb{D}\setminus\{0\}. \tag{3.18}$$

We present the analogue of Theorem 3.3 for the upper half-plane. Let us consider t_k, where $z_k = K(t_k) = \frac{i-t_k}{i+t_k}$ is the solution of the equation (3.18), and the following set of nodes on the closed upper half-plane

$$\mathbb{C}_N = \{t_k : k = 1, \ldots, N\}. \tag{3.19}$$

Let us denote by $\omega = K^{-1}(z) = i\frac{1-z}{1+z}$, $w = K^{-1}(\xi) = i\frac{1-\xi}{1+\xi}$, $a_k = K(\lambda_k) = \frac{i-\lambda_k}{i+\lambda_k}$, $z_k = K(t_k) = \frac{i-t_k}{i+t_k}$. Then

$$\overline{\left(\frac{i\frac{1-\xi}{1+\xi}-\lambda_k}{i\frac{1-\xi}{1+\xi}-\overline{\lambda_k}}\frac{|1+\lambda_k^2|}{1+\lambda_k^2}\right)}\frac{i\frac{1-z}{1+z}-\lambda_k}{i\frac{1-z}{1+z}-\overline{\lambda_k}}\frac{|1+\lambda_k^2|}{1+\lambda_k^2} = \overline{\left(\frac{\xi-a_k}{1-\overline{a_k}\xi}\right)}\frac{z-a_k}{1-\overline{a_k}z}. \tag{3.20}$$

According to (3.20) and the property $\overline{w} = K^{-1}(\xi^*)$, we get

$$\overline{\widetilde{B}_N(\overline{w})}\widetilde{B}_N(\omega) = \overline{B_N(\xi^*)}B_N(z).$$

From this and the definition of z_k, it follows that

$$\overline{\widetilde{B}_N(\overline{t}_j)}\widetilde{B}_N(t_i) = \overline{B_N(z_j^*)}B_N(z_i) = \frac{u}{u} = 1. \tag{3.21}$$

Consider the following discrete scalar product:

$$\langle F, G\rangle_N = \sum_{t\in\mathbb{C}_N} F(t)\overline{G(t)}\widetilde{\rho}_N(t).$$

Theorem 3.6 (Nagy-Csiha, Pap, [14]). *The finite collection of $\Psi_n^\lambda, (1 \leq n \leq N)$ and $\widetilde{\Psi}_n^\lambda, (0 \leq n \leq N)$ are discrete biorthogonal systems with respect to the scalar product*

$$\langle F, G\rangle_N = \sum_{t\in\mathbb{C}_N} F(t)\overline{G(t)}\widetilde{\rho}_N(t),$$

namely

$$\langle \Psi_m^\lambda, \widetilde{\Psi}_n^\lambda\rangle_N = \delta_{mn} \quad (1 \leq m, n \leq N).$$

For $\omega \in \mathbb{R}$, $\Psi_n^\lambda(\omega) = \widetilde{\Psi}_n^\lambda(\omega)$. If we choose in the proof of the theorem $u \in \mathbb{T}$, then the discretisation points are all real numbers, i.e., $t_k \in \mathbb{R}$, $k = 1, \ldots, N$, and from Theorem 3.6, we reobtain Theorem 2.2 of Eisner and Pap [4]. For the Hardy space of the upper half-plane, it is possible to introduce similar projection operators by using the biorthogonal systems $(\Psi_n, \widetilde{\Psi}_n, n \in \mathbb{N}^*)$.

They are also biorthogonal with respect to the continuous measure on the real line. Indeed, for $t \in \mathbb{R}$, we have that $\widetilde{\Psi}_k^\lambda(t) = \Psi_k^\lambda(t)$, consequently

$$\int_{-\infty}^{\infty} \Psi_n^\lambda(t)\overline{\widetilde{\Psi}_m^\lambda(t)}dt = \int_{-\infty}^{\infty} \Psi_n^\lambda(t)\overline{\Psi_m^\lambda(t)}dt = \delta_{mn}.$$

Similarly as in the case of unit disc, we can consider the following projection operators:

$$Q_N f(t) = \sum_{k=1}^{N} \langle f, \widetilde{\Psi}_k^\lambda\rangle \Psi_k^\lambda(t),\ t \in \overline{\mathbb{C}}_+,\ f \in H_2(\mathbb{C}_+),$$

$$Q_N^\bullet f(t) = \sum_{k=1}^{N} \langle f, \Psi_k^\lambda\rangle \widetilde{\Psi}_k^\lambda(t),\ t \in \mathbb{C} \setminus \mathbb{C}_+,\ f \in H_2(\mathbb{C} \setminus \mathbb{C}_+),$$

where

$$\langle f, \widetilde{\Psi}_k^\lambda\rangle = \langle f, \Psi_k^\lambda\rangle = \int_{-\infty}^{\infty} f(t)\overline{\Psi_k^\lambda(t)}dt. \tag{3.22}$$

If $t \in \mathbb{R}$, then the two projection operators are the same, $Q_N f(t) = Q_N^\bullet f(t)$, and for $t \in \mathbb{C}_+$ we have $\frac{i+t}{i+\overline{t}} Q_N f(t) = Q_N^\bullet f(\overline{t})$. In the previous section, we saw that $Q_N f(z)$ is a rational interpolant of type $(N-1, N)$ of f in λ_k, $k = 1, \dots, N$. With $Q_N^\bullet f(z)$, we can construct interpolation for the analytic continuation of f (if this exists) outside of the disc with nodes $\overline{\lambda_k}$, $k = 1, \dots, N$. In the case of the upper half-plane, the discrete projection operator is the following:

$$Q_N^\circ f(t) = \sum_{k=1}^{N} \langle f, \widetilde{\Psi}_k^\lambda\rangle_N \Psi_k^\lambda(t),$$

where

$$\langle f, \widetilde{\Psi}_k^\lambda\rangle_N = \sum_{t_j \in \mathbb{C}_N}^{N} \frac{f(t_j)\overline{\widetilde{\Psi}_k^\lambda(t_j)}}{\widetilde{K}_N(t_j, t_j)}.$$

The question, whether $Q_N^\circ f$ is an interpolation operator or not, naturally arises. In what follows, we will study the properties of this discrete projection operator.

Theorem 3.7. *Assume that $\mathbb{C}_N$ defined by (3.19) has N different elements. For every $f \in C(\overline{\mathbb{C}}_+)$, the projection operator $Q_N^\circ f$ is a Lagrange type rational interpolation operator of type $(N-1, N)$ at $t_k \in \mathbb{C}_N$, i.e.,*

$$Q_N^\circ f(t_k) = f(t_k),\ \ t_k \in \mathbb{C}_N.$$

Proof. Let us consider the kernel function of the discrete projection operator:

$$\widetilde{K}_N^\circ(z, \xi) := \overline{\sum_{k=1}^{N} \overline{\widetilde{\Psi}_k^\lambda(\xi)} \Psi_k^\lambda(z)} = \overline{\widetilde{K}_N(z, \xi)}. \tag{3.23}$$

The discrete projection can be expressed also by the kernel function, i.e.,

$$Q_N^\circ f(t) = \langle f(.), \widetilde{K}_N^\circ(t, .)\rangle_N = \sum_{t_j \in \mathbb{C}_N} \frac{\widetilde{K}_N(t, t_j)}{\widetilde{K}_N(t_j, t_j)} f(t_j).$$

Let us consider

$$q_{N,\xi}(t) = \frac{\widetilde{K}_N(t, \xi)}{\widetilde{K}_N(\xi, \xi)}.$$

If $t \neq \xi$, then

$$q_{N,\xi}(t) = \frac{\xi - i}{\bar{\xi} - i} \frac{1 - \overline{\widetilde{B}_N(\bar{\xi})}\widetilde{B}_N(t)}{\widetilde{K}_N(\xi,\xi)2i\pi(\xi - t)}.$$

If $u \neq 0$, then for $t_j, t_k \in \mathbb{C}_N$, according to (3.21), we have $q_{N,t_j}(t_k) = \delta_{ik}$, so these rational functions behave like the Lagrange interpolation polynomials. Consequently, the discrete projection operator $Q_N^\circ f$ has the following interpolation property

$$Q_N^\circ f(t_k) = \sum_{t_j \in \mathbb{C}_N} \frac{\widetilde{K}_N(t_k, t_j)}{\widetilde{K}_N(t_j, t_j)} f(t_j) = f(t_k), \quad t_k \in \mathbb{C}_N.$$

$\square$

We introduce $q_{N,\bar{t}}$, the dual of $q_{N,t}$. Similarly to the disc, a biorthogonal property and a discrete orthogonality of these functions can be proved for these functions.

Theorem 3.8. *Assume that $\mathbb{C}_N$ defined by (3.19) has N different elements. For $t_j, t_m \in \mathbb{C}_N$, the functions $q_{N,\bar{t}_m}$, and ℓ_{N,t_j} satisfy the following biorthogonality relation*

$$\langle q_{N,t_j}, q_{N,\bar{t}_m} \rangle = \int_{-\infty}^{\infty} q_{N,t_j}(t)\overline{q_{N,\bar{t}_m}(t)}dt = \frac{t_m - i}{\bar{t}_m - i} \frac{1}{\widetilde{K}(t_m, t_m)} \delta_{jm}.$$

In addition, q_{N,t_n} satisfies the following discrete orthogonality relation:

$$\langle q_{N,t_n}, q_{N,t_m} \rangle_N = \delta_{nm} \frac{1}{\widetilde{K}_N(t_n, t_n)}.$$

Proof. We have

$$\langle q_{N,t_j}, q_{N,\bar{t}_m} \rangle = \int_{-\infty}^{\infty} q_{N,t_j}(t)\overline{q_{N,\bar{t}_m}(t)}dt = \int_{-\infty}^{\infty} \frac{\widetilde{K}_N(t, t_j)}{\widetilde{K}_N(t_j, t_j)} \overline{\frac{\widetilde{K}_N(t, \bar{t}_m)}{\widetilde{K}_N(\bar{t}_m, \bar{t}_m)}} dt$$

$$= \frac{1}{\widetilde{K}_N(t_j, t_j)\overline{\widetilde{K}_N(\bar{t}_m, \bar{t}_m)}} \int_{-\infty}^{\infty} \widetilde{K}_N(t, t_j)\overline{\widetilde{K}_N(t, \bar{t}_m)}dt$$

$$= \frac{1}{\widetilde{K}_N(t_j, t_j)\overline{\widetilde{K}_N(\bar{t}_m, \bar{t}_m)}} \int_{-\infty}^{\infty} \sum_{k=0}^{N-1} \Psi_k^\lambda(t)\overline{\widetilde{\Psi}_k^\lambda(t_j)} \overline{\sum_{k'=0}^{N-1} \Psi_{k'}^\lambda(t)\overline{\widetilde{\Psi}_{k'}^\lambda(\bar{t}_m)}}dt$$

$$= \frac{1}{\widetilde{K}_N(t_j, t_j)\overline{\widetilde{K}_N(\bar{t}_m, \bar{t}_m)}} \sum_{k=0}^{N-1} \sum_{k'=0}^{N-1} \overline{\widetilde{\Psi}_k^\lambda(t_j)}\widetilde{\Psi}_{k'}^\lambda(\bar{t}_m) \int_{-\infty}^{\infty} \Psi_k^\lambda(t)\overline{\Psi_{k'}^\lambda(t)}dt$$

$$= \frac{1}{\widetilde{K}_N(t_j, t_j)\overline{\widetilde{K}_N(\bar{t}_m, \bar{t}_m)}} \sum_{k=0}^{N-1} \sum_{k'=0}^{N-1} \overline{\widetilde{\Psi}_k^\lambda(t_j)}\widetilde{\Psi}_{k'}^\lambda(\bar{t}_m)\delta_{kk'}$$

$$= \frac{1}{\widetilde{K}_N(t_j, t_j)\overline{\widetilde{K}_N(\bar{t}_m, \bar{t}_m)}} \sum_{k=0}^{N-1} \overline{\widetilde{\Psi}_k^\lambda(t_j)}\widetilde{\Psi}_k^\lambda(\bar{t}_m)$$

$$= \frac{1}{\widetilde{K}_N(t_j, t_j)\overline{\widetilde{K}_N(\bar{t}_m, \bar{t}_m)}} \frac{i + t_m}{i + \bar{t}_m} \sum_{k=0}^{N-1} \overline{\widetilde{\Psi}_k^\lambda(t_j)}\Psi_k^\lambda(t_m)$$

$$= \frac{1}{\widetilde{K}_N(t_j, t_j)\overline{\widetilde{K}_N(\bar{t}_m, \bar{t}_m)}} \frac{i + t_m}{i + \bar{t}_m} \widetilde{K}_N(t_m, t_j).$$

If $t_m \neq t_j$, then if $u \neq 0$, according to (3.21),

$$\widetilde{K}_N(t_m, t_j) = \frac{t_j - i}{\bar{t}_j - i} \frac{1 - \overline{\widetilde{B}_N(\bar{t}_j)}\widetilde{B}_N(t_m)}{2i\pi(t_j - t_m)} = 0.$$

Since

$$\begin{aligned}
\overline{\widetilde{K}_N(\bar{t}_m, \bar{t}_m)} &= \overline{\sum_{k=0}^{N-1} \Psi_k^\lambda(\bar{t}_m)\overline{\widetilde{\Psi}_k^\lambda \bar{t}_m)}} \\
&= \overline{\sum_{k=0}^{N-1} \frac{i + t_m}{i + \bar{t}_m} \widetilde{\Psi}_k^\lambda(t_m) \overline{\frac{i + t_m}{i + \bar{t}_m} \Psi_j^\lambda(t_m)}} \\
&= \frac{i + t_m}{i + \bar{t}_m} \frac{\bar{t}_m - i}{t_m - i} \sum_{k=0}^{N-1} \Psi_k^\lambda(t_m)\overline{\widetilde{\Psi}_k^\lambda(t_m)} \\
&= \frac{i + t_m}{i + \bar{t}_m} \frac{\bar{t}_m - i}{t_m - i} \widetilde{K}(t_m, t_m),
\end{aligned}$$

then for every $u \in \overline{\mathbb{D}} \setminus \{0\}$, we get

$$\int_{-\infty}^{\infty} q_{N,t_j}(t)\overline{q_{N,\bar{t}_m}(t)}dt = \frac{t_m - i}{\bar{t}_m - i} \frac{1}{\widetilde{K}(t_m, t_m)} \delta_{jm}.$$

When the nodes t_m are all on the real line, then $\bar{t}_m = t_m$, we obtain that the system $\{q_{N,t_j}(t), t_j \in \mathbb{C}_N\}$ is orthogonal, and we reobtain the result proved by Eisner, Pap in [4]

$$\langle q_{N,t_n}, q_{N,t_m} \rangle_N = \sum_{t_j \in \mathbb{C}_N} \frac{\widetilde{K}_N(t_j, t_n)}{\widetilde{K}_N(t_n, t_n)} \overline{\left(\frac{\widetilde{K}_N(t_j, t_m)}{\widetilde{K}_N(t_m, t_m)} \right)} \frac{1}{\widetilde{K}_N(t_j, t_j)} = \delta_{nm} \frac{1}{\widetilde{K}_N(t_n, t_n)}.$$

□

Beside the interpolation property $Q_N f$ and $Q_N^\circ f$ reconstruct exactly f in some cases if we measure the function in the N interpolation points. If f has the form $f(t) = \sum_{k=1}^N c_k \Psi_k^\lambda(t)$, then the continuous biorthogonality implies that

$$\langle f, \widetilde{\Psi}_j^\lambda \rangle = \sum_{k=0}^{N-1} c_k \langle \Psi_k^\lambda, \widetilde{\Psi}_j^\lambda \rangle = \sum_{k=0}^{N-1} c_k \delta_{kj} = c_j,$$

therefore we get that $f = Q_N f$. From discrete orthogonality, we get that

$$[f, \widetilde{\Psi}_j^\lambda]_{\lambda,u} = \sum_{k=0}^{N-1} c_k [\Psi_k^\lambda, \widetilde{\Psi}_j^\lambda]_{\lambda,u} = \sum_{k=0}^{N-1} c_k \delta_{kj} = c_j,$$

which implies, that

$$f = Q_N^\circ f.$$

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