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$(\mathcal{F}, h)$ Üst Sınıfı Aracılığıyla (ψ, φ) Zayıf Büzülme Dönüşümleri Üzerine Bir Çalışma

Esra YOLAÇAN*

Öne Çıkanlar:
- Çizge
- Üçgen $\alpha -$yörüngesel admissible
- Üçgen $\mu -$altyörüngesel admissible

Anahtar Kelimeler:
- $(\mathcal{F}, h)$ üst sınıfı
- Sabit nokta
- Branciari metrik uzayı

ÖZET:
Branciari, metrik uzaydaki iki terimli üçgen eşitsizliğini üç terimli dörtgen eşitsizliğiyle yer değiştirerek yeni bir mesafe fonksiyonu oluşturmak için metrik kavramını yeniden yapılandırdı. Tanımlanan bu fonksiyon literatürde dikdörtgensel metrik ya da genelleştirilmiş metrik olarak adlandırılır. Ansari tarafından ortaya konulan üst sınıf dönüşümü temel alınarak Branciari metrik uzayında üst sınıf tip II aracılığıyla zayıf büzülmeli dönüşümlerin bir genellemesi verildi. Sonraki aşamada ise bir çizge vasıtasıyla Branciari metrik uzayında grafik zayıf büzülmeli dönüşümler için yeni sabit nokta sonuçlarını ispat etmek amacıyla burada bir uygulama verildi. Son olarak çalışılan dönüşüm için ana sonuçlarımızı destekleyen bir örnek gösterildi.

A Study on (ψ, φ) Weakly Contractive Mapping via $(\mathcal{F}, h)$ Upper Class

Highlights:
- Graph
- Triangular $\alpha -$orbital admissible
- Triangular $\mu -$suborbital admissible

Keywords:
- $(\mathcal{F}, h)$ upper class
- Fixed point
- Branciari metric space

ABSTRACT:
Branciari reorganized the notion of metric to attain a novel distance function by replacing the triangular inequality with the quadrilateral inequality. The reorganized metric function was said rectangular metric in some resources, or general metric in some others. Ansari introduced a more general function so-called upper class. Inspired and motivated by this facts, we give an extansion of weakly contractive mapping via upper class type II in the setting of Branciari metric space. An application is given here to prove new fixed point results for graphic weakly contractive mappings in Branciari metric space endowed with a graph. Moreover, we derive an example in support of our main results.

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INTRODUCTION

Fixed point theory, which is an impressive combination of topology, analysis and geometry has been turned out to be a very substantial and essential tool in the survey of nonlinear phenomenon. The existence of fixed points is therefore of paramount importance in several areas of mathematics, economics, engineering, game theory and other sciences (Ruzhansky et al., 2017).

The survey of fixed and common fixed points of mappings supplying a particular metrical contractive condition has allured several researchers and promoted a fascinating research study over the last six decades. Bisht classify several contractive notions which provide the existence of the fixed point (Bisht, 2023). Chiroma et al. presented the concept of generalized quasi-weakly contractive operators in metric-like spaces, observed novel states for the existence of fixed points for such maps (Chiroma et al., 2023). Al-Khaleel et al. establised new cyclic contractions Chatterjea/Kannan type, also showed existence and uniqueness results in the setting of Branciari metric space (Al-Khaleel et al., 2023).

The purport of Branciari metric space (in short, or BMS) was originally proposed by the author (Branciari, 2000), where the triangle inequality was substituted for quadrilateral inequality. Since a few of the pioneer papers that concerned with fixed point theorem in BMS presumed that the respective topology is Hausdorff and/or that a sequence can convergence to at most one point and/or that each convergent sequence is a Cauchy sequence, these proofs has gaps which was taken away Sarmaa et al. and Samet (Sarmaa et al.,2009; Samet, 2010). Mamud and Tola furnish generalised $(\alpha, \psi)-$contraction mapping in the context of $b-$BMS (Mamud and Tola, 2022). Thereafter, many researches proved numerous available fixed point theorems in BMS, we refer to (Kadelburg and Radenovic, 2014; Abagaro et al., 2022; Arshad et al., 2016; Yolacan, 2016; Li et al., 2022; Baiya and Kaewcharoen, 2019; Patil et al., 2022) and others.

On the other side, some fixed point theorems have been lately studied by considering contractive mappings denoted by an upper class. This approach has been initiated in the inspiring article of Ansari (Ansari,2014). Ansari and Shukla presented the concept of ordered $F-(\mathcal{F}, h)-$contraction/subcontraction mappings (Ansari and Shukla, 2016). Their results are a widening and generalisation of many available conclusions in the litterateur. Huang et al. established the notions of rational/subrational type contractive mappings (Huang et al.,2017). Ansari and Abodayeh studied the partial $S-$metric spaces for upper class functions (Ansari and Abodayeh, 2020). Following Ansari's approach, recently a consistent literature on fixed point for C−class and pair upper class functions, in various ambient spaces, has been promoted, see (Ansari and Tomar, 2021; Ansari et al. 2022). Starting from this background, the goal of this writing is to obtain some constructive fixed point theorems for weakly contractive mapping endowed with upper class.

Definition 1. Let $\xi \neq \emptyset$ and let $d_B: \xi \times \xi \to [0, \infty)$ satisfy the following conditions for $\forall w, u \in \xi$ and entire distinct $x, y \in \xi$ each of them dissimilar to w and u. (i) $d_B(w, u) = 0$ iff $w = u$, (ii) $d_B(w, u) = d_B(u, w)$, (iii) $d_B(w, u) \leq d_B(w, x) + d_B(x, y) + d_B(y, u)$. In turn, d_B is stated a Branciari metric. Hereby, the pairwise (ξ, d_B) is said BMS (Branciari, 2000).

Definition 2. Let (ξ, d_B) a BMS and $\{w_n\} \subseteq \xi$. (i) $\{w_n\}$ is said BMS convergent to a limit w iff $d_B(w_n, w) \to 0$ as $n \to \infty$. (ii) $\{w_n\}$ is said BMS Cauchy sequence $\Leftrightarrow$ there is $N(\varepsilon) \in \mathbb{Z}^+$ for $\forall \varepsilon > 0$ such that $d_B(w_n, w_m) < \varepsilon$ for whole $N(\varepsilon) < m < n$. (iii) BMS (ξ, d_B) is be termed complete if any BMS Cauchy sequence is BMS convergent. (iv) The map $S: (\xi, d_B) \to (\xi, d_B)$ is continuous if for $\{w_n\} \subseteq \xi$ such that $d_B(w_n, w) \to 0$ as $n \to \infty$, we hold $d_B(Sw_n, Sw) \to 0$ as $n \to \infty$ (Branciari, 2000).

Lemma 1. Let (ξ,d_B) be a BMS, $\{w_n\}$ be a Cauchy sequence on ξ such that $w_n\neq w_m$ whensoever $n\neq m$. In turn, $\{w_n\}$ may converge to at most one point (Kadelburg and Radenovic, 2014).

Lemma 2. Let (ξ,d_B) be a BMS, $\{w_n\}\subseteq\xi$ with $w_n\neq w_m$ for $n\neq m$. Assume that $\lim_{n\to\infty} d_B(w_{n+1},w_n)=\lim_{n\to\infty} d_B(w_{n+2},w_n)=0$ and that $\{w_n\}$ is not a Cauchy sequence. In turn there consists $\varepsilon>0$, also two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $k<m_k<n_k$ and sequences below

$d_B(w_{m_k},w_{n_k}), d_B(w_{m_k},w_{n_{k+1}}), d_B(w_{m_{k-1}},w_{n_k}), d_B(w_{m_{k-1}},w_{n_{k+1}})$

tend to ε as $k\to\infty$ (Kadelburg and Radenovic, 2014).

Definition 3. Let $\alpha:\xi\times\xi\to[0,\infty)$ be a function, $S:\xi\to\xi$ be a map. We state that S is $\alpha-$orbital admissible if $\alpha(w,Sw)\geq 1\Rightarrow\alpha(Sw,S^2w)\geq 1$. Furthermore, S is said to be triangular is $\alpha-$orbital admissible (in short, or $T\alpha-OA$) if S is $\alpha-$orbital admissible and $\alpha(w,Sw)\geq 1$ and $\alpha(u,Su)\geq 1$ imply $\alpha(w,Su)\geq 1$ (Popescu, 2014).

Lemma 3. Let $S:\xi\to\xi$ be a $T\alpha-OA$. Suppose that there exists $w_1\in\xi$ such that $\alpha(w_1,Sw_1)\geq 1$. Describe a sequence $\{w_n\}$ by $w_{n+1}=Sw_n$. In turn we for $\forall m,n\in\mathbb{N}$; $\alpha(w_n,w_m)\geq 1$ with $m>n$ (Popescu, 2014).

Definition 4. Let $\mu:\xi\times\xi\to[0,\infty)$ be a function, $S:\xi\to\xi$ be a map. We call that S is $\mu-$subadmissible if $w,u\in\xi$, $\mu(w,u)\leq 1$ implies that $\mu(Sw,Su)\leq 1$ (Salimi et al., 2013).

Definition 5. A map $S:\xi\to\xi$ is called to be triangular $\mu-$subadmissible if:

(S1) S is $\mu-$ subadmissible,

(S2) $\mu(w,x)\leq 1$ and $\mu(x,u)\leq 1$ implies $\mu(w,u)\leq 1$ for $w,x,u\in\xi$ (Karapinar et al., 2013).

Example 1. Let $\xi=\mathbb{R}$, $Sw=w^5$ and $\mu(w,u)=e^{w-u}$ then S is a triangular $\mu-$subadmissible mapping. In fact, if $\mu(w,u)=e^{w-u}\leq 1$ then $w-u\leq 0$ which implies $Sw\leq Su$. In other words, $\mu(Sw,Su)=e^{Sw-Su}\leq 1$. Again, if $\mu(w,x)\leq 1$ and $\mu(x,u)\leq 1$, therefore $w\leq x$ and $x\leq u$. That is to say, $w\leq u$ then $\mu(w,u)=e^{w-u}\leq 1$.

Inspired and motivated by this facts, we introduce new notions as shown below.

Definition 6. Let $\mu:\xi\times\xi\to[0,\infty)$ be a function, $S:\xi\to\xi$ be a map. Here S is called to be $\mu-$suborbital admissible if

(S3) $w\in\xi,\mu(w,Sw)\leq 1$ implies $\mu(Sw,S^2w)\leq 1$.

Example 2. Let $\xi=\{1,2,3,4,5,6,7,8\}$. We describe a mapping $S:\xi\to\xi$ such that

$S1=1, S3=2, S5=6, S7=8$

$S2=3, S4=4, S6=5, S8=7.$

Furthermore, we define $\mu:\xi\times\xi\to[0,\infty)$ such that

$$\mu(w,u)=\begin{cases}1 & if\ (w,u)\in\{(1,2)(1,3),(2,2),(3,3),(2,3),(3,2),(2,4),(3,4),(4,5)\},\\ 8 & otherwise.\end{cases}$$

Since $\mu(2,S2)=\mu(2,3)=1$ and $\mu(3,S3)=\mu(3,2)=1$, S is $\mu-$suborbital admissible. On the other side, we have $\mu(4,5)=1$, but $\mu(S4,S5)=\mu(4,6)=8\nleq 1$. Thus, S is not $\mu-$ subadmissible.

Definition 7. Let $\mu:\xi\times\xi\to[0,\infty)$ be a function, $S:\xi\to\xi$ be a map. Here S is called to be triangular $\mu-$suborbital admissible (in short, or $T\mu-SA$) if $\mu-$suborbital admissible and

(S4) $w,u\in\xi,\mu(w,u)\leq 1$ and $\mu(u,Su)\leq 1$ implies $\mu(w,Su)\leq 1$.

Example 3. Let $\xi=\{1,2,3,4\}$, $S:\xi\to\xi$ such that

$$S1=1, S3=2,$$
$$S2=3, S4=4,$$

and $\mu:\xi\times\xi\to[0,\infty)$,

$$\mu(w,u)=\begin{cases}1 & if\ (w,u)\in\Theta,\\ 3 & otherwise,\end{cases}$$

where

$\Theta=\{(1,2)(1,3),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$.

Clearly, S is $\mu-$suborbital admissible, S is $T\mu-SA$. But $\mu(1,2)=\mu(2,4)=1,\mu(1,4)=3\nleq 1$, so S is not triangular $\mu-$subadmissible.

Lemma 4. Let $S:\xi\to\xi$ be $T\mu-SA$. Suppose that there exists $w_1\in\xi$ such that $\mu(w_1,Sw_1)\leq 1$. Describe a sequence $\{w_n\}$ by $w_{n+1}=Sw_n$. In turn, we for $\forall m,n\in\mathbb{N}$; $\mu(w_n,w_m)\leq 1$ with $n<m$.

Proof. As S is $\mu-$suborbital admissible and $\mu(w_1,Sw_1)\leq 1$ for $w_1\in\xi$, we conclude that $\mu(w_2,w_3)\leq 1$. By inductive, we have $\mu(w_n,w_{n+1})\leq 1$ for all n. Assume that $\mu(w_n,w_m)\leq 1$ and prove that $\mu(w_n,w_{m+1})\leq 1$, where $n<m$. Because S is $T\mu-SA$ and $\mu(w_m,w_{m+1})\leq 1$, we obtain that $\mu(w_n,w_{n+1})\leq 1$. Thus, we get that $\mu(w_n,w_m)\leq 1$ for $\forall m,n\in\mathbb{N}$, where $n<m$.

Definition 8. Let $\xi=[0,\infty)$ and $h:\xi\times\xi\times\xi\to\mathbb{R}$. We call that h is a function of subclass of type II, if $w,u\geq 1$, then $h(1,1,z)\leq h(w,u,z)$ for $\forall z\in[0,\infty)$ (Ansari, 2014; Ansari and Shukla, 2016).

Example 4. Describe $h:\xi\times\xi\times\xi\to\mathbb{R}$ for $w,u,z\in\mathbb{N}$ by (i) $h(w,u,z)=w^{\zeta}u^{\tau}z^{\varsigma}, for\ \zeta,\tau,\varsigma\in\mathbb{N}$; (ii) $h(w,u,z)=z^{\eta}3^{-1}\left(u^{\omega}w^{\tau}y+u^{\delta}+w^{\varrho}\right)$ for $\omega,\tau,\delta,\varrho\in\mathbb{N}$. In turn, h is a function of subclass of type II (Ansari, 2014; Ansari and Shukla, 2016).

Definition 9. Let $h:\xi\times\xi\times\xi\to\mathbb{R}$ and $\mathcal{F}:\xi\times\xi\to\mathbb{R}$. We call that the pairwise $(\mathcal{F},h)$ is an upper class of type II; if (i) h is a subclass of type II; (ii) $1\geq k\geq 0\Rightarrow\mathcal{F}(k,l)\leq\mathcal{F}(1,l)$; (iii) $h(1,1,z)\leq\mathcal{F}(k,l)$ for $\forall k,l,z\in[0,\infty)$ (Ansari, 2014; Ansari and Shukla, 2016).

Example 5. Define $h:\xi\times\xi\times\xi\to\mathbb{R}$ and $\mathcal{F}:\xi\times\xi\to\mathbb{R}$ by (i) $h(w,u,z)=(wu+a)^{z},1<a,\mathcal{F}(k,l)=(1+a)^{kl}$; (iii) $h(w,u,z)=w^{\zeta}u^{\tau}z^{\varsigma}, for\ \zeta,\tau,\varsigma\in\mathbb{N},\mathcal{F}(k,l)=k^{\gamma}l^{\gamma}$ for $w,u,z,k,l\in[0,\infty),\gamma\in\mathbb{N}$. In turn, the pairwise $(\mathcal{F},h)$ is an upper class of type II (Ansari, 2014; Ansari and Shukla, 2016).

Let Φ^* and Ψ be set functions defined by $\varphi,\psi:[0,\infty)\to[0,\infty)$ such that φ is lower semi-continuous and $\varphi(\kappa)=0\Leftrightarrow\kappa=0$, ψ is nondecreasing, continuous and $\psi(\kappa)=0\Leftrightarrow\kappa=0$.

MATERIALS AND METHODS

In this section, we state some results for (ψ,φ) weakly contractive mapping satisfying $(\mathcal{F},h)$ Upper class defined on a BMS. Moreover, we illustrate the following result by the Example 6, which additionally shows that a complete BMS may not be a metric space. Now, we are ready to state and prove our main result.

Theorem 1. Let (ξ,d_B) be a BMS, S be a map. Supposing for $\psi\in\Psi,\varphi\in\Phi^*,\forall w,u\in\xi$,

$$h\left(\alpha(w,Sw),\alpha(u,Su),\psi\left(d_B(Sw,Su)\right)\right)$$
$$\leq\mathcal{F}\left(\mu(w,Sw).\mu(u,Su),\psi\left(K(w,u)\right)-\varphi\left(K(w,u)\right)\right)\qquad(1)$$

and

$$K(w,u)=max\left\{d_B(w,u),d_B(w,Sw),d_B(u,Su),\frac{d_B(w,Sw).d_B(u,Su)}{1+d_B(w,u)},\frac{d_B(w,Sw).d_B(u,Su)}{1+d_B(Sw,Su)}\right\}.\qquad(2)$$

Here $(\mathcal{F},h)$ is an upper class of type II, moreover assume that the undermentioned assertions have:

(i) S is $T\alpha-OA$ and $T\mu-SA$;

(ii) there is $w_0\in\xi$ such that $\alpha(w_0,Sw_0)\geq 1,\mu(w_0,Sw_0)\leq 1$ and $\alpha(w_0,S^2w_0)\geq 1,\mu(w_0,S^2w_0)\leq 1$;

(iii) (1) S is continuous, or

(2) for any sequence $\{w_n\}\subseteq\xi$ with $\alpha(w_n,w_{n+1})\geq 1,\mu(w_n,w_{n+1})\leq 1$, and $w_n\to a$ when $n\to\infty$, one get $\alpha(a,Sa)\geq 1,\mu(a,Sa)\leq 1$.

Then S hold a fixed point $a_*\in\xi$, also $d_B(a_*,a_*)=0$.

Proof. Let $w_0\in\xi$ be such that $\alpha(w_0,Sw_0)\geq 1,\mu(w_0,Sw_0)\leq 1$ and $\alpha(w_0,S^2w_0)\geq 1,\mu(w_0,S^2w_0)\leq 1$. We express that sequence $\{w_n\}\subseteq\xi$ as $w_n=S^nw_0$ for $\forall n$. Openly, if $w_{n_0}=w_{n_0+1}$ for $n_0\geq 1$, and so $w_{n_0}$is a fixed point of S. Therefore, we presume that $w_{n_0}\neq w_{n_0+1}$ for $\forall n$. Next, from Lemma 3 and Lemma 4, we have

$$\alpha(w_n,w_{n+1})\geq 1,\mu(w_n,w_{n+1})\leq 1 \text{ for } \forall n, \tag{3}$$

and

$$\alpha(w_n,w_{n+2})\geq 1,\mu(w_n,w_{n+2})\leq 1 \text{ for } \forall n. \tag{4}$$

By (1), (2) and (3), for $\forall n$, we write

$$\begin{aligned}
&h\left(1,1,\psi\big(d_B(w_n,w_{n+1})\big)\right)\\
&=h\left(1,1,\psi\big(d_B(Sw_{n-1},Sw_n)\big)\right)\\
&\leq h\left(\alpha(w_{n-1},Sw_{n-1}),\alpha(w_n,Sw_n),\psi\big(d_B(Sw_{n-1},Sw_n)\big)\right)\\
&\leq \mathcal{F}\left(\mu(w_{n-1},Sw_{n-1}).\mu(w_n,Sw_n),\psi\big(K(w_{n-1},w_n)\big)\right.\\
&\qquad\qquad \left.-\varphi\big(K(w_{n-1},w_n)\big)\right)
\end{aligned} \tag{5}$$

which implies that

$$\begin{aligned}
K(w_{n-1},w_n)&=max\left\{\begin{matrix} d_B(w_{n-1},w_n),d_B(w_n,Sw_n),d_B(w_{n-1},Sw_{n-1}),\\ \dfrac{d_B(w_{n-1},Sw_{n-1}).d_B(w_n,Sw_n)}{1+d_B(w_{n-1},w_n)},\dfrac{d_B(w_{n-1},Sw_{n-1}).d_B(w_n,Sw_n)}{1+d_B(Sw_{n-1},Sw_n)}\end{matrix}\right\}\\
&=max\{d_B(w_{n-1},w_n),d_B(w_n,w_{n+1})\}.
\end{aligned}$$

If $K(w_{n-1},w_n)=d_B(w_n,w_{n+1})$, then we have

$$h\left(1,1,\psi\big(d_B(w_n,w_{n+1})\big)\right)\leq\mathcal{F}(\mu(w_{n-1},Sw_{n-1}).\mu(w_n,Sw_n),\psi\big(d_B(w_n,w_{n+1})\big)-\varphi\big(d_B(w_n,w_{n+1})\big))$$

$$\leq\mathcal{F}(1,\psi\big(d_B(w_n,w_{n+1})\big)-\varphi\big(d_B(w_n,w_{n+1})\big))$$

$$\Longrightarrow$$

$$\psi\big(d_B(w_n,w_{n+1})\big)\leq\psi\big(d_B(w_n,w_{n+1})\big)-\varphi\big(d_B(w_n,w_{n+1})\big), \tag{6}$$

which implies $\varphi\big(d_B(w_n,w_{n+1})\big)>0$, and hence (6) becomes

$$\psi\big(d_B(w_n,w_{n+1})\big)\leq\psi\big(d_B(w_n,w_{n+1})\big),$$

which is a contradiction. Hence $max\{d_B(w_n,w_{n+1}),d_B(w_{n-1},w_n)\}=d_B(w_{n-1},w_n)$ for $\forall n$. Therefore, from (5), we get

$$\begin{aligned}
h\left(1,1,\psi\big(d_B(w_n,w_{n+1})\big)\right)&=h\left(1,1,\psi\big(d_B(Sw_{n-1},Sw_n)\big)\right)\\
&\leq h(\alpha(w_{n-1},Sw_{n-1}),\alpha(w_n,Sw_n),\psi\big(d_B(Sw_{n-1},Sw_n)\big))
\end{aligned}$$

$$\leq \mathcal{F}(\mu(w_{n-1}, Sw_{n-1}).\mu(w_n, Sw_n), \psi(d_B(w_{n-1}, w_n)) - \varphi(d_B(w_{n-1}, w_n)))$$
$$\leq \mathcal{F}(1, \psi(d_B(w_{n-1}, w_n)) - \varphi(d_B(w_{n-1}, w_n)))$$
$$\Longrightarrow$$
$$\psi(d_B(w_n, w_{n+1})) \leq \psi(d_B(w_{n-1}, w_n)) - \varphi(d_B(w_{n-1}, w_n)). \quad (7)$$

In view of the monotone property of ψ and $0 < \varphi(d_B(w_{n-1}, w_n))$, then $d_B(w_{n-1}, w_n) > d_B(w_n, w_{n+1})$ for $\forall n$, that is the sequence $\{d_B(w_n, w_{n+1})\}$ is nonincreasing. Hence, there exists $\kappa \geq 0$ such that $\lim_{n\to\infty} d_B(w_n, w_{n+1}) = 0$. Taking $limsup$ when $n \to \infty$ in (5) and owing to feature of ψ and φ, and so $\psi(\kappa) \leq \psi(\kappa) - \varphi(\kappa)$, which indicates that $\varphi(\kappa)$ iff $\kappa = 0$. Thus, we hold

$$d_B(w_n, w_{n+1}) \to 0 \quad when\, n \to \infty. \quad (8)$$

In a similar manner, using (1), (2) and (4), one can prove

$$d_B(w_n, w_{n+2}) \to 0 \quad when\, n \to \infty. \quad (9)$$

Suppose that $\{w_n\}$ is a sequence of distinct points, in other words, $w_n \neq w_m$ whensoever $n \neq m$ and prove $\{w_n\}$ is BMS Cauchy sequence. Supposing $\{w_n\}$ is not BMS Cauchy sequence. Next, from Lemma 2, using (8) and (9), we deduce there is $\varepsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that $k < m_k < n_k$,

$$\lim_{n\to\infty} d_B(w_{m_k}, w_{n_k}) = \lim_{n\to\infty} d_B(w_{m_k}, w_{n_{k+1}}) = \lim_{n\to\infty} d_B(w_{m_{k-1}}, w_{n_k}) = \lim_{n\to\infty} d_B(w_{m_{k-1}}, w_{n_{k+1}}) = \varepsilon. \quad (10)$$

Now we substitute $w = w_{m_{k-1}}$ and $u = w_{n_k}$ in (1) and (2),

$$h\left(1,1,\psi\left(d_B(w_{m_k}, w_{n_{k+1}})\right)\right)$$
$$= h\left(1,1,\psi\left(d_B(Sw_{m_{k-1}}, Sw_{n_k})\right)\right)$$
$$\leq h\left(\alpha(w_{m_{k-1}}, Sw_{m_{k-1}}), \alpha(w_{n_k}, Sw_{n_k}), \psi\left(d_B(Sw_{m_{k-1}}, Sw_{n_k})\right)\right)$$
$$\leq \mathcal{F}(\mu(w_{m_{k-1}}, Sw_{m_{k-1}}).\mu(w_{n_k}, Sw_{n_k}), \psi\left(K(w_{m_{k-1}}, w_{n_k})\right) - \varphi\left(K(w_{m_{k-1}}, w_{n_k})\right))$$
$$\leq \mathcal{F}(1, \psi\left(K(w_{m_{k-1}}, w_{n_k})\right) - \varphi\left(K(w_{m_{k-1}}, w_{n_k})\right))$$

where

$$K(w_{m_{k-1}}, w_{n_k}) = max\left\{\begin{array}{c} d_B(w_{m_{k-1}}, w_{n_k}), d_B(w_{n_k}, Sw_{n_k}), d_B(w_{m_{k-1}}, Sw_{m_{k-1}}), \\ \dfrac{d_B(w_{n_k}, Sw_{n_k}).d_B(w_{m_{k-1}}, Sw_{m_{k-1}})}{1 + d_B(w_{m_{k-1}}, w_{n_k})}, \dfrac{d_B(w_{n_k}, Sw_{n_k}).d_B(w_{m_{k-1}}, Sw_{m_{k-1}})}{1 + d_B(Sw_{m_{k-1}}, Sw_{n_k})} \end{array}\right\}.$$

Obviously, when $k \to \infty$ we have $K(w_{m_{k-1}}, w_{n_k}) \to \varepsilon$ in view of (8) and (10). Taking limit when $k \to \infty$ in above expression, we obtain

$$h(1,1,\psi(\varepsilon)) \leq \mathcal{F}(1, \psi(\varepsilon)) - liminf_{d_B(w_{m_{k-1}}, w_{n_k}) \to \varepsilon^+} \varphi(\varepsilon)) \Rightarrow \psi(\varepsilon) \leq \psi(\varepsilon) - liminf_{d_B(w_{m_{k-1}}, w_{n_k}) \to \varepsilon^+} \varphi(\varepsilon) < \psi(\varepsilon),$$

which is a contradiction. Thereof, $\{w_n\}$ is BMS Cauchy sequence.

Case $(iii)_1$: Because S is continuous, it follows that $Sa_*=a_*$, as a Cauchy sequence with distinct elements in ξ may not own two limits from Lemma 1.

Case $(iii)_2$: Since $w_n\to a_*$ when $n\to\infty$ and by (3), we have $\alpha(a_*,Sa_*)\geq 1$, $\mu(a_*,Sa_*)\leq 1$. Let $w=w_n$ and $u=a_*$ in (1) and (2). Thus one write

$$h\left(1,1,\psi\big(d_B(w_{n+1},Sa_*)\big)\right)=h\left(1,1,\psi\big(d_B(Sw_n,Sa_*)\big)\right)$$
$$\leq h\big(\alpha(w_n,Sw_n),\alpha(a_*,Sa_*),\psi\big(d_B(Sw_n,Sa_*)\big)\big)$$
$$\leq \mathcal{F}\big(\mu(w_n,Sw_n).\mu(a_*,Sa_*),\psi\big(K(w_n,a_*)\big)-\varphi\big(K(w_n,a_*)\big)\big)$$
$$\leq \mathcal{F}\big(1,\psi\big(K(w_n,a_*)\big)-\varphi\big(K(w_n,a_*)\big)\big)$$
$$\Longrightarrow$$
$$\psi\big(d_B(w_{n+1},Sa_*)\big)\leq\psi\big(K(w_n,a_*)\big)-\varphi\big(K(w_n,a_*)\big)\leq\psi(K(w_n,a_*))$$

where

$$K(w_n,a_*)=\left\{\begin{matrix} d_B(w_n,a_*),d_B(a_*,Sa_*),d_B(w_n,Sw_n),\\ \dfrac{d_B(a_*,Sa_*).d_B(w_n,Sw_n)}{1+d_B(w_n,a_*)},\dfrac{d_B(a_*,Sa_*).d_B(w_n,Sw_n)}{1+d_B(Sw_n,Sa_*)}\end{matrix}\right\}.$$

Since $d_B(w_n,a_*)\to 0$ and $d_B(w_n,Sw_n)\to 0$ when $n\to\infty$, thus we get that $K(w_n,a_*)\to d_B(a_*,Sa_*)$ when $n\to\infty$. It follows that $limsup_{n\to\infty}d_B(w_{n+1},Sa_*)\leq d_B(a_*,Sa_*)$, so

$$d_B(a_*,Sa_*)\leq d_B(a_*,w_n)+d_B(w_n,w_{n+1})+d_B(w_{n+1},Sa_*). \tag{11}$$

Taking limit when $n\to\infty$ in (11) and by (8) and $d_B\big(w_{n,}a_*\big)\to 0$ when $n\to\infty$,
$d_B(a_*,Sa_*)\leq limsup_{n\to\infty}d_B(w_{n+1},Sa_*)\leq d_B(a_*,Sa_*)$.

Thus, $d_B(a_*,Sa_*)=0$, which ensures $a_*=Sa_*$. Thereat, S own a fixed point $a_*\in\xi$ and $d_B(a_*,a_*)=0$.

Next, in support of the proven conclusions, we furnish an instance which is motivated by Example 20 of Arshad et al. (Arshad et al., 2016).

Example 6. Let $\xi=[-2,-1]\cup\{0\}\cup[1,2]$. Determine $d_B:\xi\times\xi\to[0,\infty)$ as follows:
$d_B(1,1)=d_B(-2,-2)=d_B(0,0)=d_B(-1,-1)=d_B(2,2)=0$,
$d_B(2,-1)=d_B(-1,1)=d_B(-1,2)=d_B(1,-1)=1$,
$d_B(2,1)=d_B(1,2)=3, d_B(w,u)=|w-u|, otherwise$.

Obviously, (ξ,d_B) is complete BMS, however, it is not metric space in that d_B does not supply the triangle inequality. In fact,
$d_B(1,2)=3>d_B(1,-1)+d_B(-1,2)=2$.

Let $S:\xi\to\xi$ be the mapping defined by $Sw=\begin{cases}-w & if\ w\in[-2,-1)\cup(1,2],\\ 0 & otherwise\end{cases}$. Let $\alpha:\xi\times\xi\to[0,\infty)$ be given by $\alpha(w,u)=\begin{cases}1 & if\ wu\geq 0,\\ 0 & otherwise,\end{cases}$ and $\mu:\xi\times\xi\to[0,\infty)$ be given by $\mu(w,u)=\begin{cases}1 & if\ wu\geq 0,\\ 9 & otherwise\end{cases}$. Define the functions $h:\xi\times\xi\times\xi\to\mathbb{R}$ and $\mathcal{F}:\xi\times\xi\to\mathbb{R}$ by

$h(w,u,z)=wuz$ and $\mathcal{F}(k,l)=kl$,

for all $w,u,z,k,l\in\xi$. Hence, the pairwise $\mathcal{F}(k,l)$ is an upper class of type II. Define also the mappings $\varphi,\psi:[0,\infty)\to[0,\infty)$ by $\psi(\kappa)=3\kappa,\varphi(\kappa)=\frac{\kappa}{3}$. Fairly, S is $T\alpha-OA$ and $T\mu-SA$. The assumptions of Theorem 1 are ensured by S, thus S has a fixed point $0\in\xi$.

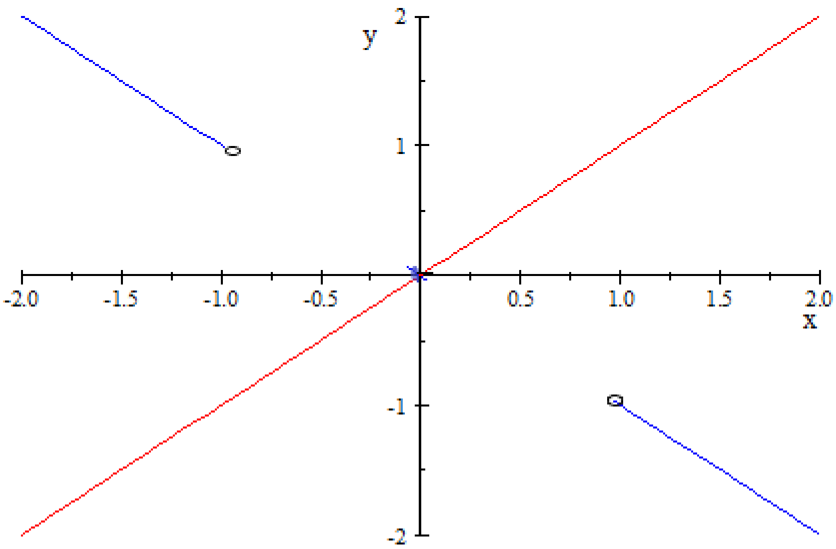

Figure 1. Plot showing fixed point of S (blue line)

RESULTS AND DISCUSSION

Application

Throughout this section, let (ξ, d_G) be a BMS, Δ be a diagonal of $\xi \times \xi$, and G be a graph with no parallel edges such that the set $V(G)$ of its vertices corresponds to the points of ξ and $\Delta \subseteq E(G)$, here $E(G)$ is the set edges of G. Videlicet, G remark as $\big(V(G), E(G)\big)$. Furthermore, one can handle G as a weighted graph by allocating to every edge the interval among its vertices. If w and u be vertices of G, then a path on G from w through u of length N is a sequence $\{w_i\}_{i=0}^{N}$ of $N+1$ vertices such that $w_0 = w, w_N = u$ and for $i = \overline{1,N}$, $(w_{i-1}, w_i) \in E(G)$. G is a connected if there exists a path among any two vertices (Jachymski, 2008).

Definition 10. A map $S: \xi \to \xi$ is said $G-$continuous if determined $w \in \xi$ and sequence $\{w_n\}$

$$\lim_{n\to\infty} w_n = w, (w_n, w_{n+1}) \in E(G)\ \forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} Sw_n = Sw \text{ (Jachymski, 2008).}$$

Definiton 11. A map $S: \xi \to \xi$ is said $G-$contraction if S preserves edges of G, namely,

$\forall w, u \in \xi;\ (w,u) \in E(G) \Rightarrow (Sw, Su) \in E(G)$,

and S decreases weights of edges of G as below:

$\exists\ 0 < \theta < 1, \forall w, u \in \xi;\ (w,u) \in E(G) \Rightarrow d_G(Sw, Su) \leq \theta d_G(w,u)$ (Jachymski, 2008).

Definiton 12. Let (ξ, d_G) be a BMS via G and $S: \xi \to \xi$ a map. If there is $\psi \in \Psi, \varphi \in \Phi^*$ such that

$\forall w, u \in \xi, (w,u) \in E(G) \Rightarrow (Sw, Su) \in E(G)$,

$\forall w, u \in \xi, (w,u) \in E(G)$,

$\Rightarrow$

$$h\Big(\alpha(w,Sw), \alpha(u,Su), \psi\big(d_G(Sw,Su)\big)\Big) \leq \mathcal{F}\Big(\mu(w,Sw).\mu(u,Su), \psi\big(K(w,u)\big) - \varphi\big(K(w,u)\big)\Big)$$

where

$$K(w,u) = max\left\{d_G(w,u), d_G(w,Sw), d_G(u,Su), \frac{d_G(w,Sw).d_G(u,Su)}{1+d_G(w,u)}, \frac{d_G(w,Sw).d_G(u,Su)}{1+d_G(Sw,Su)}\right\}.$$

Here $(\mathcal{F},h)$ is an upper class of type II, then S is said $(\psi,\varphi)-$graphic weakly contractive mapping.

Theorem 2. Let (ξ, d_G) be a BMS via G and $S: \xi \to \xi$ a $(\psi,\varphi)-$graphic weakly contractive mapping satisfying the undermentioned assertions:

(i) there exists $w_0 \in \xi$ such that $(w_0, Sw_0) \in E(G)$;

(ii) S is G −continuous or if $\{w_n\} \subseteq \xi$ such that

$$(w_n, w_{n+1}) \in E(G) \; and \; \lim_{n\to\infty} w_n = w \;\Rightarrow \forall n \in \mathbb{N} \, (w_n, w) \in E(G);$$

(iii) $(w,u),(u,z) \in E(G) \Rightarrow (w,z) \in E(G)$ for $\forall w,u,z \in \xi$.

Then S hold a fixed point.

Proof. Determine $\alpha: \xi \times \xi \to [0,\infty)$ as $\alpha(w,u) = \begin{cases} 1 & if\ (w,u) \in E(G), \\ 0 & otherwise \end{cases}$. At first we show that S is $T\alpha - OA$. Let

$\alpha(w,u) \geq 1 \Rightarrow (w,u) \in E(G)$.

As S is (ψ,φ) −graphic weakly contractive mapping, we hold $(Sw,Su) \in E(G)$, in other words, $\alpha(Sw,Su) \geq 1$. Farther, let

$\alpha(w,z) \geq 1, \alpha(z,u) \geq 1 \Rightarrow (w,z),(z,u) \in E(G)$.

So, by (iii), we have $(w,u) \in E(G)$. Nominately, $\alpha(w,u) \geq 1$.

Define $\mu: \xi \times \xi \to [0,\infty)$ by $\mu(w,u) = \begin{cases} 1 & if\ (w,u) \in E(G), \\ 8 & otherwise \end{cases}$. Secondly we claim that S is triangular μ −subadmissible mapping. Let

$\mu(w,u) \leq 1 \Rightarrow (w,u) \in E(G)$.

As again S is (ψ,φ) −graphic weakly contractive mapping, we hold $(Sw,Su) \in E(G)$, in other words, $\mu(Sw,Su) \leq 1$. Farther, let

$\mu(\mathrm{w},\mathrm{z}) \leq 1, \mu(\mathrm{z},\mathrm{u}) \leq 1 \Rightarrow (\mathrm{w},\mathrm{z}),(\mathrm{z},\mathrm{u}) \in \mathrm{E(G)}$.

Thus, by the condition (iii), we attain $(w,u) \in E(G)$, that is, $\mu(\mathrm{w},\mathrm{u}) \leq 1$. Hence, we obtain that S is triangular μ −subadmissible mapping. Let S is G −continuous on (ξ, d_G). In the present case we have

$\lim_{n\to\infty} w_n = w, (w_n, w_{n+1}) \in E(G) \; \forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} Sw_n = Sw,$

$\lim_{n\to\infty} w_n = w, \alpha(w_n, w_{n+1}) \geq 1 \Rightarrow \lim_{n\to\infty} Sw_n = Sw, \forall n \in \mathbb{N},$

which states that S is G −continuous (see, Hussain et al., 2013). Then, $\forall n \in \mathbb{N} \, (w_n, w_{n+1}) \in E(G)$

and $\lim_{n\to\infty} w_n = w$. Therefore, from (ii), we get $\forall n \in \mathbb{N} \, (w_n, w_{n+1}) \in E(G)$. Finally, by (iii), we have

there exists $w_0 \in \xi$ such that $(w_0, Sw_0) \in E(G)$.

Using an analog argument, we can indicate $\mu(w_0, Sw_0) \leq 1$. If $\alpha(w,u) \geq 1, \mu(w,u) \leq 1$, then $(w,u) \in E(G)$. Therefore, overall circumstances of Theorem 1 are fulfilled, so S hold a fixed point.

If G is a connected graph, then assertion (iii) of Theorem 2 is naturally derived. Hence, we get the conclusion below.

Corrollary 1. Let (ξ, d_G) be a BMS via G and $S: \xi \to \xi$ a (ψ,φ) −graphic weakly contractive mapping satisfying the undermentioned assertions:

(i) there exists $w_0 \in \xi$ such that $(w_0, Sw_0) \in E(G)$;

(ii) S is G −continuous or if $\{w_n\} \subseteq \xi$ such that

$$(w_n, w_{n+1}) \in E(G) \; and \; \lim_{n\to\infty} w_n = w \;\Rightarrow \forall n \in \mathbb{N} \, (w_n, w) \in E(G);$$

(iii) G is a connected graph.

Then S hold a fixed point.

CONCLUSION

In the this writing, we present extension of a (ψ, φ) −weakly contractive mapping involving $(\mathcal{F}, h)$ upper class on BMS and fixed point outcomes for a (ψ, φ) −graphic weakly contractive mapping. Our deductions conduce a more general approximation to such a contractions engendered by Hussain et al. and Huang et al. (Hussain et al. 2013 and Huang et al. 2017).

Conflict of Interest

The article author declare that there is no conflict of interest.

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