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ALTERED NUMBERS OF LUCAS NUMBER SQUARED

Fikri KÖKEN[1,*], Emre KANKAL[2]

[1]Necmettin Erbakan University, Seydisehir Ahmet Cengiz Faculty of Engineering, Department of Computer Engineering, Konya, TURKEY. fkoken@erbakan.edu.tr, ORCID: 0000-0002-8304-9525
[2]Necmettin Erbakan University, Institute of Science and Technology, Department of Mathematics, Konya, TURKEY. kangalemre56@gmail.com, ORCID: 0000-0002-2707-5323

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ABSTRACT

We investigate two types altered Lucas numbers denoted $G^{(2)}_{L(n)}(a)$ and $H^{(2)}_{L(n)}(a)$ defined by adding or subtracting a value $\{a\}$ from the square of the n^{th} Lucas numbers. We achieve these numbers form as the consecutive products of the Fibonacci numbers. Therefore, consecutive sum-subtraction relations of altered Lucas numbers and their Binet-like formulas are given by using some properties of the Fibonacci numbers. Also, we explore the gcd sequences of *r*–successive terms of altered Lucas numbers denoted $\left\{G^{(2)}_{L(n),r}(a)\right\}$ and $\left\{H^{(2)}_{L(n),r}(a)\right\}$, $r=1,2$, $a\in\{1,9\}$ according to the greatest common divisor (gcd) properties of consecutive terms of the Fibonacci numbers. We show that these sequences are periodic or Fibonacci sequences.

Keywords: *Altered Lucas numbers, Greatest common divisor (gcd) sequences, Fibonacci sequence*.

1. INTRODUCTION

One can produce the Lucas sequence by using a recurrence relation $L_n = L_{n-1} + L_{n-2}$ $n \geq 2$ with initial conditions $L_0 = 2$ and $L_1 = 1$. The Lucas sequence $\{L_n\}_{n=0}^{\infty}$ consists of the numbers $\{2,1,3,4,7,11,18,...\}$ (Lucas numbers are sequence number A000032 in OEIS [1]). Also, the n^{th} Lucas number can be presented with the Binet formula $L_n = \alpha^n + \beta^n$, $\alpha,\beta = \left(1 \pm \sqrt{5}\right)/2$, $n \in Z^+$. The Binet formula is used to generalize indices from $n \in Z^+$ to $n \in Z$ such as $L_{-n} = (-1)^n L_n$, and to prove some properties of the Lucas numbers, such as the Cassini identity $L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n-1}$, subscript

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sum $F_{m+1}L_n + F_mL_{n-1} = L_{m+n}$, and subscript subtraction $(-1)^n (F_{m+1}L_n - F_mL_{n+1}) = L_{m-n}$ identities. Similarly, let $F_0 = 0$ and $F_1 = 1$ be initial conditions, then a n^{th} Fibonacci number is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n \in Z$. The Fibonacci sequence $\{F_n\}_{-\infty}^{\infty}$ consists of numbers $\{..,2,-1,1,0,1,1,2,,..\}$ (A147316). The n^{th} Fibonacci number is given with the Binet formula $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, $\alpha, \beta = (1 \pm \sqrt{5})/2$, $n \in Z$. In addition, the following equations that can act as any bridge between the Fibonacci F_n and Lucas L_n numbers, $F_{n+1} + F_{n-1} = L_n$, $2F_{n+m} = F_mL_n + F_nL_m$, $L_{n+1} + L_{n-1} = 5F_n$, $2L_{n+m} = L_mL_n + 5F_nF_m$ are valid as well-known properties in the literature. The proof of many equations belonging to the Fibonacci and Lucas numbers can be given by using the Fibonacci and Lucas Binet formulas [2].

Now, we give a lot of sum properties as examples of sequences produced from the Lucas numbers. A sum of the Lucas numbers is $\sum_{i=1}^{n} L_i = L_{n+2} - 3$ (Concerned with sequence A027961 in OEIS [1]). A sum of single-indices Lucas numbers is found as $\sum_{i=1}^{n} L_{2i-1} = L_{2n} - 2$ (A004146). A sum of the even-indices Lucas numbers is $\sum_{i=1}^{n} L_{2i} = L_{2n+1} - 1$. A sum of the square of the Lucas numbers is $\sum_{i=1}^{n} L_i^2 = L_nL_{n+1} - 2$ (A005970) [2]. In [3,4], the authors consider these results as any sequence, and these sequences are studied as altered Lucas sequences.

In [3], the author defined the shifted Lucas numbers $\{L_n + a\}_{n\geq 0}$ derived from the Lucas sequences and established a gcd sequence denoted $\{l_n(a)\}_{n\geq 0} = \{gcd(L_n + a, L_{n+1} + a)\}_{n\geq 0}$ by taking their greatest common divisor of them. The sequence $\{l_n(a)\}_{n\geq 0}$ is bounded by its values $\{|a^2 \pm 5|\}$ as $l_{2n-1}(a) \leq a^2 + 5$, $l_{2n}(a) \leq |a^2 - 5|$. When $a = 1$, the sequence $\{l_n(1)\}_{n\geq 0}$ is a periodic sequence that appears to take the following values $l_{4n-1}(1) = \{3,1,6,1,3,2\}$, $n \in Z_6$; $l_{4n}(1) = \{1,4,1\}$, $l_{4n+1}(1) = \{2,1,1\}$, $l_{4n+2}(1) = \{1,1,4\}$, $n \in Z_3$. He compared the bounded inequalities according to the values found for the sequence $\{l_n(1)\}$.

F. Koken study on the altered sequences $\{L_n^+\}_{n>0}$ and $\{L_n^-\}_{n>0}$; these consist of numbers L_n^+ and L_n^-, are defined as when n is odd, $L_n^+ = L_n - 1$ and $L_n^- = L_n + 1$; when n is even, $L_n^+ = L_n + 3$ and $L_n^- = L_n - 3$. Let L_n^+ be the n^{th} altered numbers, $L_{4k}^+ = 5F_{2k+1}F_{2k-1}$, $L_{4k+1}^+ = 5F_{2k+1}F_{2k}$, $L_{4k+2}^+ = L_{2k+2}L_{2k}$

and $L^{+}_{4k+3}=L_{2k+2}L_{2k+1}$ are given. The entities of the $\{L^{-}_{n}\}$ have shown the numbers $L^{-}_{4k}=L_{2k+1}L_{2k-1}$, $L^{-}_{4k+1}=L_{2k+1}L_{2k}$, $L^{-}_{4k+2}=5F_{2k+2}F_{2k}$ and $L^{-}_{4k+3}=5F_{2k+2}F_{2k+1}$. In addition, let $L^{\pm}_{n,r}=\left(L^{\pm}_{n},L^{\pm}_{n+r}\right)$ denote *r*-successive gcd numbers, the sequence $\{L^{+}_{4k,1}\}_{k\geq1}$ is equal to the subsequence $\{5F_{2k+1}\}_{k\geq1}$, and the $\{L^{+}_{4k-2,1}\}_{k\geq1}$ is equal to the subsequence $\{L_{2k}\}_{k\geq1}$. Also, the numbers $L^{-}_{n,1}$ has been given with equalities $L^{-}_{4k,1}=L_{2k+1}$ and $L^{-}_{4k+2,1}=5F_{2k+2}$. Also, according to values $r=2,3,4$, the gcd numbers $L^{+}_{n,r}$ and $L^{-}_{n,r}$ are obtained in [4].

We establish this paper as follows. In Section 2, we give a brief overview of necessary definitions and identities. In Section 3.1, we define two altered sequences, and explore properties of sums, difference, Binet's formula and closed forms for the numbers $G^{(2)}_{L(n)}(a)$ and $H^{(2)}_{L(n)}(a)$. In Section 3.2, we establish two types r-successive altered Lucas gcd sequences denoted with $G^{(2)}_{L(n),r}(a)$ and $H^{(2)}_{L(n),r}(a)$ for the values $G^{(2)}_{L(n)}(a)$ and $H^{(2)}_{L(n)}(a)$, and investigate these sequences according to the cases $r=1,2$.

2. MATERIAL AND METHOD

The gcd property of integer sequences can be given as $(F_m,F_n)=(F_n,F_r)$ for $m=qn+r$ all $m,n,r,q\in N$, where F_n is the n^{th} Fibonacci number. Thus, it is seen that the greatest common divisor of two Fibonacci numbers is a Fibonacci number such as $(F_m,F_n)=F_{(m,n)}$. For example, two successive Fibonacci numbers are relatively prime, $(F_n,F_{n+1})=(F_n,F_{n+2})=1$ in [1,2].

According to whether n is odd or even in Lucas identities known as the Cassini identity $L^2_n-5(-1)^n=L_{n+1}L_{n-1}$ and $L^2_n-4(-1)^n=5F^2_n$, we can obtain the equations $L^2_{2k+1}+5=L_{2k+2}L_{2k}$, $L^2_{2k}-5=L_{2k+1}L_{2k-1}$, $L^2_{2k+1}+4=5F^2_{2k+1}$ and $L^2_{2k}-4=5F^2_{2k}$ [2]. We inspire by these equations for this question, "Can any altered Lucas sequences such as $\{L^2_n\pm a\}$ be defined?".

Also, in the literature, there have been a great many papers studying sums of *1*-consecutive products of the Lucas numbers; $\sum_{i=1}^{2n}L_iL_{i+1}=L^2_{2n+1}-1$ or $\sum_{i=0}^{2n}L_iL_{i+1}=L^2_{2n+1}+1$ and $\sum_{i=1}^{2n+1}L_iL_{i+1}=L^2_{2n+2}-6$ or $\sum_{i=0}^{2n+1}L_iL_{i+1}=L^2_{2n+2}-4$ [2,5,6]. We can consider the results of these sums as altered Lucas numbers motivated by these sums.

Now, we will develop a theory using the following equations:

$$L_{m+n+1}^2 + L_{m-n}^2 = 5F_{2m+1}F_{2n+1}, \tag{1}$$

$$L_{m+n}^2 - L_{m-n}^2 = 5F_{2m}F_{2n}. \tag{2}$$

The identities in Eq. 1 and Eq. 2 can be proved using Binet's formula. We have mainly used the identities in Eq. 1 and Eq. 2 to obtain the following equations, but one can use Binet's formula for their proofs.

Lemma 1. Let F_n and L_n be the n^{th} Fibonacci and Lucas number, then

$$L_{2k}^2 + 1 = 5F_{2k-1}F_{2k+1}, \tag{3}$$

$$L_{2k+1}^2 - 1 = 5F_{2(k+1)}F_{2k}, \tag{4}$$

$$L_{2k+1}^2 + 9 = 5F_{2k+3}F_{2k-1}, \tag{5}$$

$$L_{2k}^2 - 9 = 5F_{2(k+1)}F_{2(k-1)}. \tag{6}$$

Proof: For $m = k+1$ and $n = k$ in Eq. 1, we have obtained $L_{2k+2}^2 + L_1^2 = 5F_{2k+3}F_{2k+1}$. Let $m = k+2$ and $n = k$ in Eq. 2, then we have achieved $L_{2k+2}^2 - L_2^2 = 5F_{2k+4}F_{2k}$. The others are given in similar ways. ■

In [7], [8], the identities in Eq. 3 and Eq. 4 given within the preliminary information section are again shown in Lemma 1 with a different proof method. In [7], [8], the authors have investigated solutions of the diophantine equation of the form $A_{n_1}A_{n_2}...A_{n_k} \pm 1 = B_m^2$, where A_n and B_n are either the n^{th} Fibonacci number or Lucas number.

The problem of finding all integral solutions to this diophantine equation is known as the Brocard–Ramanujan problem. These studies show that altered Lucas numbers $\{L_n^2 \pm 1\}$ will play a significant part in the Diophantine equations applications of the numbers theory. That is, one can explore solutions of some diophantine equations of form $A_{n_1}A_{n_2}...A_{n_k} \pm a = L_m^2$.

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3. ALTERED SEQUENCES OF LUCAS NUMBERS SQUARED

In this section, let's define two types of altered numbers derived from the n^{th} Lucas number squared for a value $\{a\}$ according to whether their indices are even or old, respectively.

3.1. $G_{L(n)}^{(2)}(a)$ and $H_{L(n)}^{(2)}(a)$ Altered Lucas Numbers

Let L_n be the n^{th} Lucas number. Altered Lucas numbers are defined as

$$G_{L(n)}^{(2)}(a)=L_n^2+(-1)^n a, \tag{7}$$

$$H_{L(n)}^{(2)}(a)=L_n^2-(-1)^n a, \tag{8}$$

and also, the altered Lucas sequences are denoted as $\left\{G_{L(n)}^{(2)}(a)\right\}_{n=0}^{\infty}$ and $\left\{H_{L(n)}^{(2)}(a)\right\}_{n=0}^{\infty}$.

For example, the numbers $G_{L(n)}^{(2)}(1)=H_{L(n)}^{(2)}(-1)$ and $H_{L(n)}^{(2)}(9)=G_{L(n)}^{(2)}(-9)$ are given in Table 1.

Table 1. $G_{L(n)}^{(2)}(1)$ and $H_{L(n)}^{(2)}(9)$, altered Lucas numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$G_{L(n)}^{(2)}(1)$	5	0	10	15	50	120	325	840	2210	5775	15130	39600	103685
$H_{L(n)}^{(2)}(9)$	-5	10	0	25	40	130	315	850	2200	5785	15120	39610	103675

Table 1 shows that they are any increasing sequences with special values except for the first values, and also, these numbers are divisible by the Fibonacci number $F_5=5$. Thus, some sums of *1*-consecutive products of the Lucas numbers are divisible by $F_5=5$ such as $\sum_{i=1}^{2n} L_i L_{i+1}=G_{L(2n+1)}^{(2)}(1)$, $\sum_{i=2}^{2n+1} L_i L_{i+1}=H_{L(2n+2)}^{(2)}(9)$ and $\sum_{i=0}^{2n+1} L_i L_{i+1}=H_{L(2n+2)}^{(2)}(4)$. It is clearly seen from the Fibonacci identities $L_n^2-5(-1)^n=L_{n+1}L_{n-1}$ and $L_n^2-4(-1)^n=5F_n^2$, we have

$$H_{L(n)}^{(2)}(4)=G_{L(n)}^{(2)}(-4)=5F_n^2, \tag{9}$$

$$H_{L(n)}^{(2)}(5)=G_{L(n)}^{(2)}(-5)=L_{n+1}L_{n-1}. \tag{10}$$

But, we give the closed forms of the altered sequences $\left\{G_{L(n)}^{(2)}(1)\right\}$ and $\left\{H_{L(n)}^{(2)}(9)\right\}$ as follows.

Theorem 1. Let $G_{L(n)}^{(2)}(1)$ and $H_{L(n)}^{(2)}(9)$ denote the n^{th} altered numbers of the Lucas numbers squared, then they are valid:

$$G_{L(n)}^{(2)}(1)=5F_{n+1}F_{n-1}, \tag{11}$$

$$H_{L(n)}^{(2)}(9)=5F_{n+2}F_{n-2}. \tag{12}$$

Proof: If we use the identity given in Eq. 3 for $a=1$ and $n=2k$ at the definition in Eq. 7 then $G_{L(2k)}^{(2)}(1)$ is given as $G_{L(2k)}^{(2)}(1)=5F_{2k-1}F_{2k+1}$, and if we use the Eq. 4 for $a=1$ and $n=2k+1$ in Eq. 7, $G_{L(2k+1)}^{(2)}(1)$ is given $G_{L(2k+1)}^{(2)}(1)=5F_{2(k+1)}F_{2k}$. Therefore, the number $G_{L(n)}^{(2)}(1)=5F_{n+1}F_{n-1}$ is obtainedby considering according to $n=2k$ and $n=2k+1$ situations. If we use the Eq. 5 for $a=9$ and $n=2k+1$ at the definition in Eq. 8, then $H_{L(2k+1)}^{(2)}(9)$ equal $5F_{2k+3}F_{2k-1}$. And if we use the identity in Eq. 6 for $a=9$ and $n=2k$ in Eq. 8, $H_{L(2k)}^{(2)}(9)$ equal $5F_{2(k+1)}F_{2(k-1)}$. We have $H_{L(n)}^{(2)}(9)=5F_{n+2}F_{n-2}$ is seen from $n=2k$ and $n=2k+1$ situation. ■

Now, let's research about some sum and subtraction identities of the numbers $G_{L(n)}^{(2)}(1)$ and $H_{L(n)}^{(2)}(9)$

Theorem 2. $G_{L(n)}^{(2)}(1)$ and $H_{L(n)}^{(2)}(9)$ are the n^{th} altered numbers of the Lucas numbers squared, then

$$G_{L(n)}^{(2)}(1)+G_{L(n+1)}^{(2)}(1)=H_{L(n)}^{(2)}(9)+H_{L(n+1)}^{(2)}(9)=5F_{2n+1}, \tag{13}$$

$$G_{L(n+1)}^{(2)}(1)-G_{L(n-1)}^{(2)}(1)=H_{L(n+1)}^{(2)}(9)-H_{L(n-1)}^{(2)}(9)=5F_{2n}, \tag{14}$$

$$2G_{L(n+1)}^{(2)}(1)+G_{L(n)}^{(2)}(1)-G_{L(n-1)}^{(2)}(1)=5F_{2n+2}, \tag{15}$$

$$2H_{L(n+1)}^{(2)}(9)+H_{L(n)}^{(2)}(9)-H_{L(n-1)}^{(2)}(9)=5F_{n+1}L_{n+1}. \tag{16}$$

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Proof If we have rewritten identities in Eq. 13 and Eq. 14 using the identities in Eq. 11 and Eq. 12, then, $G_{L(n)}^{(2)}+G_{L(n+1)}^{(2)}=5\left(F_{n+1}\left(F_{n-1}+F_n\right)+F_n^2\right)=5F_{2n+1}$ and $H_{L(n+1)}^{(2)}-H_{L(n-1)}^{(2)}=5F_n\left(F_{n+1}+F_{n-1}\right)=5F_nL_n$ are obtained by the identities $F_n^2+F_{n+1}^2=F_{2n+1}$ and $F_nL_n=F_{2n}$. Since the other relations are made similarly, they are not given for brevity. If we sum identities in Eq. 13 and Eq. 14 side-to-side collection, we get identities in Eq. 15 and Eq16. ∎

As a result, the sum of two successive altered Lucas numbers equals the Fibonacci number. Using the Fibonacci Binet formula, a Binet-like formula for the numbers $G_{L(n)}^{(2)}(1)$ and $H_{L(n)}^{(2)}(9)$ can be obtained.

Theorem 3. Let $G_{L(n)}^{(2)}(1)$ and $H_{L(n)}^{(2)}(9)$ be the n^{th} altered numbers of the Lucas numbers squared, then

$$G_{L(n)}^{(2)}(1)=\left(\alpha^{n+1}-\beta^{n+1}\right)\left(\alpha^{n-1}-\beta^{n-1}\right), \tag{17}$$

$$H_{L(n)}^{(2)}(9)=\left(\alpha^{n+2}-\beta^{n+2}\right)\left(\alpha^{n-2}-\beta^{n-2}\right). \tag{18}$$

Proof: They appear as an application of the Fibonacci Binet formula from closed forms in Eq. 11 and Eq. 12. ∎

The identities in Eq. 16 and Eq. 17 are referred to as Binet-like formulas for the numbers $G_{L(n)}^{(2)}(1)$ and $H_{L(n)}^{(2)}(9)$. They can be utilized to establish various properties of the numbers $G_{L(n)}^{(2)}(1)$ and $H_{L(n)}^{(2)}(9)$. Additional information and applications of these formulas in sequences $a(n)=F_nF_{n+2}$ and $b(n)=F_nF_{n+4}$ can be found in the sequences (A059929) and (A192883).

Theorem 4. Let $G_{L(n)}^{(2)}\left(L_t^2\right)$ and $H_{L(n)}^{(2)}\left(L_t^2\right)$ be the n^{th} altered numbers of the Lucas numbers squared, then

$$G_{L(n)}^{(2)}\left(L_t^2\right)=5F_{n+t}F_{n-t},\quad t\, is\, odd\ , \tag{19}$$

$$H_{L(n)}^{(2)}\left(L_t^2\right)=5F_{n+t}F_{n-t},\quad t\, is\, even\ , \tag{20}$$

where L_t^2 is the square of the t^{th} Lucas numbers used in place of $\{a\}$.

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Proof. If we have rewritten values of $m=k+(t+1)/2$ and $n=k-(t-1)/2$ in Eq. 1 for t is odd, then $G^{(2)}_{L(2k+1)}(L_t^2)=5F_{2k+t+1}F_{2k-t+1}$ is given with according to $a=L_t^2$ and $n=2k+1$ in Eq. 7. Also, if they are taken $m=k+(t+1)/2$ and $n=k-(t-1)/2$ in Eq. 2, the $G^{(2)}_{L(2k)}(L_t^2)=5F_{2k+t}F_{2k-t}$ is $a=F_t^2$ and $n=2k$ in Eq. 7.

Similarly, if we consider values of $m=k+t/2$ and $n=k-t/2$ in Eq. 1 and Eq. 2 when t is even, according to $a=L_t^2$ in Eq. 8, they are obtained as the $H^{(2)}_{L(2k+1)}(L_t^2)$ and $H^{(2)}_{L(2k)}(L_t^2)$, which are produce the identity in Eq. 20. ∎

Also, the general terms of the altered sequences $\{G^{(2)}_{L(n)}(L_2^2)\}$ and $\{H^{(2)}_{L(n)}(L_1^2)\}$ can be given by the Fibonacci identities as $G^{(2)}_{L(n)}(9)=5F_n^2+13(-1)^n$ and $H^{(2)}_{L(n)}(1)=5F_n^2+3(-1)^n=F_{3m}/F_m$ (A047946). But, they are the form of other altered Fibonacci sequences. In addition, they could not be generalized as the product of Fibonacci or Lucas numbers.

3.2. $G^{(2)}_{L(n),r}(1)$ and $H^{(2)}_{L(n),r}(9)$ Altered Lucas Gcd Sequences

We have examined properties related to the greatest common divisor (gcd) of two numbers whose indices differ r from the altered sequences, definitions of whose are given

$$G^{(2)}_{L(n),r}(a)=\left(G^{(2)}_{L(n)}(a),G^{(2)}_{L(n+r)}(a)\right), \tag{21}$$

$$H^{(2)}_{L(n),r}(a)=\left(H^{(2)}_{L(n)}(a),H^{(2)}_{L(n+r)}(a)\right), \tag{22}$$

where $G^{(2)}_{L(n)}(a)$ and $H^{(2)}_{L(n)}(a)$ be the n^{th} altered Lucas numbers. The sequences $\{G^{(2)}_{L(n),r}(a)\}$ and $\{H^{(2)}_{L(n),r}(a)\}$ formed by these numbers are called the r-successive altered Lucas gcd sequences.

Now, the numbers $G^{(2)}_{L(n),1}(1)$ and $H^{(2)}_{L(n),1}(9)$ are sampled in Table 2.

Table 2. $G^{(2)}_{L(n),1}(1)$ and $H^{(2)}_{L(n),1}(9)$, *1*-successive altered Lucas gcd numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$G^{(2)}_{L(n),1}(1)$	5	10	5	5	10	5	5	10	5	5	10	5	5

$H^{(2)}_{L(n),1}(9)$	5	10	25	5	10	5	5	50	5	5	10	5	25

The special values in Table 2 show that the sequences $\left\{G^{(2)}_{L(n),1}(1)\right\}$ and $\left\{H^{(2)}_{L(n),1}(9)\right\}$ are not increasing or decreasing. But, they can be periodic. Thus, we have studied whether or not the *1*-successive altered Lucas gcd sequences take special values in certain periods.

Theorem 5. Let $G^{(2)}_{L(n),1}(1)$ and $H^{(2)}_{L(n),1}(9)$ be the n^{th} *1*-successive altered Lucas gcd numbers, then

$$G^{(2)}_{L(n),1}(1)=\begin{cases}10, & n\equiv 1(\text{mod}\,3)\\ 5, & otherwise\end{cases}, \tag{23}$$

$$H^{(2)}_{L(n),1}(9)=\begin{cases}50, & n\equiv 7(\text{mod}\,15)\\ 25 & k\equiv 2,12(\text{mod}\,15)\\ 10, & k\equiv 1,4,10,13(\text{mod}\,15)\\ 5, & otherwise\end{cases}. \tag{24}$$

Proof: We have rewritten the number $G^{(2)}_{L(2k),1}(1)=\left(G^{(2)}_{L(2k)}(1),G^{(2)}_{L(2k+1)}(1)\right)=5\left(F_{2k+1}F_{2k-1},F_{2k+2}F_{2k}\right)$, according to the closed form in Eq. 11 and the definition in Eq. 21. By using the property $(F_n,F_{n+1})=1$, we have $(F_{2k+1},F_{2k+2})=(F_{2k+1},F_{2k})=(F_{2k-1},F_{2k})=1$. So, we should examine the situation (F_{2k-1},F_{2k+2}). By using the property $2|F_{3n}$, if we have $2k-1\equiv 0(\text{mod}\,3)$ and $2k+2\equiv 0(\text{mod}\,3)$ then $k\equiv 2(\text{mod}\,3)$ $\Leftrightarrow$ $(F_{2k-1},F_{2k+2})=2$. Otherwise, $(F_{2k-1},F_{2k+2})=1$. It is seen that

$$G^{(2)}_{L(2k),1}(1)=5\left(F_{2k+1}F_{2k-1},F_{2k+2}F_{2k}\right)=\begin{cases}10, & k\equiv 2(\text{mod}\,3)\\ 5, & otherwise\end{cases}. \tag{25}$$

Also, we have $G^{(2)}_{L(2k+1),1}(1)=5\left(F_{2(k+1)}F_{2k},F_{2k+1}F_{2k+3}\right)$, according to the identities in Eq. 11 and Eq. 21. Since $(F_n,F_{n+1})=1$, we can write $(F_{2k+2},F_{2k+3})=(F_{2k+2},F_{2k+1})=(F_{2k},F_{2k+1})=1$. So, we should examine the situation (F_{2k},F_{2k+3}). So, if we have $2k\equiv 0(\text{mod}\,3)$ and $2k+3\equiv 0(\text{mod}\,3)$ then $k\equiv 0(\text{mod}\,3)$ $\Leftrightarrow$ $(F_{2k},F_{2k+3})=2$ by using the property $2|F_{3n}$. Orherwise, $(F_{2k},F_{2k+3})=1$. It is obtained as

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$$G^{(2)}_{L(2k+1),1}(1)=5\left(F_{2(k+1)}F_{2k},F_{2k+1}F_{2k+3}\right)=\begin{cases}10, & k\equiv 0(\bmod 3)\\ 5, & otherwise\end{cases}. \tag{26}$$

Whether the index is even or odd from the identities found in Eq. 25 and Eq. 26, it is seen that $k\equiv 2(\bmod 3)$ for $n=2k$; and $k\equiv 0(\bmod 3)$ for $n=2k+1$. Thus, we find $G^{(2)}_{L(n),1}(1)=10$ for $n\equiv 1(\bmod 3)$. In the other cases, then $G^{(2)}_{L(n),1}(1)=5$.

Similarly, we have $H^{(2)}_{L(2k),1}(9)=\left(H^{(2)}_{L(2k)}(9),H^{(2)}_{L(2k+1)}(9)\right)=5\left(F_{2k+2}F_{2k-2},F_{2k+3}F_{2k-1}\right)$, according to identity in Eq. 12 and the definition in Eq. 22. We consider $H^{(2)}_{L(2k),1}(9)=5\left(F_{2k-2},F_{2k+3}\right)\left(F_{2k+2},F_{2k-1}\right)$ since $\left(F_{2k+2},F_{2k+3}\right)=\left(F_{2k-2},F_{2k-1}\right)=1$. Using the property $\left(F_x,F_y\right)=F_{(x,y-x)}$, we rewrite their identities

$$\left(F_{2k-2},F_{2k+3}\right)=F_{(2k-2,2k+3)}=F_{(2k-2,5)}=F_5,\quad 2k-2\equiv 0(\bmod 5), \tag{27}$$

$$\left(F_{2k+2},F_{2k-1}\right)=F_{(2k+2,2k-1)}=F_{(3,2k-1)}=F_3,\quad 2k-1\equiv 0(\bmod 3). \tag{28}$$

It is seen if $k\equiv 1(\bmod 5)$, then $\left(F_{2k-2},F_{2k+3}\right)=5$; and if $k\equiv 2(\bmod 3)$ then $\left(F_{2k+2},F_{2k-1}\right)=2$. According to the Chinese remainder theorem, we obtain $H^{(2)}_{L(2k),1}(9)=50$ for $k\equiv 11(\bmod 15)$. The desired results for the products of the two expressions in their possible cases are obtained as

$$H^{(2)}_{L(2k),1}(9)=5\left(F_{2k-2},F_{2k+3}\right)\left(F_{2k+2},F_{2k-1}\right)=\begin{cases}50, & k\equiv 11(\bmod 15)\\ 25 & k\equiv 1,6(\bmod 15)\\ 10, & k\equiv 2,5,8,14(\bmod 15)\\ 5, & otherwise\end{cases}. \tag{29}$$

Same way, according to identities in Eq. 12 and Eq. 22, we have $H^{(2)}_{L(2k+1),1}(9)=5\left(F_{2k+3}F_{2k-1},F_{2k+4}F_{2k}\right)$. Because of $\left(F_{2k+3},F_{2k+4}\right)=\left(F_{2k-1},F_{2k}\right)=1$, we can rewrite $H^{(2)}_{L(2k+1),1}(9)=5\left(F_{2k+3},F_{2k}\right)\left(F_{2k-1},F_{2k+4}\right)$. Using the properties $\left(F_x,F_y\right)=F_{(x,y-x)}$, we have $\left(F_{2k},F_{2k+3}\right)=F_{(2k,3)}=F_3$, $2k\equiv 0(\bmod 3)$ and $\left(F_{2k+4},F_{2k-1}\right)=F_{(5,2k-1)}=F_5$, $2k-1\equiv 0(\bmod 5)$. It is seen that if $k\equiv 0(\bmod 3)$ then $\left(F_{2k},F_{2k+3}\right)=2$; and if $k\equiv 3(\bmod 5)$ then $\left(F_{2k+4},F_{2k-1}\right)=5$. According to the Chinese remainder theorem, we obtain as $H^{(2)}_{L(2k+1),1}(9)=50$ for $k\equiv 3(\mathrm{mod}15)$. The desired results for the products of the two expressions in their possible cases are obtained as

$$H^{(2)}_{L(2k+1),1}(9)=5(F_{2k},F_{2k+3})(F_{2k+4},F_{2k-1})=\begin{cases}50, & k\equiv 3(\mathrm{mod}15)\\ 25, & k\equiv 8,13(\mathrm{mod}15)\\ 10, & k\equiv 0,6,9,12(\mathrm{mod}15)\\ 5, & otherwise\end{cases}. \tag{30}$$

According to whether the indices are $n=2k$ and $n=2k+1$ even or odd from the values found in Eq. 29 and Eq. 30, respectively we consider $H^{(2)}_{L(2k),1}(9)=50$ for $k\equiv 11(\mathrm{mod}15)$ and $H^{(2)}_{L(2k+1),1}(9)=50$ for $k\equiv 3(\mathrm{mod}15)$. Thus, we find $H^{(2)}_{L(n),1}=50$ for $n\equiv 7(\mathrm{mod}15)$. When it's appropriate case in Eq. 29 and Eq. 30, it is follow $k\equiv 1,6(\mathrm{mod}15)$ for $n=2k$ and $k\equiv 8,13(\mathrm{mod}15)$ for $n=2k+1$, it is $H^{(2)}_{L(n),1}=25$, $n\equiv 2,12(\mathrm{mod}15)$. If the other cases are written in their place, desired results are obtained similar way. ■

For terms of the *2*-successive altered gcd sequences, let's create Table 3:

Table 3. $G^{(2)}_{L(n),2}(1)$ and $H^{(2)}_{L(n),2}(9)$, *2*-successive altered Lucas gcd numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$G^{(2)}_{L(n),2}(1)$	5	15	10	15	25	120	65	105	170	825	445	720	1165
$H^{(2)}_{L(n),2}(9)$	5	5	40	5	5	10	5	5	40	5	5	10	5

In Table 3, it is seen that the *2*-successive altered Lucas gcd sequence $\{G^{(2)}_{L(n),2}(1)\}$, $n\geq 2$ takes values according to a specific increasing sequence. The sequence $\{H^{(2)}_{L(n),2}(9)\}$ is periodic. Now let's give the properties of these sequences.

Theorem 6. Let $G^{(2)}_{L(n),2}(1)$ and $H^{(2)}_{L(n),2}(9)$ be the n^{th} *2*-successive altered Lucas gcd numbers, then

$$G^{(2)}_{L(n),2}(1)=\begin{cases}15F_{n+1}, & n\equiv 1(\mathrm{mod}\,4)\\ 5F_{n+1}, & otherwise\end{cases},\quad H^{(2)}_{L(n),2}(9)=\begin{cases}40, & k\equiv 2(\mathrm{mod}\,6)\\ 10, & k\equiv 5(\mathrm{mod}\,6)\\ 5, & otherwise\end{cases}. \tag{31}$$

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Proof: According to the identity in Eq. 11, we write $G^{(2)}_{L(n),2}(1)=5F_{n+1}(F_{n-1},F_{n+3})$. So, we have $(F_{n-1},F_{n+3})=F_{(n-1,4)}=F_4$, $n\equiv 1(\mathrm{mod}\,4)$ by using the property $(F_x,F_y)=F_{(x,y-x)}$ and $G^{(2)}_{L(n),2}(1)=5F_4F_{n+1}=15F_{n+1}$ for $n\equiv 1(\mathrm{mod}\,4)$. Otherwise, $(F_{n-1},F_{n+3})=F_{(n-1,4)}=F_2$ or F_1. Since $F_1=F_2=1$, we have $G^{(2)}_{L(n),2}(1)=5F_{n+1}$ for $n\not\equiv 1(\mathrm{mod}\,4)$.

According to the identity in Eq. 12, we write $H^{(2)}_{L(n),2}(9)=5(F_{n+2}F_{n-2},F_{n+4}F_n)$. Since $(F_{2k+2},F_{2k})=(F_{2k+2},F_{2k+4})=(F_{2k-2},F_{2k})=1$, we can take as $H^{(2)}_{L(n),2}(9)=5(F_{n-2},F_{n+4})$. Thus, we get $H^{(2)}_{L(n),2}(9)=5F_{(n-2,6)}=5F_6$, $n\equiv 2(\mathrm{mod}\,6)$. Otherwise, the others are $H^{(2)}_{L(n),2}(9)=5F_{(n-2,6)}=5F_3$, $n\equiv 5(\mathrm{mod}\,6)$; or $5F_2$, $n\equiv 0,4(\mathrm{mod}\,6)$; or $5F_1$, $n\equiv 1,3(\mathrm{mod}\,6)$. ■

Theorem 7. Let $G^{(2)}_{L(n),2}$ be the n^{th} *2*-successive altered Lucas gcd number, then

$$G^{(2)}_{L(n+1),2}(1)+G^{(2)}_{L(n),2}(1)=\begin{cases}5(F_{n+1}+L_{n+2}), & n\equiv 1(\mathrm{mod}\,4)\\ 5L_{n+3}, & n\equiv 0(\mathrm{mod}\,4)\\ 5F_{n+3} & \mathit{otherwise}\end{cases}. \tag{32}$$

Proof: We know the number $G^{(2)}_{L(n),2}(1)=15F_{n+1}$ for $n\equiv 1(\mathrm{mod}\,4)$, otherwise it is $5F_{n+1}$. Thus,

$$G^{(2)}_{L(n+1),2}(1)+G^{(2)}_{L(n),2}(1)=\begin{cases}5(F_{n+2}+3F_{n+1}), & n\equiv 1(\mathrm{mod}\,4)\\ 5(3F_{n+2}+F_{n+1}), & n\equiv 0(\mathrm{mod}\,4)\\ 5(F_{n+1}+F_{n+2}), & \mathit{otherwise}\end{cases} \tag{33}$$

By using the identity $F_{n+1}+F_{n-1}=L_n$, we have

$$G^{(2)}_{L(n+1),2}(1)+G^{(2)}_{L(n),2}(1)=\begin{cases}5(F_{n+3}+2F_{n+1}), & n\equiv 1(\mathrm{mod}\,4)\\ 5(F_{n+2}+F_{n+4}), & n\equiv 0(\mathrm{mod}\,4)\\ 5F_{n+3}, & \mathit{otherwise}\end{cases}. \tag{34}$$

The desired result is achieved. ■

We will continue our work according to the particular values of these numbers given in Table 4, since closed-form expressions cannot be found for the numbers $G_{L(n)}^{(2)}(9)=H_{L(n)}^{(2)}(-9)$ and $H_{L(n)}^{(2)}(1)=G_{L(n)}^{(2)}(-1)$ for the value of $a=\{9,1\}$ in identities given Eq. 7 and Eq. 8, respectively.

Table 4. $G_{L(n)}^{(2)}(9)$ and $H_{L(n)}^{(2)}(1)$, altered Lucas numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$G_{L(n)}^{(2)}(9)$	13	-8	18	7	58	112	333	832	2218	5767	15138	39592	103693
$H_{L(n)}^{(2)}(1)$	3	2	8	17	48	122	323	842	2208	5777	15128	39602	103683

Table 4 shows that the sequences $G_{L(n),1}^{(2)}(9)=H_{L(n),1}^{(2)}(1)=\{1,2,1\}$, $n\in Z_3$; $H_{L(n),2}^{(2)}(1)=\{1,1,8,1,1,2\}$, $n\in Z_6$; $G_{L(n),3}^{(2)}(9)=\{1,2,2\}$, $n\in Z_3$ and $H_{L(n),3}^{(2)}(1)=\{1,2,2,17,2,2,1,2,2\}$, $n\in Z_9$ are periodic [9]. But, the proofs for these values have not been provided. Thus, these values have been determined through a computer program up to 100 for the numbers $G_{L(n),r}^{(2)}(9)$ and $H_{L(n),r}^{(2)}(1)$, $r=1,2,3$. In [9], it is the numbers $G_{L(n),2}^{(2)}(9)=\{1,1,2,7,1,16,1,1,2,1,1,56,1,1,2,1,1,16,1,7,2,1,1,8\}, n\in Z_{24}$.

4. CONCLUSION AND RECOMMENDATIONS

In our study, two types of altered Lucas numbers denoted $G_{L(n)}^{(2)}(a)$ and $H_{L(n)}^{(2)}(a)$ are derived with values $\{a\}$. We have shown that the numbers $G_{L(n)}^{(2)}(1)$ and $H_{L(n)}^{(2)}(9)$ equal some consecutive products of the Fibonacci numbers. Thus, *r*-successive altered Lucas gcd sequences $\{G_{L(n),r}^{(2)}(1)\}$ and $\{H_{L(n),r}^{(2)}(9)\}$ are studied for $r=1,2$. We have obtained these sequences are either periodic and bounded or primefree and unbounded. Also, we have generalized the value $\{a\}$ as the square of Lucas number such as

$$G_{L(n)}^{(2)}(L_t^2)=5F_{n+t}F_{n-t},\quad t\ is\ odd\ , \tag{35}$$

$$H_{L(n)}^{(2)}(L_t^2)=5F_{n+t}F_{n-t},\quad t\ is\ even\ . \tag{36}$$

Other properties of these sequences and their *r*-successive gcd sequences are left to the interested readers for future research. Nevertheless, we will consider some matrix and graph theory applications in the next articles.

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REFERENCES

[1] OEIS Foundation Inc. (2011). The on-line encyclopedia of integer sequences. http://oeis.org.

[2] Koshy T. (2001). Fibonacci and Lucas numbers with applications. John Wiley and Sons Inc, NY.

[3] Chen, K.W. (2011). Greatest common divisors in shifted Fibonacci sequences. Journal of Integer Sequences, 14, 11.4.7.

[4] Koken, F. (2020). The gcd sequences of the altered Lucas sequences. Annales Mathematicae Silesianae, 34, 2, 222-240. DOI: 10.2478/amsil-2020-0005.

[5] Cerin, Z. (2013). On factors of sums of consecutive Fibonacci and Lucas numbers. Annales Mathematicae et Informaticae, 41, 19–25.

[6] Tekcan, A., Gezer, B. and Bizim, O. (2007). Some relations on Lucas numbers and their sums. Advanced Studies in Contemporary Mathematics, 15, 195–211.

[7] Szalay, L. (2012). Diophantine equations with binary recurrences associated to the Brocard–Ramanujan problem. Portugaliae Mathematica, 69, 3, 213–220.

[8] Pongsriiam, P. (2017). Fibonacci and Lucas numbers associated with Brocard-Ramanujan equation. Communications of the Korean Mathematical Society, 32, 3, 511-522.

[9] Kankal E. (2023). The thesis of master of science, Necmettin Erbakan University. The Graduate School of Natural And Applied Science, Konya.