http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 72, Number 3, Pages 587–605 (2023)
DOI:10.31801/cfsuasmas.1252070
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: February 16, 2023; Accepted: March 28, 2023

TESTING EQUALITY OF MEANS IN ONE-WAY ANOVA USING THREE AND FOUR MOMENT APPROXIMATIONS

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Abstract. In this study, we focus on two test statistics for testing the equality of treatment means in one-way analysis of variance (ANOVA). The first one is the well known Cochran (C_{LS}) test statistic based on least squares (LS) estimators and the second one is robust version of it (RC_{MML}) based on modified maximum likelihood (MML) estimators. These two test statistics are asymptotically distributed as chi-square. However, distributions of them are unknown for small samples. Therefore, three-moment chi-square and four moment F approximations to the null distributions of C_{LS} and RC_{MML} are derived inspired by Tiku and Wong [19]. To investigate the small and moderate sample properties of these tests based on the mentioned approximations, an extensive Monte-Carlo simulation study is performed when the underlying distribution is long-tailed symmetric (LTS). Simulation results show that four-moment F approximation provides better approximation than the three-moment chi-square approximation for both C_{LS} and RC_{MML} tests. Therefore, the simulated Type I error rates and powers of the C_{LS} and RC_{MML} test statistics are calculated using four-moment F approximation. According to simulation results, RC_{MML} test is more powerful than the corresponding C_{LS} test.

1. Introduction

Testing the equality of treatment means in one-way analysis of variance (ANOVA) is one of the oldest problems in theoretical and applied statistics. The problem of interest can be stated in the following hypothesis

$$\begin{aligned} &H_0 : \mu_1 = \mu_2 = \cdots = \mu_a = \mu \quad vs. \\ &H_1 : \mu_i \neq \mu_j \quad \text{for some} \quad i \neq j. \end{aligned} \tag{1}$$

2020 *Mathematics Subject Classification.* 62F03, 62F05, 62F35.
Keywords. Cochran test statistic, three moment chi-square approximation, four-moment F approximation, Monte Carlo simulation.

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Classical F test based on least squares (LS) estimators is appropriate for testing the null hypothesis in (1) when the usual ANOVA assumptions such as independent and identically distributed normal error terms with constant variance are satisfied. Although the F test is relatively robust in terms of the size performance, it may lose power under assumption violations, see Gamage and Weerahandi [4], Hampel [6], Schrader and Hettmansperger [14] and Şenoğlu and Tiku [15] etc. There is an extensive literature focusing on one-way ANOVA under normality and heterogeneiy of variances assumptions. Therefore, a variety of tests have been developed and compared, see for example Brown and Forsythe [2], Cochran [3], James [8], Krishnamoorthy et al. [9], Li et al. [10], Mehrotra [11], Weerahandi [22], Welch [23], etc. for detailed information.

In this study, we are interested in Cochran [3] test statistic based on least squares (LS) estimators, denoted as C_{LS}. The reason of why we focus on this statistic is that many tests available in the literature are based on the C_{LS}. For example, Welch test is a modification of Cochran's test. In addition, C_{LS} is often used as the standard test for testing homogeneity in meta-analysis, see Hartung et al. [7]. As it is well known that this test statistic is proposed under normality and heterogeneity of variances assumptions. However, nonnormal distributions are encountered more frequently in practice. Therefore, Guven et al. [5] considered robust version of the Cochran test statistic based on modified maximum likelihood (MML) estimators, denoted as RC_{MML}, and fiducial based test using RC_{MML} for testing the equality of means when the underlying distribution is long-tailed symmetric (LTS). MML estimators proposed by Tiku [16,17] are asymptotically equivalent to the maximum likelihood (ML) estimators and more efficient than the LS estimators under nonnormality. Also, MML estimators are robust to the outliers, see Aydogdu et al. [1], Tiku et al. [20] and references therein.

It should be noted that C_{LS} and RC_{MML} test statistics have asymptotic chi-square distribution with $a-1$ degrees of freedom under H_0. Here, a denotes the number of treatments. However, their null distributions are difficult to obtain for small samples, even at moderate sample sizes. If one uses asymptotic distribution in small samples this results in highly liberal tests. To deal with this problem, in this study, two useful moment approximations for the small sample distributions of the C_{LS} and RC_{MML} test statistics are derived by inspiring the Tiku and Wong [19]. The former is based on the first three moments of the chi-square distribution and the latter is based on the first four moments of the F distribution. To the best of our knowledge, this is the first study using three-moment chi-square and four moment F approximations to test the equality of treatment means in one-way ANOVA under heteroscedasticity and nonnormality. These approximations are applied to the various problems in the literature. For example, Tiku and Wong [19] used three-moment chi-square and four moment F approximations for testing a unit root in an AR(1) model. Sürücü and Sazak [13] studied the three-parameter Weibull distribution to monitor reliability. Also, they provided reasonably accurate results to the percentage points of the distribution of cumulative time between failures by using two and three moment approximations. Purutcuoğlu [12] extended Tiku and Wong's [19] work to skewed distributions, namely, gamma and generalized logistic.

The outline of this study is organized as follows. In Section 2, C_{LS} and RC_{MML} test statistics are reviewed. In Section 3, a brief description of the three moment chi-square and the four moment F approximations are given. In section 4, results of the simulation study are presented. Concluding remarks are given in Section 5.

2. Test Statistics

In this section, we briefly review the well known C_{LS} test based on LS estimators and RC_{MML} test based on MML estimators.

2.1. **Cochran Test.** Let $Y_{i1}, Y_{i2}, ..., Y_{in_i}$ be a random sample from $N(\mu_i, \sigma_i^2)$, $i = 1, 2, ..., a$ distribution.

C_{LS} test proposed by Cochran in 1937, which is also referred to as natural test statistic in the literature is defined as follows

$$C_{LS} = \sum_{i=1}^{a} \frac{n_i}{S_i^2} \left(\bar{Y}_i - \frac{\sum\limits_{i=1}^{a} n_i \bar{Y}_i / S_i^2}{\sum\limits_{i=1}^{a} n_i / S_i^2} \right)^2 . \tag{2}$$

Here, $\bar{Y}_i$ and S_i^2 are LS estimators of μ_i and σ_i^2, respectively and formulated as follows

$$\bar{Y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij} \quad \text{and} \quad S_i^2 = \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_i)^2 / (n_i - 1). \tag{3}$$

2.2. **Robust Cochran Test.** Let $Y_{i1}, Y_{i2}, ..., Y_{in_i}$ be a random sample from LTS(p, μ_i, σ_i), $(i = 1, ..., a)$ distribution.

The probability density function (pdf) of LTS distribution is

$$f(y) = \frac{1}{\sqrt{k}\beta(1/2, p - 1/2)\sigma} \left(1 + \frac{(y-\mu)^2}{k\sigma^2} \right)^{-p}, -\infty < y < \infty; -\infty < \mu < \infty; \sigma > 0; p \geq 2 \tag{4}$$

where μ is location, σ is scale, p is shape parameter and $k = 2p - 3$, see [18]. It should be noted LTS distribution is used for modeling outlier(s) in data. It has a long tail when the shape parameter p is small and reduces to the normal distribution when p goes to infinity. If a random variable Y is distributed as LTS(p, μ, σ), then $t = \sqrt{(\nu/k)}((Y - \mu)/\sigma)$ is distributed as Student's t with $\nu = 2p - 1$ degrees of freedom.

RC_{MML} test proposed by Güven et al. in 2019 is given as follows

$$RC_{MML} = \sum_{i=1}^{a} \frac{M_i}{\hat{\sigma}_i^2} \left[\hat{\mu}_i - \frac{\sum_{i=1}^{a} M_i \hat{\mu}_i / \hat{\sigma}_i^2}{\sum_{i=1}^{a} M_i / \hat{\sigma}_i^2} \right]^2. \tag{5}$$

Here, $\hat{\mu}_i$ and $\hat{\sigma}_i^2$ are MML estimators of μ_i and σ_i^2, respectively and formulated as follows

$$\hat{\mu}_i = \frac{\sum_{j=1}^{n_i} \beta_{ij} y_{i(j)}}{m_i} \quad \text{and} \quad \hat{\sigma}_i = \frac{B_i + \sqrt{B_i^2 + 4A_i C_i}}{2\sqrt{A_i(A_i - 1)}}. \tag{6}$$

In Eq. (6), $A_i = n_i$, $B_i = \frac{2p}{k} \sum_{j=1}^{n_i} \alpha_{ij}(y_{i(j)} - \hat{\mu}_i)$, $C_i = \frac{2p}{k} \sum_{j=1}^{n_i} \beta_{ij}(y_{i(j)} - \hat{\mu}_i)^2$, $m_i = \sum_{j=1}^{n_i} \beta_{ij}$. $M_i = 2pm_i/k$ and

$$\alpha_{ij} = \frac{(2/k)t_{i(j)}^3}{\left(1 + (1/k)t_{i(j)}^2\right)^2} \quad \text{and} \quad \beta_{ij} = \frac{1 - (1/k)t_{i(j)}^2}{\left(1 + (1/k)t_{i(j)}^2\right)^2}.$$

It should be noted that $y_{i(j)}$, $i = 1, 2, ..., a$ and $j = 1, 2, ..., n_i$ are the ordered observations of a sample. The approximate values of the expected values of the ordered statistics, i.e, $t_{i(j)}$=$E(y_{i(j)})$ values are computed from the following equality

$$\int_{-\infty}^{t_{i(j)}} f(z)dz = \frac{j}{n_i + 1}.$$

Remark 1. *C_{LS} test statistic given in* (2) *and RC_{MML} test statistic given in* (5) *are asymptotically distributed as chi-square with $a - 1$ degrees of freedom, see [5, 9] for details. However, as mentioned earlier, the null distribution of these test statistics are unknown for small and moderate samples. To deal with this problem two approximations that can be used to calculate critical values are given.*

3. Moment Approximations

In this section, we briefly mentioned three moment chi-square and four-moment F approximations derived by Tiku and Wong [19].

3.1. **Three-moment chi-square approximation.** Let X^* be a random variable and

$$W_1 = \frac{X^* + a}{b}. \tag{7}$$

Here, W_1 has the central chi-square distribution with ν degrees of freedom. The values of a, b and ν are obtained by equating the first three moments on both sides of (7):

$$\nu = \frac{8}{\beta_1^*} \quad b = \sqrt{\frac{\mu_2}{2\nu}} \quad \text{and} \quad a = b\nu - \mu_1^{'} \tag{8}$$

where $\beta_1^* = \mu_3^2/\mu_2^3$ ($\mu_3 > 0$), $\mu_1^{'}$ is the mean of a random variable X^*, μ_2 is the variance of a random variable X^* and μ_3 is the third central moment of a random variable X^*.

It should be noted that for (7) to be valid β_1^* and β_2^* values of X^* should satisfy the following condition:

$$E = |\beta_2^* - (3 + 1.5\beta_1^*)| \leq 0.5 \tag{9}$$

where $\beta_2^* = \mu_4/\mu_2^2$ and μ_4 is the fourth central moment of a random variable X^*.

Realize that $\beta_2^* = 3 + 1.5\beta_1^*$ is called the Type III line for a chi-square distribution, see Tiku and Yip [21] and references therein.

3.2. **Four-moment F approximation.** Let X^* be a random variable and

$$W_2 = \frac{X^* + g}{h}. \tag{10}$$

Here, W_2 has the central F distribution with (ν_1, ν_2) degrees of freedom. The values of ν_1, ν_2, g and h are obtained by equating the four moments on both sides of (10):

$$\begin{aligned}
\nu_2 &= 2\left[3 + \frac{\beta_2^* + 3}{\beta_2^* - (3 + 1.5\beta_1^*)}\right] \\
\nu_1 &= \frac{1}{2}(\nu_2 - 2)\left[-1 + \sqrt{1 + \frac{32(\nu_2 - 4)/(\nu_2 - 6)^2}{\beta_1^* - 32(\nu_2 - 4)/(\nu_2 - 6)^2}}\right] \\
h &= \sqrt{\left\{\frac{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)}{2\nu_2^2(\nu_1 + \nu_2 - 2)}\right\}\mu_2} \\
g &= \frac{\nu_2}{\nu_2 - 2}h - \mu_1^{'}.
\end{aligned} \tag{11}$$

Here, $\beta_1^* = \mu_3^2/\mu_2^3$ ($\mu_3 > 0$), $\beta_2^* = \mu_4/\mu_2^2$, $\mu_1^{'}$ is the mean of a random variable X^*, μ_2 is the variance of a random variable X^*, μ_3 is the third central moment of a random variable X^* and μ_4 is the fourth central moment of a random variable X^*.

It should be noted that for (10) to be valid (β_1^*, β_2^*) values of X^* should satisfy the following conditions:

$$\beta_1^* > C_1 \quad \text{and} \quad \beta_2^* > C_2. \tag{12}$$

where $C_1 = \frac{32(\nu_2-4)}{(\nu_2-6)^2}$ and $C_2 = 3 + 1.5\beta_1^*$.

Realize that the inequalities in (12) determine the F region in the (β_1^*, β_2^*)-plane bounded by the χ^2-line and the reciprocal χ^2-line, see [12].

4. Monte Carlo Simulation Study

In this section, the performances of the RC_{MML} and C_{LS} test statistics based on approximations are compared when the underlying population distributions are LTS. Throughout the simulation study, the following parameter settings are used:

- Number of treatments: $a = 3$,
- Shape parameter: $p = 2,\ 2.5,\ 3.5$ and 5,
- Sample sizes: $(n_1, n_2, n_3) = (6, 6, 6),\ (6, 9, 12),\ (12, 12, 12),\ (12, 15, 18)$ and $(20, 20, 20)$,
- Variances: $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1, 1, 1),\ (1, 1.5, 2.5)$ and $(1, 3, 5)$.

Based on the parameter settings, random samples with sample size (n_1, n_2, n_3) were generated from the $LTS(p, \mu_i, \sigma_i)$ distributions. Since it is very difficult to obtain the distribution of RC_{MML} and C_{LS} test statistics or their moments, we simulated (from 10,000 runs) their first four moments. The simulated mean, variance, β_1^* and β_2^* values of the test statistics RC_{MML} and C_{LS} are given in Table 1. In addition, the values for inequalities in (9) and (12) are also included in Table 1, to see whether the three-moment chi square and four-moment F approximations are applicable or not. If the condition in (9) is satisfied, then three-moment chi-square approximation provides accurate values for the percentage points of X^*. Thus, distributions belonging to the Type III region are approximated by this method. In other words, the $100(1-\alpha)\%$ point of X^* is approximately $b\chi^2_{(1-\alpha)}(\nu) - a$ where $\chi^2_{(1-\alpha)}(\nu)$ is the $100(1-\alpha)\%$ point of central chi-square distribution with ν degrees of freedom. Similarly, if the conditions in (12) are satisfied, then four-moment F approximation provides accurate values for the percentage points of X^*. Thus, distributions belonging to the F-region are approximated by this method. In other words, the $100(1-\alpha)\%$ point of X^* is approximately $hF_{(1-\alpha)}(\nu_1, \nu_2) - g$ where $F_{(1-\alpha)}(\nu_1, \nu_2)$ is the $100(1-\alpha)\%$ point of central central F distribution with (ν_1, ν_2) degrees of freedom.

According to the results given in Table 1, condition (9) is satisfied when the sample sizes are $(n_1, n_2, n_3) = (12, 12, 12), (12, 15, 18)$ and $(20, 20, 20)$ for all values of p except $p = 2$. However, when $p = 2$, if sample sizes are $(n_1, n_2, n_3) = (12, 15, 18)$ and $(20, 20, 20)$, then this condition is satisfied. It should be noted that

(β_1^*, β_2^*) values of RC_{MML} and C_{LS} test statistics satisfy the conditions in (12) for all sample sizes and p values. In other words, four moment F approximation is applicable for all parameter settings. Therefore, 95% points of the Eq. (10) and simulated type I error rates and powers of both tests are computed using four-moment F approximation. To illustrate the accuracy of four moment F approximation, the simulated values of the probabilities (based on 10,000 Monte Carlo runs) formulated as

$$P_1 = P(RC_{MML} \geq c_{MML}\,|H_0) \quad \text{and} \quad P_2 = P(C_{LS} \geq c_{LS}\,|H_0) \tag{13}$$

are given in Table 2. Here, c_{MML} and c_{LS} are the 95% points as determined by (10). The simulated values of the probabilities (based on 10,000 Monte Carlo runs)

$$P_3 = P(RC_{MML} \geq c\,|H_0) \quad \text{and} \quad P_4 = P(C_{LS} \geq c\,|H_0) \tag{14}$$

are also calculated and included in Table 2. Here, c is the 95% point of the chi-square distribution with $a-1$ degrees of freedom. The purpose here is to show that both test statistics are not distributed as chi-quare with $a-1$ degrees of freedom when the sample sizes are small and moderate.

As it is known that simulated values of the probabilities given in (13) and (14) are Type I error rates of the test statistics. According to Table 2, Type I error rates of both tests are very close to the nominal level $\alpha = 0.05$ based on the probabilities in (13). Therefore, four-moment F approximation performs quite well.

It should be noted that μ_i's $i = 1, 2, 3$ are taken to be 0 for calculating the Type I error rates. The simulated power values are presented in Table 3. They are obtained by substracting and adding a constant s to the observations in the first and third group, respectively.

From Table 3, it can be seen that RC_{MML} test is more powerful than the C_{LS} test. RC_{MML} test outperforms the C_{LS} test especially when $p = 2$ and 2.5. According to the results, it is clear that powers of two tests become very close to each other as expected as the shape parameter p increases, i.e. when the distribution converges to normal.

Table 1 Simulated values of the mean, variance, β_1^* and β_2^* of RC_{MML} and C_{LS} test statistics.

$p = 2$								
$(\sigma_1^2, \sigma_2^2, \sigma_3^2)$		Mean	Variance	β_1^*	β_2^*	C_1	C_2	E
(n_1, n_2, n_3)=(6,6,6)								
$(1, 1, 1)$	RC_{MML}	2.3096	8.0560	13.0149	27.7682	3.1930	22.5223	5.2459
	C_{LS}	2.5299	8.0138	11.7138	26.8661	4.0835	20.5707	6.2954
$(1, 1.5, 2.5)$	RC_{MML}	2.3589	8.3187	11.6877	24.9991	2.9603	20.5315	4.4676
	C_{LS}	2.5722	8.1623	9.1842	20.0265	2.5773	16.7763	3.2503
$(1, 3, 5)$	RC_{MML}	2.3634	8.1424	9.6312	19.8070	1.8271	17.4468	2.3602
	C_{LS}	2.5762	8.1160	8.1244	16.9847	1.5691	15.1866	1.7981
(n_1, n_2, n_3)=(6,9,12)								
$(1, 1, 1)$	RC_{MML}	2.1631	6.0217	8.4289	18.8903	2.7253	15.6434	3.2469
	C_{LS}	2.3272	6.0408	8.0179	19.9702	4.1843	15.0269	4.9433
$(1, 1.5, 2.5)$	RC_{MML}	2.1308	5.7908	7.6515	15.9360	1.3274	14.4773	1.4587
	C_{LS}	2.3022	5.5273	5.8677	12.9254	1.2088	11.8016	1.1238
$(1, 3, 5)$	RC_{MML}	2.0855	5.3528	6.3573	13.0810	0.5607	12.5359	0.5451
	C_{LS}	2.2886	5.3462	5.0975	11.5050	1.0033	10.6463	0.8587
(n_1, n_2, n_3)=(12,12,12)								
$(1, 1, 1)$	RC_{MML}	2.0800	5.4741	7.9669	17.6418	2.3582	14.9504	2.6914
	C_{LS}	2.2032	5.1416	5.9498	14.0425	2.2354	11.9247	2.1179
$(1, 1.5, 2.5)$	RC_{MML}	2.0619	5.0819	6.7256	14.0429	0.9464	13.0884	0.9546
	C_{LS}	2.2280	4.9446	5.2560	12.1676	1.4686	10.8840	1.2836
$(1, 3, 5)$	RC_{MML}	2.1180	5.6276	7.0758	13.9387	0.3129	13.6137	0.3250
	C_{LS}	2.2368	5.2004	5.4540	11.6230	0.4983	11.1810	0.4420
(n_1, n_2, n_3)=(12,15,18)								
$(1, 1, 1)$	RC_{MML}	2.0499	5.0062	6.2150	12.5231	0.2095	12.3224	0.2006
	C_{LS}	2.1970	4.8792	5.0956	10.9974	0.4149	10.6434	0.3541
$(1, 1.5, 2.5)$	RC_{MML}	2.0358	4.7405	5.2598	11.0816	0.2209	10.8897	0.1918
	C_{LS}	2.1673	4.5687	4.3740	9.9229	0.4607	9.5610	0.3620
$(1, 3, 5)$	RC_{MML}	2.0321	4.8645	5.9746	12.2518	0.3099	11.9619	0.2899
	C_{LS}	2.1854	4.7859	4.7202	10.5556	0.5806	10.0804	0.4753
(n_1, n_2, n_3)=(20,20,20)								
$(1, 1, 1)$	RC_{MML}	2.0226	4.4795	5.3256	11.3624	0.4275	10.9883	0.3740
	C_{LS}	2.1341	4.2243	3.9059	9.0935	0.3165	8.8588	0.2347
$(1, 1.5, 2.5)$	RC_{MML}	2.0160	4.5365	5.2599	11.3575	0.5381	10.8899	0.4676
	C_{LS}	2.1090	4.2262	3.7145	8.7742	0.2798	8.5717	0.2024
$(1, 3, 5)$	RC_{MML}	2.0648	4.9787	6.0703	12.5036	0.4215	12.1054	0.3982
	C_{LS}	2.1520	4.4884	4.0370	9.0617	0.0082	9.0556	0.0061

Table 1 Continued

$p = 2.5$								
$(\sigma_1^2, \sigma_2^2, \sigma_3^2)$		Mean	Variance	β_1^*	β_2^*	C_1	C_2	E
(n_1, n_2, n_3)=(6,6,6)								
$(1, 1, 1)$	RC_{MML}	2.4218	8.4302	10.6901	22.3395	2.3585	19.0352	3.3043
	C_{LS}	2.5581	8.5239	9.2394	18.9141	1.6411	16.8591	2.0550
$(1, 1.5, 2.5)$	RC_{MML}	2.4228	8.6027	12.0572	24.8101	2.4296	21.0858	3.7242
	C_{LS}	2.5597	8.7233	10.5542	21.6902	2.0672	18.8312	2.8590
$(1, 3, 5)$	RC_{MML}	2.4826	9.2056	12.0715	24.4862	2.2087	21.1072	3.3789
	C_{LS}	2.6070	9.1769	10.8083	22.3328	2.2136	19.2124	3.1204
(n_1, n_2, n_3)=(6,9,12)								
(1,1,1)	RC_{MML}	2.2667	7.0193	10.1246	22.4666	3.1407	18.1869	4.2798
	C_{LS}	2.3759	6.9486	8.7447	19.1618	2.5003	16.1170	3.0448
(1,1.5,2.5)	RC_{MML}	2.2354	6.5174	8.4731	17.5403	1.5531	15.7097	1.8306
	C_{LS}	2.3585	6.4990	7.2715	15.9467	1.9077	13.9072	2.0395
(1,3,5)	RC_{MML}	2.2131	5.9959	7.5144	16.2003	1.7687	14.2716	1.9287
	C_{LS}	2.3379	6.0546	6.2077	13.6899	1.4305	12.3115	1.3783
(n_1, n_2, n_3)=(12,12,12)								
(1,1,1)	RC_{MML}	2.1299	5.4463	6.2480	12.6131	0.2510	12.3720	0.2412
	C_{LS}	2.2468	5.4260	5.1322	10.9339	0.2751	10.6982	0.2356
(1,1.5,2.5)	RC_{MML}	2.1374	5.5258	6.1414	12.3430	0.1377	12.2121	0.1309
	C_{LS}	2.2439	5.5119	5.5494	11.5547	0.2575	11.3241	0.2306
(1,3,5)	RC_{MML}	2.1556	5.5529	6.0006	12.4211	0.4479	12.0009	0.4202
	C_{LS}	2.2780	5.4755	4.7076	10.1637	0.1253	10.0614	0.1023
(n_1, n_2, n_3)=(12,15,18)								
(1,1,1)	RC_{MML}	2.0891	4.8593	5.4115	11.5966	0.5427	11.1172	0.4794
	C_{LS}	2.2011	4.8299	4.6185	10.3151	0.4790	9.9278	0.3873
(1,1.5,2.5)	RC_{MML}	2.0681	4.9123	5.7607	11.7959	0.1693	11.6410	0.1549
	C_{LS}	2.1738	4.9129	5.0528	10.9016	0.3797	10.5792	0.3224
(1,3,5)	RC_{MML}	2.0624	4.7238	5.0612	11.0668	0.5584	10.5919	0.4749
	C_{LS}	2.1714	4.7558	4.2012	9.4379	0.1768	9.3019	0.1360
(n_1, n_2, n_3)=(20,20,20)								
(1,1,1)	RC_{MML}	2.0612	4.7198	4.9356	10.6220	0.2610	10.4033	0.2187
	C_{LS}	2.1347	4.6836	4.5503	10.3010	0.5924	9.8255	0.4755
(1,1.5,2.5)	RC_{MML}	2.0534	4.6072	4.7783	10.1984	0.0377	10.1674	0.0310
	C_{LS}	2.1400	4.6175	4.3784	9.9652	0.5058	9.5676	0.3977
(1,3,5)	RC_{MML}	2.0755	4.8576	5.3440	11.1513	0.1544	11.0161	0.1352
	C_{LS}	2.1376	4.8247	5.0309	10.9896	0.5230	10.5464	0.4433

Table 1 Continued

$p = 3.5$								
$(\sigma_1^2, \sigma_2^2, \sigma_3^2)$		Mean	Variance	β_1^*	β_2^*	C_1	C_2	E
(n_1, n_2, n_3)=(6,6,6)								
(1, 1, 1)	RC_{MML}	2.4878	9.2504	12.7133	25.9098	2.4074	22.0700	3.8398
	C_{LS}	2.5587	9.4876	12.5679	25.3752	2.2334	21.8518	3.5234
(1,1.5,2.5)	RC_{MML}	2.5112	9.8580	14.3304	28.8503	2.4866	24.4957	4.3546
	C_{LS}	2.5792	10.0060	14.0845	28.6982	2.6402	24.1268	4.5714
(1,3,5)	RC_{MML}	2.6010	10.9686	13.7113	27.6405	2.4100	23.5669	4.0736
	C_{LS}	2.6614	11.0235	12.7401	25.2558	1.9796	22.1101	3.1457
(n_1, n_2, n_3)=(6,9,12)								
(1,1,1)	RC_{MML}	2.3729	7.3840	9.1295	18.6714	1.5930	16.6942	1.9772
	C_{LS}	2.4405	7.4683	8.2328	16.7771	1.2386	15.3492	1.4279
(1,1.5,2.5)	RC_{MML}	2.3098	6.7497	7.8150	16.4073	1.5096	14.7225	1.6848
	C_{LS}	2.3768	6.8880	7.6406	16.3291	1.6958	14.4609	1.8681
(1,3,5)	RC_{MML}	2.3133	6.5897	6.8536	14.1312	0.8341	13.2804	0.8508
	C_{LS}	2.3764	6.6548	6.7169	14.1695	1.0846	13.0753	1.0941
(n_1, n_2, n_3)=(12,12,12)								
(1,1,1)	RC_{MML}	2.1419	5.5832	6.3234	12.7305	0.2535	12.4850	0.2454
	C_{LS}	2.2150	5.7049	6.0440	12.4456	0.4029	12.0660	0.3797
(1,1.5,2.5)	RC_{MML}	2.2279	5.8083	5.4341	11.2752	0.1403	11.1511	0.1241
	C_{LS}	2.2801	5.8356	5.1763	10.9279	0.1901	10.7644	0.1635
(1,3,5)	RC_{MML}	2.2066	5.7402	5.9609	12.4132	0.5049	11.9413	0.4719
	C_{LS}	2.2648	5.8047	5.5062	11.7500	0.5500	11.2593	0.4907
(n_1, n_2, n_3)=(12,15,18)								
(1,1,1)	RC_{MML}	2.1052	4.9854	5.6536	11.8726	0.4331	11.4803	0.3923
	C_{LS}	2.1725	4.9796	4.9204	10.8199	0.5248	10.3805	0.4393
(1,1.5,2.5)	RC_{MML}	2.1222	5.3348	5.9678	11.9782	0.0284	11.9517	0.0265
	C_{LS}	2.1881	5.3890	5.6574	11.7592	0.3015	11.4861	0.2730
(1,3,5)	RC_{MML}	2.1788	5.2685	5.2153	11.2398	0.4822	10.8229	0.4169
	C_{LS}	2.2412	5.3480	4.8351	10.2684	0.0190	10.2527	0.0157
(n_1, n_2, n_3)=(20,20,20)								
(1,1,1)	RC_{MML}	2.0968	4.8573	5.0836	10.6636	0.0449	10.6254	0.0382
	C_{LS}	2.1408	4.8398	4.8967	10.5150	0.2038	10.3450	0.1700
(1,1.5,2.5)	RC_{MML}	2.0911	4.8850	5.2287	10.9304	0.1009	10.8431	0.0873
	C_{LS}	2.1524	4.9565	4.8703	10.3574	0.0625	10.3055	0.0520
(1,3,5)	RC_{MML}	2.0905	4.9111	5.2425	11.2253	0.4169	10.8637	0.3615
	C_{LS}	2.1339	4.9440	4.7707	10.2893	0.1621	10.1560	0.1333

Table 1 Continued

$p = 5$								
$(\sigma_1^2, \sigma_2^2, \sigma_3^2)$		Mean	Variance	β_1^*	β_2^*	C_1	C_2	E
(n_1, n_2, n_3)=(6,6,6)								
(1, 1, 1)	RC_{MML}	2.5931	10.1696	10.4780	19.8989	0.8685	18.7169	1.1819
	C_{LS}	2.6224	10.2105	10.1684	19.3852	0.8505	18.2526	1.1326
(1,1.5,2.5)	RC_{MML}	2.5824	9.2375	8.2641	16.3544	0.8314	15.3961	0.9583
	C_{LS}	2.6117	9.3139	8.2007	16.2315	0.8116	15.3010	0.9305
(1,3,5)	RC_{MML}	2.6785	11.3840	12.1254	22.8392	1.0877	21.1881	1.6510
	C_{LS}	2.7104	11.4115	11.6371	22.0232	1.0651	20.4556	1.5675
(n_1, n_2, n_3)=(6,9,12)								
(1,1,1)	RC_{MML}	2.4202	7.8965	8.2753	16.1644	0.6520	15.4129	0.7515
	C_{LS}	2.4478	7.8960	8.0382	15.8409	0.6931	15.0573	0.7835
(1,1.5,2.5)	RC_{MML}	2.3891	7.3910	7.6100	15.2101	0.7291	14.4150	0.7951
	C_{LS}	2.4176	7.4448	7.5063	15.1283	0.8036	14.2594	0.8689
(1,3,5)	RC_{MML}	2.3322	6.7876	6.8458	14.3607	1.0698	13.2686	1.0920
	C_{LS}	2.3615	6.8576	6.6484	13.8872	0.9135	12.9726	0.9146
(n_1, n_2, n_3)=(12,12,12)								
(1,1,1)	RC_{MML}	2.2527	6.0280	6.1148	12.5301	0.3773	12.1722	0.3579
	C_{LS}	2.2829	6.0599	5.8781	12.1358	0.3440	11.8171	0.3187
(1,1.5,2.5)	RC_{MML}	2.2251	5.7954	5.7693	12.0057	0.3840	11.6539	0.3518
	C_{LS}	2.2525	5.8256	5.5710	11.8138	0.5092	11.3565	0.4573
(1,3,5)	RC_{MML}	2.2721	6.3965	6.7848	13.6628	0.4799	13.1771	0.4857
	C_{LS}	2.2959	6.3694	6.6065	13.4051	0.4977	12.9098	0.4953
(n_1, n_2, n_3)=(12,15,18)								
(1,1,1)	RC_{MML}	2.1660	5.1636	5.0322	10.8800	0.3916	10.5482	0.3318
	LS	2.1890	5.2156	4.9121	10.5098	0.1695	10.3682	0.1416
(1,1.5,2.5)	RC_{MML}	2.1914	5.2469	4.9893	10.7124	0.2711	10.4839	0.2285
	C_{LS}	2.2216	5.3025	4.7285	10.2224	0.1584	10.0927	0.1297
(1,3,5)	RC_{MML}	2.1852	5.3835	5.4095	11.5094	0.4477	11.1142	0.3952
	C_{LS}	2.2144	5.3921	5.1696	11.0698	0.3667	10.7544	0.3154
(n_1, n_2, n_3)=(20,20,20)								
(1,1,1)	RC_{MML}	2.1482	4.9962	4.9164	10.3884	0.0164	10.3747	0.0137
	C_{LS}	2.1717	5.0489	5.0828	10.9542	0.3872	10.6243	0.3299
(1,1.5,2.5)	RC_{MML}	2.0912	4.7621	5.1844	11.1729	0.4599	10.7766	0.3963
	C_{LS}	2.1220	4.8095	4.9371	10.6404	0.2800	10.4057	0.2347
(1,3,5)	RC_{MML}	2.1392	5.2311	5.4792	11.3733	0.1739	11.2188	0.1545
	C_{LS}	2.1688	5.3126	5.2535	10.9122	0.0368	10.8803	0.0320

Table 2 Simulated critical values and the probabilities
$P_1 = P(RC_{MML} \geq c_{MML} \,|H_0\,)$, $P_2 = P(C_{LS} \geq c_{LS} \,|H_0\,)$,
$P_3 = P(RC_{MML} \geq c \,|H_0\,)$, $P_4 = P(C_{LS} \geq c \,|H_0\,)$ and $c = 5.9915$.

			$p=2$			$p=2.5$		
(n_1,n_2,n_3)	$(\sigma_1^2,\sigma_2^2,\sigma_3^2)$		critical value	P_1 P_2	P_3 P_4	critical value	P_1 P_2	P_3 P_4
(6,6,6)	(1,1,1)	c_{MML}	7.6647	0.0488	0.0823	7.9867	0.0484	0.0907
		c_{LS}	7.8448	0.0478	0.0899	8.2283	0.0474	0.0956
	(1,1.5,2.5)	c_{MML}	7.8346	0.0489	0.0864	8.0174	0.0471	0.0894
		c_{LS}	8.0483	0.0493	0.0948	8.2436	0.0467	0.0963
	(1,3,5)	c_{MML}	7.8853	0.0495	0.0896	8.2841	0.0442	0.0930
		c_{LS}	8.1243	0.0469	0.0994	8.4219	0.0452	0.0993
(6,9,12)	(1,1,1)	c_{MML}	6.8603	0.0518	0.0708	7.3034	0.0487	0.0780
		c_{LS}	6.9498	0.0494	0.0728	7.4359	0.0467	0.0818
	(1,1.5,2.5)	c_{MML}	6.8381	0.0506	0.0674	7.2065	0.0485	0.0745
		c_{LS}	6.9046	0.0476	0.0707	7.2969	0.0495	0.0814
	(1,3,5)	c_{MML}	6.6798	0.0504	0.0650	6.9674	0.0492	0.0711
		c_{LS}	6.8224	0.0497	0.0721	7.1381	0.0484	0.0765
(12,12,12)	(1,1,1)	c_{MML}	6.5818	0.0522	0.0643	6.7971	0.0473	0.0648
		c_{LS}	6.5636	0.0513	0.0648	6.8911	0.0490	0.0715
	(1,1.5,2.5)	c_{MML}	6.5034	0.0481	0.0595	6.8512	0.0463	0.0667
		c_{LS}	6.5510	0.0483	0.0635	6.9328	0.0487	0.0701
	(1,3,5)	c_{MML}	6.8559	0.0453	0.0625	6.8449	0.0486	0.0701
		c_{LS}	6.7639	0.0468	0.0643	6.9529	0.0495	0.0758
(12,15,18)	(1,1,1)	c_{MML}	6.5290	0.0460	0.0562	6.4603	0.0495	0.0619
		c_{LS}	6.5855	0.0479	0.0620	6.5512	0.0493	0.0631
	(1,1.5,2.5)	c_{MML}	6.3847	0.0508	0.0603	6.5065	0.0461	0.0578
		c_{LS}	6.3940	0.0487	0.0594	6.5804	0.0488	0.0619
	(1,3,5)	c_{MML}	6.4353	0.0483	0.0588	6.3653	0.0525	0.0614
		c_{LS}	6.5075	0.0516	0.0655	6.5095	0.0484	0.0634
(20,20,20)	(1,1,1)	c_{MML}	6.2296	0.0483	0.0532	6.3909	0.0492	0.0599
		c_{LS}	6.1992	0.0516	0.0576	6.4054	0.0520	0.0620
	(1,1.5,2.5)	c_{MML}	6.2378	0.0485	0.0551	6.3530	0.0500	0.0597
		c_{LS}	6.1721	0.0523	0.0568	6.3847	0.0497	0.0599
	(1,3,5)	c_{MML}	6.5082	0.0477	0.0581	6.4863	0.0496	0.0617
		c_{LS}	6.3803	0.0482	0.0573	6.4893	0.0481	0.0607

Table 2 Continued

(n_1, n_2, n_3)	$(\sigma_1^2, \sigma_2^2, \sigma_3^2)$		$p = 3.5$ critical value	P_1 P_2	P_3 P_4	$p = 5$ critical value	P_1 P_2	P_3 P_4
(6,6,6)	(1,1,1)	c_{MML}	8.2775	0.0462	0.0920	8.8412	0.0479	0.0999
		c_{LS}	8.4364	0.0459	0.0950	8.8920	0.0475	0.1011
	(1,1.5,2.5)	c_{MML}	8.4453	0.0460	0.0924	8.5783	0.0472	0.1070
		c_{LS}	8.5552	0.0438	0.0963	8.6353	0.0465	0.1080
	(1,3,5)	c_{MML}	8.8812	0.0467	0.1031	9.2216	0.0461	0.1086
		c_{LS}	9.0109	0.0453	0.1068	9.2778	0.0453	0.1109
(6,9,12)	(1,1,1)	c_{MML}	7.6554	0.0452	0.0830	7.9833	0.0464	0.0899
		c_{LS}	7.7920	0.0454	0.0874	8.0084	0.0462	0.0907
	(1,1.5,2.5)	c_{MML}	7.3755	0.0485	0.0781	7.7680	0.0476	0.0851
		c_{LS}	7.4786	0.0497	0.0797	7.8084	0.0471	0.0892
	(1,3,5)	c_{MML}	7.3827	0.0487	0.0802	7.4531	0.0491	0.0837
		c_{LS}	7.4452	0.0495	0.0819	7.5242	0.0485	0.0848
(12,12,12)	(1,1,1)	c_{MML}	6.8673	0.0465	0.0661	7.1474	0.0486	0.0765
		c_{LS}	6.9734	0.0483	0.0688	7.1926	0.0493	0.0776
	(1,1.5,2.5)	c_{MML}	7.0540	0.0521	0.0742	7.0208	0.0493	0.0732
		c_{LS}	7.1076	0.0507	0.0773	7.0444	0.0491	0.0744
	(1,3,5)	c_{MML}	6.9678	0.0488	0.0703	7.3045	0.0465	0.0766
		c_{LS}	7.0428	0.0485	0.0744	7.3154	0.0469	0.0769
(12,15,18)	(1,1,1)	c_{MML}	6.5469	0.0466	0.0583	6.6820	0.0500	0.0645
		c_{LS}	6.5912	0.0474	0.0645	6.7506	0.0493	0.0678
	(1,1.5,2.5)	c_{MML}	6.7658	0.0482	0.0669	6.7563	0.0496	0.0682
		c_{LS}	6.8207	0.0487	0.0674	6.8186	0.0497	0.0704
	(1,3,5)	c_{MML}	6.7338	0.0496	0.0674	6.7962	0.0504	0.0694
		c_{LS}	6.8771	0.0492	0.0702	6.8344	0.0492	0.0715
(20,20,20)	(1,1,1)	c_{MML}	6.5162	0.0484	0.0599	6.6309	0.0479	0.0632
		c_{LS}	6.5309	0.0481	0.0602	6.6387	0.0488	0.0637
	(1,1.5,2.5)	c_{MML}	6.5188	0.0487	0.0607	6.4234	0.0473	0.0583
		c_{LS}	6.6109	0.0472	0.0626	6.4906	0.0485	0.0600
	(1,3,5)	c_{MML}	6.4954	0.0503	0.0620	6.7159	0.0476	0.0642
		c_{LS}	6.5732	0.0519	0.0655	6.7944	0.0478	0.0653

Table 3 Simulated powers of the RC_{MML} and C_{LS} tests based on four-moment F approximation.

$p = 2$									
(n_1, n_2, n_3)	$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1, 1, 1)$			$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1, 1.5, 2.5)$			$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1, 3, 5)$		
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
(6,6,6)	0.00	0.0488	0.0478	0.00	0.0489	0.0493	0.00	0.0495	0.0469
	0.15	0.08	0.08	0.18	0.08	0.08	0.22	0.08	0.07
	0.30	0.19	0.19	0.36	0.18	0.18	0.44	0.16	0.16
	0.45	0.37	0.36	0.54	0.35	0.34	0.66	0.32	0.32
	0.60	0.58	0.57	0.72	0.55	0.53	0.88	0.52	0.50
	0.75	0.75	0.73	0.90	0.71	0.68	1.10	0.69	0.66
	0.90	0.87	0.84	1.08	0.84	0.82	1.32	0.82	0.79
	1.05	0.93	0.91	1.26	0.92	0.90	1.54	0.90	0.88
	1.20	0.97	0.95	1.44	0.96	0.94	1.76	0.95	0.93
	1.35	0.98	0.97	1.62	0.98	0.96	1.98	0.97	0.96
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
(6,9,12)	0.00	0.0518	0.0494	0.00	0.0506	0.0476	0.00	0.0504	0.0497
	0.11	0.08	0.07	0.14	0.08	0.08	0.17	0.08	0.08
	0.22	0.17	0.17	0.28	0.17	0.16	0.34	0.18	0.17
	0.33	0.32	0.30	0.42	0.33	0.31	0.51	0.34	0.32
	0.44	0.50	0.47	0.56	0.53	0.49	0.68	0.53	0.48
	0.55	0.68	0.64	0.70	0.71	0.66	0.85	0.72	0.66
	0.66	0.81	0.77	0.84	0.84	0.80	1.02	0.84	0.79
	0.77	0.91	0.87	0.98	0.92	0.87	1.19	0.93	0.88
	0.88	0.95	0.92	1.12	0.96	0.93	1.36	0.96	0.93
	0.99	0.98	0.96	1.26	0.98	0.96	1.53	0.98	0.96
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
(12,12,12)	0.00	0.0522	0.0513	0.00	0.0481	0.0483	0.00	0.0453	0.0468
	0.09	0.08	0.08	0.11	0.08	0.07	0.14	0.07	0.07
	0.18	0.16	0.15	0.22	0.15	0.14	0.28	0.15	0.14
	0.27	0.31	0.27	0.33	0.29	0.26	0.42	0.29	0.27
	0.36	0.52	0.46	0.44	0.49	0.43	0.56	0.48	0.42
	0.45	0.70	0.62	0.55	0.66	0.59	0.70	0.66	0.59
	0.54	0.85	0.77	0.66	0.82	0.73	0.84	0.82	0.73
	0.63	0.92	0.86	0.77	0.91	0.84	0.98	0.91	0.84
	0.72	0.96	0.92	0.88	0.96	0.90	1.12	0.96	0.91
	0.81	0.99	0.96	0.99	0.98	0.94	1.26	0.98	0.95
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
(12,15,18)	0.00	0.0460	0.0479	0.00	0.0508	0.0487	0.00	0.0483	0.0516
	0.08	0.07	0.07	0.10	0.07	0.07	0.13	0.07	0.07
	0.16	0.15	0.13	0.20	0.16	0.14	0.26	0.17	0.15
	0.24	0.30	0.26	0.30	0.31	0.27	0.39	0.34	0.29
	0.32	0.51	0.43	0.40	0.50	0.43	0.52	0.54	0.46
	0.40	0.69	0.59	0.50	0.70	0.59	0.65	0.74	0.64
	0.48	0.83	0.73	0.60	0.83	0.73	0.78	0.87	0.77
	0.56	0.92	0.84	0.70	0.92	0.84	0.91	0.94	0.86
	0.64	0.97	0.91	0.80	0.97	0.91	1.04	0.98	0.92
	0.72	0.99	0.95	0.90	0.98	0.95	1.17	0.99	0.96
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
(20,20,20)	0.00	0.0483	0.0516	0.00	0.0485	0.0523	0.00	0.0477	0.0482
	0.06	0.07	0.07	0.08	0.07	0.07	0.11	0.07	0.06
	0.12	0.14	0.12	0.16	0.14	0.12	0.22	0.16	0.14
	0.18	0.26	0.22	0.24	0.28	0.22	0.33	0.31	0.25
	0.24	0.43	0.35	0.32	0.45	0.36	0.44	0.51	0.41
	0.30	0.60	0.50	0.40	0.64	0.51	0.55	0.70	0.58
	0.36	0.75	0.64	0.48	0.79	0.65	0.66	0.86	0.74
	0.42	0.87	0.76	0.56	0.89	0.78	0.77	0.94	0.85
	0.48	0.94	0.85	0.64	0.95	0.86	0.88	0.98	0.92
	0.54	0.98	0.92	0.72	0.98	0.92	0.99	0.99	0.96

Table 3 Continued

$p = 2.5$									
(n_1, n_2, n_3)	$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1, 1, 1)$			$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1, 1.5, 2.5)$			$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1, 3, 5)$		
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
	0.00	0.0484	0.0474	0.00	0.0471	0.0467	0.00	0.0442	0.0452
	0.16	0.08	0.08	0.20	0.08	0.07	0.25	0.07	0.07
	0.32	0.17	0.17	0.40	0.17	0.16	0.50	0.15	0.15
	0.48	0.32	0.32	0.60	0.31	0.30	0.75	0.30	0.30
(6,6,6)	0.64	0.52	0.50	0.80	0.50	0.49	1.00	0.50	0.49
	0.80	0.69	0.68	1.00	0.67	0.66	1.25	0.68	0.67
	0.96	0.83	0.81	1.20	0.82	0.81	1.50	0.81	0.80
	1.12	0.92	0.90	1.40	0.91	0.90	1.75	0.91	0.90
	1.28	0.96	0.95	1.60	0.96	0.95	2.00	0.95	0.94
	1.44	0.98	0.98	1.80	0.98	0.98	2.25	0.98	0.97
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
	0.00	0.0487	0.0467	0.00	0.0485	0.0495	0.00	0.0492	0.0484
	0.13	0.0799	0.0809	0.16	0.07	0.08	0.19	0.07	0.07
	0.26	0.1685	0.1638	0.32	0.16	0.16	0.38	0.16	0.15
	0.39	0.3213	0.3076	0.48	0.31	0.30	0.57	0.32	0.30
(6,9,12)	0.52	0.5239	0.4966	0.64	0.50	0.48	0.76	0.50	0.48
	0.65	0.7009	0.6701	0.80	0.69	0.66	0.95	0.69	0.65
	0.78	0.8425	0.8117	0.96	0.83	0.80	1.14	0.83	0.80
	0.91	0.9268	0.9009	0.12	0.92	0.89	1.33	0.92	0.88
	1.04	0.9714	0.9525	1.28	0.96	0.94	1.52	0.96	0.94
	1.17	0.9875	0.9760	1.44	0.99	0.97	1.71	0.99	0.97
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
	0.00	0.0473	0.0490	0.00	0.0463	0.0487	0.00	0.0486	0.0495
	0.11	0.07	0.07	0.13	0.07	0.07	0.16	0.08	0.07
	0.22	0.17	0.16	0.26	0.16	0.15	0.32	0.15	0.14
	0.33	0.34	0.31	0.39	0.28	0.26	0.48	0.29	0.27
(12,12,12)	0.44	0.54	0.50	0.52	0.49	0.45	0.64	0.48	0.44
	0.55	0.74	0.68	0.65	0.67	0.62	0.80	0.66	0.61
	0.66	0.88	0.83	0.78	0.82	0.77	0.96	0.81	0.76
	0.77	0.95	0.91	0.91	0.92	0.87	1.12	0.91	0.86
	0.88	0.98	0.96	1.04	0.96	0.93	1.28	0.96	0.93
	0.99	0.99	0.98	1.17	0.99	0.97	1.44	0.99	0.97
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
	0.00	0.0495	0.0493	0.00	0.0461	0.0488	0.00	0.0525	0.0484
	0.09	0.08	0.08	0.11	0.07	0.07	0.13	0.07	0.07
	0.18	0.16	0.15	0.22	0.15	0.14	0.26	0.14	0.13
	0.27	0.29	0.27	0.33	0.28	0.25	0.39	0.28	0.25
(12,15,18)	0.36	0.49	0.44	0.44	0.47	0.42	0.52	0.44	0.39
	0.45	0.68	0.62	0.55	0.65	0.58	0.65	0.63	0.55
	0.54	0.83	0.76	0.66	0.81	0.74	0.78	0.78	0.71
	0.63	0.92	0.87	0.77	0.90	0.85	0.91	0.89	0.83
	0.72	0.97	0.93	0.88	0.96	0.92	1.04	0.96	0.91
	0.81	0.99	0.97	0.99	0.99	0.96	1.17	0.98	0.95
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
	0.00	0.0492	0.0520	0.00	0.0500	0.0497	0.00	0.0496	0.0481
	0.07	0.07	0.07	0.09	0.07	0.07	0.11	0.07	0.07
	0.14	0.14	0.13	0.18	0.14	0.13	0.22	0.13	0.12
	0.21	0.25	0.22	0.27	0.26	0.23	0.33	0.24	0.22
(20,20,20)	0.28	0.42	0.37	0.36	0.43	0.37	0.44	0.40	0.35
	0.35	0.60	0.53	0.45	0.61	0.54	0.55	0.58	0.52
	0.42	0.76	0.69	0.54	0.78	0.69	0.66	0.75	0.67
	0.49	0.88	0.81	0.63	0.89	0.81	0.77	0.88	0.80
	0.56	0.95	0.89	0.72	0.95	0.90	0.88	0.94	0.88
	0.63	0.98	0.95	0.81	0.98	0.95	0.99	0.98	0.94

Table 3 Continued

$p = 3.5$									
(n_1, n_2, n_3)	$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1, 1, 1)$			$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1, 1.5, 2.5)$			$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1, 3, 5)$		
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
(6,6,6)	0.00	0.0462	0.0459	0.00	0.0460	0.0438	0.00	0.0467	0.0453
	0.17	0.07	0.07	0.21	0.07	0.07	0.27	0.07	0.07
	0.34	0.16	0.16	0.42	0.15	0.15	0.54	0.15	0.14
	0.51	0.31	0.31	0.63	0.28	0.28	0.81	0.28	0.28
	0.68	0.50	0.49	0.84	0.47	0.46	1.08	0.47	0.47
	0.85	0.68	0.67	1.05	0.66	0.65	1.35	0.66	0.65
	1.02	0.82	0.81	1.26	0.81	0.80	1.62	0.81	0.80
	1.19	0.91	0.91	1.47	0.90	0.89	1.89	0.90	0.89
	1.36	0.96	0.95	1.68	0.96	0.95	2.16	0.96	0.95
	1.53	0.99	0.98	1.89	0.98	0.98	2.43	0.98	0.97
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
(6,9,12)	0.00	0.0452	0.0454	0.00	0.0485	0.0497	0.00	0.0487	0.0495
	0.13	0.07	0.07	0.16	0.07	0.07	0.20	0.07	0.07
	0.26	0.14	0.13	0.32	0.14	0.13	0.40	0.15	0.13
	0.39	0.27	0.26	0.48	0.27	0.27	0.60	0.28	0.27
	0.52	0.44	0.43	0.64	0.44	0.44	0.80	0.45	0.44
	0.65	0.63	0.61	0.80	0.63	0.62	1.00	0.65	0.63
	0.78	0.78	0.77	0.96	0.78	0.77	1.20	0.80	0.79
	0.91	0.89	0.87	0.12	0.88	0.87	1.40	0.90	0.89
	1.04	0.95	0.94	1.28	0.94	0.93	1.60	0.96	0.95
	1.17	0.98	0.98	1.44	0.98	0.97	1.80	0.98	0.97
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
(12,12,12)	0.00	0.0465	0.0483	0.00	0.0521	0.0507	0.00	0.0488	0.0485
	0.11	0.07	0.07	0.13	0.06	0.06	0.16	0.07	0.07
	0.22	0.15	0.14	0.26	0.13	0.12	0.32	0.13	0.13
	0.33	0.29	0.28	0.39	0.24	0.23	0.48	0.24	0.23
	0.44	0.48	0.47	0.52	0.41	0.40	0.64	0.40	0.39
	0.55	0.66	0.64	0.65	0.59	0.57	0.80	0.58	0.56
	0.66	0.83	0.80	0.78	0.76	0.73	0.96	0.75	0.73
	0.77	0.92	0.90	0.91	0.87	0.85	1.12	0.87	0.85
	0.88	0.97	0.96	1.04	0.94	0.93	1.28	0.94	0.92
	0.99	0.99	0.98	1.17	0.98	0.97	1.44	0.98	0.97
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
(12,15,18)	0.00	0.0466	0.0474	0.00	0.0482	0.0487	0.00	0.0496	0.0492
	0.09	0.07	0.07	0.12	0.07	0.07	0.15	0.07	0.07
	0.18	0.14	0.13	0.24	0.14	0.13	0.30	0.14	0.13
	0.27	0.26	0.24	0.36	0.27	0.26	0.45	0.27	0.26
	0.36	0.42	0.41	0.48	0.45	0.43	0.60	0.47	0.44
	0.45	0.60	0.58	0.60	0.64	0.62	0.75	0.66	0.62
	0.54	0.76	0.73	0.72	0.78	0.76	0.90	0.81	0.78
	0.63	0.88	0.85	0.84	0.91	0.88	1.05	0.91	0.88
	0.72	0.95	0.93	0.96	0.96	0.94	1.20	0.97	0.95
	0.81	0.98	0.97	1.08	0.99	0.97	1.35	0.99	0.97
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
(20,20,20)	0.00	0.0484	0.0481	0.00	0.0487	0.0472	0.00	0.0503	0.0519
	0.08	0.07	0.07	0.10	0.07	0.07	0.13	0.07	0.07
	0.16	0.13	0.13	0.20	0.14	0.13	0.26	0.15	0.14
	0.24	0.27	0.26	0.30	0.26	0.24	0.39	0.28	0.27
	0.32	0.45	0.43	0.40	0.44	0.41	0.52	0.47	0.43
	0.40	0.63	0.60	0.50	0.63	0.59	0.65	0.65	0.62
	0.48	0.80	0.77	0.60	0.78	0.75	0.78	0.81	0.78
	0.56	0.91	0.88	0.70	0.90	0.87	0.91	0.92	0.89
	0.64	0.96	0.94	0.80	0.96	0.94	1.04	0.97	0.95
	0.72	0.99	0.98	0.90	0.98	0.97	1.17	0.99	0.98

Table 3 Continued

$p = 5$									
(n_1, n_2, n_3)	$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1, 1, 1)$			$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1, 1.5, 2.5)$			$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1, 3, 5)$		
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
(6,6,6)	0.00	0.0479	0.0475	0.00	0.0472	0.0465	0.00	0.0461	0.0453
	0.17	0.07	0.07	0.22	0.07	0.07	0.27	0.07	0.07
	0.34	0.14	0.14	0.44	0.15	0.14	0.54	0.14	0.14
	0.51	0.28	0.28	0.66	0.29	0.29	0.81	0.26	0.26
	0.68	0.44	0.44	0.88	0.47	0.47	1.08	0.43	0.42
	0.85	0.64	0.64	1.10	0.66	0.65	1.35	0.62	0.61
	1.02	0.78	0.78	1.32	0.82	0.81	1.62	0.78	0.77
	1.19	0.89	0.89	1.54	0.91	0.90	1.89	0.89	0.88
	1.36	0.96	0.95	1.76	0.96	0.96	2.16	0.96	0.95
	1.53	0.98	0.98	1.98	0.99	0.98	2.43	0.98	0.98
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
(6,9,12)	0.00	0.0464	0.0462	0.00	0.0476	0.0471	0.00	0.0491	0.0485
	0.13	0.06	0.06	0.17	0.07	0.07	0.21	0.07	0.07
	0.26	0.13	0.13	0.34	0.14	0.14	0.42	0.15	0.14
	0.39	0.25	0.24	0.51	0.26	0.26	0.63	0.29	0.28
	0.52	0.40	0.39	0.68	0.45	0.44	0.84	0.47	0.46
	0.65	0.58	0.57	0.85	0.63	0.62	1.05	0.66	0.65
	0.78	0.74	0.74	1.02	0.79	0.78	1.26	0.82	0.81
	0.91	0.86	0.85	1.19	0.90	0.89	1.47	0.92	0.91
	1.04	0.93	0.93	1.36	0.96	0.95	1.68	0.96	0.96
	1.17	0.98	0.97	1.53	0.99	0.98	1.89	0.99	0.98
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
(12,12,12)	0.00	0.0486	0.0493	0.00	0.0493	0.0491	0.00	0.0465	0.0469
	0.11	0.07	0.07	0.14	0.07	0.07	0.17	0.07	0.07
	0.22	0.14	0.14	0.28	0.14	0.14	0.34	0.13	0.13
	0.33	0.27	0.26	0.42	0.26	0.26	0.51	0.25	0.25
	0.44	0.44	0.43	0.56	0.45	0.44	0.68	0.41	0.40
	0.55	0.62	0.61	0.70	0.63	0.63	0.85	0.59	0.58
	0.66	0.79	0.78	0.84	0.79	0.78	1.02	0.76	0.75
	0.77	0.90	0.89	0.98	0.90	0.89	1.19	0.87	0.86
	0.88	0.96	0.95	1.12	0.96	0.95	1.36	0.95	0.94
	0.99	0.99	0.98	1.26	0.99	0.98	1.53	0.98	0.98
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
(12,15,18)	0.00	0.0500	0.0493	0.00	0.0496	0.0497	0.00	0.0504	0.0492
	0.10	0.07	0.07	0.13	0.07	0.07	0.15	0.07	0.07
	0.20	0.14	0.14	0.26	0.15	0.14	0.30	0.13	0.13
	0.30	0.29	0.28	0.39	0.29	0.28	0.45	0.27	0.26
	0.40	0.47	0.46	0.52	0.49	0.48	0.60	0.44	0.43
	0.50	0.66	0.64	0.65	0.68	0.66	0.75	0.63	0.62
	0.60	0.82	0.80	0.78	0.84	0.82	0.90	0.79	0.77
	0.70	0.92	0.91	0.91	0.93	0.92	1.05	0.90	0.89
	0.80	0.97	0.96	1.04	0.98	0.97	1.20	0.96	0.95
	0.90	0.99	0.98	1.17	0.99	0.98	1.35	0.99	0.98
	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}	s	RC_{MML}	C_{LS}
(20,20,20)	0.00	0.0479	0.0488	0.00	0.0473	0.0485	0.00	0.0476	0.0478
	0.08	0.07	0.07	0.11	0.08	0.08	0.13	0.07	0.06
	0.16	0.13	0.13	0.22	0.16	0.15	0.26	0.14	0.13
	0.24	0.25	0.24	0.33	0.30	0.29	0.39	0.26	0.25
	0.32	0.41	0.39	0.44	0.48	0.47	0.52	0.42	0.41
	0.40	0.60	0.58	0.55	0.68	0.67	0.65	0.62	0.60
	0.48	0.77	0.75	0.66	0.84	0.82	0.78	0.78	0.76
	0.56	0.88	0.87	0.77	0.93	0.92	0.91	0.89	0.88
	0.64	0.95	0.94	0.88	0.98	0.97	1.04	0.96	0.95
	0.72	0.98	0.99	0.99	0.99	0.99	1.17	0.99	0.98

5. Conclusion

This study examined small and moderate sample properties of the C_{LS} and RC_{MML} tests proposed in the literature for testing the equality of treatment means in one-way ANOVA when the underlying distribution is long tailed symmetric using three moment chi-square and four moment F approximations. Although the asymptotic distributions of the C_{LS} and RC_{MML} test statistics are known in large samples, the null distributions of both test statistics are not known for small and moderate sample sizes. This is the reason why three moment chi-square and four moment F approximations are needed. An extensive Monte Carlo simulation study is conducted to see whether two approximations are applicable to the test statistics or not and to compare the performances of the test statistics in terms of the Type I error rates and power. According to simulation results four moment F approximation is applicable to the C_{LS} and RC_{MML} test statistics regardless of the sample sizes and p values. Three moment chi-square approximation applicable when sample sizes are moderate. Also, using asymptotic distribution results in inflated type I error rates when sample sizes are small and moderate while Type I errror rates of the tests using F approximation are very close to the nominal level. Therefore, this approximation performs very well for C_{LS} and RC_{MML} test statistics. RC_{MML} test is more powerful than the C_{LS} especially when the shape parameter $p = 2$ and 2.5. Note also that, when the values of the shape parameter greater and equal 3.5 and 5 the RC_{MML} test is slightly more powerful than C_{LS} test.

Declaration of Competing Interests The author declares that there is no competing interest regarding the publication of this paper.

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