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Research Article

Beyond Descartes' rule of signs

VLADIMIR PETROV KOSTOV*

ABSTRACT. We consider real univariate polynomials with all roots real. Such a polynomial with c sign changes and p sign preservations in the sequence of its coefficients has c positive and p negative roots counted with multiplicity. Suppose that all moduli of roots are distinct; we consider them as ordered on the positive half-axis. We ask the question: If the positions of the sign changes are known, what can the positions of the moduli of negative roots be? We prove several new results which show how far from trivial the answer to this question is.

Keywords: Real polynomial in one variable, hyperbolic polynomial, sign pattern, Descartes' rule of signs.

2020 Mathematics Subject Classification: 26C10.

1. INTRODUCTION

In the present paper, we study a problem related to a generalization of Descartes' rule of signs formulated in [5]. About this rule see [1], [2], [3], [4], [7], [9], [10], [16] or [17]. For its tropical analog see [6]. A related problem concerning polynomials in one variable is considered in [15]. A degree d real polynomial $Q := \sum_{j=0}^{d} a_j x^j$ is *hyperbolic* if all its roots are real. Suppose that all coefficients a_j are non-zero. For such a polynomial, Descartes' rule of signs implies that it has c positive and p negative roots (counted with multiplicity, so $c + p = d$), where c is the number of sign changes and p the number of sign preservations in the sequence of coefficients of Q. The signs of these coefficients define the *sign pattern* $(\mathrm{sgn}(a_d), \mathrm{sgn}(a_{d-1}), \ldots, \mathrm{sgn}(a_0))$. We deal mainly with monic polynomials in which case sign patterns begin with a $+$. In this case, we can use instead of and equivalently to a sign pattern the corresponding *change-preservation pattern* which is a d-vector and (by some abuse of notation) whose jth component equals c if $a_{d-j+1}a_{d-j} < 0$ and p if $a_{d-j+1}a_{d-j} > 0$. One can consider also the moduli of the roots of a hyperbolic polynomial defining a given sign pattern. We study the generic case when all moduli are distinct. A natural question to ask is:

Question 1.1. *When these moduli are ordered on the real positive half-axis, at which positions can the moduli of the negative roots be?*

Descartes' rule of signs provides no hint for the answer to this question. In the present paper, we recall known and we introduce new results in this direction which show how far from trivial the situation is.

Notation 1.1. *(1) We denote by $0 < \alpha_1 < \cdots < \alpha_c$ the positive and by $0 < \gamma_1 < \cdots < \gamma_p$ the moduli of the negative roots of a hyperbolic polynomial. We explain the notation of the order of*

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*Corresponding author: Vladimir Petrov Kostov; vladimir.kostov@unice.fr
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these moduli on the positive half-axis by an example. Suppose that $d = 6$, $c = 2$, $p = 4$ *and*

$$\alpha_1 < \gamma_1 < \gamma_2 < \alpha_2 < \gamma_3 < \gamma_4 \ .$$

Then for the order of moduli we write $PNNPNN$, *i. e. the letters* P *and* N *denote the relative positions of the moduli of the positive and negative roots.*

(2) *A sign pattern beginning with* i_1 *signs* $+$ *followed by* i_2 *signs* $-$ *followed by* i_3 *signs* $+$ *etc. is denoted by* $\Sigma_{i_1,i_2,i_3,\ldots}$.

In what follows, we consider for each given degree d couples of the form (change-preservation pattern, order of moduli) (called *couples* for short). Such a couple is *compatible* with Descartes' rule of signs if the number of components c (resp. p) of the change-preservation pattern is equal to the number of components P (resp. N) of the order of moduli. A couple is called *realizable* if there exists a polynomial defining the change-preservation pattern of the couple and whose moduli of roots define the given order.

Remark 1.1. *For fixed* d *and* c, *there are* $\binom{d}{c}$ *change-preservation patterns and* $\binom{d}{c}$ *orders of moduli hence* $\binom{d}{c}^2$ *compatible couples. Thus for a given degree* d, *the total number of compatible couples is*

$$\chi(d) := \sum_{c=0}^{d} \binom{d}{c}^2 = \sum_{c=0}^{d} \binom{d}{c}\binom{d}{d-c} = \binom{2d}{d} \ . \tag{1.1}$$

This is the coefficient of x^d *in the polynomial* $(x+1)^d(x+1)^d = (x+1)^{2d}$. *Using Stirling's formula* $n! \sim \sqrt{2\pi n}(n/e)^n$, *one concludes that* $\chi(d) \sim 2^{2d}/\sqrt{\pi d}$.

Example 1.1. (1) *For* $d = 1$, *the only compatible couples are* $(c,\ P)$ *and* $(p,\ N)$. *They are realizable respectively by the polynomials* $x - 1$ *and* $x + 1$.

(2) *For* $d = 2$, *there are* $\binom{4}{2} = 6$ *compatible couples. Out of these, the couples* $(cp,\ PN)$ *and* $(pc,\ NP)$ *are not realizable. Indeed, for a hyperbolic polynomial* $x^2 - ux - v$ *(resp.* $x^2 + ux - v$*),* $u > 0$, $v > 0$, *one has the order of moduli* NP *(resp.* PN*). The remaining* 4 *couples are realizable. To see this one can consider the family of polynomials* $x^2 + a_1x + a_0$. *In the plane of the variables* (a_1, a_0) *the domain of hyperbolic polynomials is the one below the parabola* $\mathcal{P} : a_0 = a_1^2/4$. *We list the realizable couples and the open domains in which they are realizable:*

$$(cc,\ PP) \quad \{a_1 < 0,\ 0 < a_0 < a_1^2/4\} \ , \qquad (pp,\ NN) \quad \{a_1 > 0,\ 0 < a_0 < a_1^2/4\} \ ,$$

$$(cp,\ NP) \quad \{a_1 < 0,\ a_0 < 0\} \ , \qquad (pc,\ PN) \quad \{a_1 > 0,\ a_0 < 0\} \ .$$

We can make Question 1.1 more precise:

Question 1.2. *For a given degree* d, *which compatible couples are realizable?*

The above example answers this question for $d = 1$ and 2. For $d = 3$, 4 and 5, the exhaustive answer is given in Section 3.

Remark 1.2. *There exist two commuting involutions acting on the set of degree* d *polynomials with non-vanishing coefficients. These are*

$$i_m \ : \ Q(x) \mapsto (-1)^d Q(-x) \quad \text{and} \quad i_r \ : \ Q(x) \mapsto x^d Q(1/x)/Q(0) \ .$$

The role of the factors $(-1)^d$ *and* $1/Q(0)$ *is to preserve the set of monic polynomials. When acting on a couple, the involution* i_m *changes the components* c *to* p, P *to* N *and vice versa while the involution* i_r *reads the vectors of a given couple from the right. A given couple is realizable or not simultaneously with all other couples from its orbit under the action of* i_m *and* i_r. *An orbit consists of four or two couples.*

Notation 1.2. *For a sign pattern σ, we denote by $k^*(\sigma)$ the number of orders of moduli with which σ is realizable. For an order of moduli Ω, we denote by $l_*(\Omega)$ the number of sign patterns realizable with Ω. For a given d, we denote by $\tilde{r}^*(d)$ the ratio between the numbers of realizable and of all compatible couples.*

Example 1.2. *(1) For the sign pattern $\Sigma_{3,3,1}$ one has $k^*(\Sigma_{3,3,1}) = 6$. Indeed, consider the polynomial*

$$\begin{aligned}&(x-1)(x+1)^4(x-b)\\ =&x^6 + (3-b)x^5 + (2-3b)x^4 + (-2b-2)x^3 + (2b-3)x^2 + (3b-1)x + b\,.\end{aligned}$$

For $b > 0$ sufficiently small, it defines the sign pattern $\Sigma_{3,3,1}$. One can perturb its 4-fold root at -1 to obtain polynomials with the same sign pattern and with exactly k moduli of negative roots which are > 1 and $4 - k$ moduli which are < 1, where $k = 0, 1, \ldots, 4$; these moduli are close to 1. On the other hand, the only other realizable order with this sign pattern is

$$\gamma_1 < \alpha_1 < \alpha_2 < \gamma_2 < \gamma_3 < \gamma_4\ , \quad \text{i. e.} \quad NPPNNN\ ,$$

see [11, Theorems 3 and 4], *which makes a total of 6 orders of moduli realizable with $\Sigma_{3,3,1}$.*

(2) For $m \geq 1$, $n \geq 1$, one has $k^*(\Sigma_{m,n}) = 2\min(m,n) - 1$, see [11, Theorem 1 and Corollary 1].

Our first result is the following theorem:

Theorem 1.1. *(1) For $d \geq 1$, the only orders realizable with all compatible change-preservation patterns are $PP\ldots P$ and $NN\ldots N$. The corresponding change-preservation patterns are $cc\ldots c$ and $pp\ldots p$.*

(2) For any $d \geq 1$, there exist sign patterns realizable with all compatible orders. For $d \geq 5$, there exist sign patterns with $c = 2$ which are realizable with all $\binom{d}{2}$ compatible orders.

(3) There exists no sign pattern σ such that $k^*(\sigma) = 2$.

(4) The only sign patterns σ with $k^*(\sigma) = 3$ are the ones of the form $\Sigma_{2,d-1}$, $i_r(\Sigma_{2,d-1})$, $i_m(\Sigma_{2,d-1})$ and $i_r i_m(\Sigma_{2,d-1})$.

(5) For any $\ell \in \mathbb{N}^*$, there exist a degree d and an order Ω such that $l_*(\Omega) = \ell$.

The theorem is proved in Section 4. In Section 2, we recall some notions and known results and we continue the formulation of the new ones. In particular, for each of the 6 classes of non-realizable couples introduced in Section 2, we compare the number of couples which it contains with the number of all compatible couples, see (1.1). In all 6 cases, the limit of their ratio as $d \to \infty$ is 0 (see part (2) of Remarks 2.3, part (2) of Remarks 2.4, Remark 2.5, Remark 2.6, Remark 2.7 and part (4) of Theorem 2.3). On the other hand, when considering the cases $d = 3$, 4 and 5 in Section 3, we arrive to the conclusion that it is plausible to have $\lim_{d\to\infty} \tilde{r}^*(d) = 0$ (see Notation 1.2). This however cannot be explained by the presence of the 6 classes of non-realizable couples, so for the moment it is not evident what the exhaustive answer to Question 1.2 should be.

We finish this section by a result of geometric nature. Consider the space of coefficients $Oa_{d-1}\cdots a_0 \cong \mathbb{R}^d$. The *hyperbolicity domain* is the set of values of $(a_{d-1}, \ldots, a_0)$ for which the corresponding monic polynomial Q is hyperbolic. The resultant $R := \mathrm{Res}(Q(x), (-1)^d Q(-x), x)$ vanishes exactly when Q has two opposite roots or a root at 0. When the coefficients a_j are real, the polynomials $Q(x)$ and $Q(-x)$ have a root in common either when $Q(0) = 0$ or when Q has two opposite real non-zero roots or when Q has a pair of purely imaginary roots.

Example 1.3. *For $d = 1$, 2 and 3, one obtains $R = -2a_0$, $R = 4a_0a_1^2$ and $R = -8a_0(a_2a_1 - a_0)^2$, respectively.*

We denote by [.] the integer part and we set

$$Q^1 := x^{[d/2]} + a_{d-2}x^{[d/2]-1} + a_{d-4}x^{[d/2]-2} + \cdots \ ,$$

$$Q^2 := a_{d-1}x^{[(d-1)/2]} + a_{d-3}x^{[(d-1)/2]-1} + a_{d-5}x^{[(d-1)/2]-2} + \cdots \quad \text{and}$$

$$R_0 := \mathrm{Res}(Q^1(x), Q^2(x), x)) \ .$$

Theorem 1.2. *(1) One has* $R = (-1)^{[d/2]+1}2^{d-[(d+1)/2]+1}a_0R_0^2$.
(2) The quantity R_0 *is an irreducible polynomial in the variables* a_j.

The theorem is proved in Section 5. Properties of the set $\{R_0 = 0\}$ and its pictures for $d \leq 4$ can be found in [8].

2. Canonical sign patterns, rigid orders of moduli and further results

Definition 2.1. *For a given change-preservation pattern, the corresponding canonical order is obtained by reading the pattern from the right and by replacing each component* c *(resp.* p*) by* P *(resp. by* N*). E. g., the canonical order corresponding to the pattern* $ccpcp$ *is* $NPNPP$*. This definition allows to define the canonical order corresponding to each given sign pattern beginning with* $+$.

Each sign or change-preservation pattern is realizable with its canonical order, see [12, Proposition 1].

Definition 2.2. *(1) A sign pattern (or equivalently a change-preservation pattern) realizable only with its corresponding canonical order is called* canonical.
(2) If all monic hyperbolic polynomials having a given order of moduli define one and the same sign pattern, then the order is called rigid.

Remark 2.3. *(1) It is shown in [13] that canonical are exactly these sign patterns which have no four consecutive signs equal to*

$$(+,+,-,-,) \ , \quad (-,-,+,+) \ , \quad (+,-,-,+) \quad \text{or} \quad (-,+,+,-) \ .$$

Hence canonical are these change-preservation patterns having no isolated sign changes and no isolated sign preservations, i. e. having no three consecutive components cpc *or* pcp.
(2) In the proof of Proposition 10 in [13], the set of all *canonical change-preservation patterns is represented as union of four subsets, namely of patterns beginning with a single* p *or* c*, patterns ending by a single* p *or* c*, patterns both beginning and ending by a single* p *or* c *and patterns whose two first letters are equal and whose last two letters are also equal. For* $d \geq 100$*, the number of patterns in each of these sets can be majorized by* $2 \cdot [d/2] \cdot 2^{d-[0.26d]-1}$*. Hence the number of* all *canonical sign-preservation patterns is* $\leq \tau(d) := 8 \cdot [d/2] \cdot 2^{d-[0.26d]-1}$ *and for large* d*, the number of all non-realizable couples with canonical sign-preservation patterns is*

$$\leq \tau(d) \sum_{c=0}^{d} \binom{d}{c} = 8 \cdot [d/2] \cdot 2^{2d-[0.26d]-1} < 2^{2d}/\sqrt{\pi d} \sim \chi(d) \ ,$$

see Remark 1.1; we majorize one of the factors $\binom{d}{c}$ *in (1.1) by* $\tau(d)$.

Remark 2.4. *(1) It is proved in [14] that rigid are the orders of moduli* $PP\ldots P$, $NN\ldots N$ *(defining the change-preservation patterns* $cc\ldots c$ *and* $pp\ldots p$*, the two corresponding couples are realizable by any polynomials having distinct positive or distinct negative roots) and also*

$$P_N := PNPNPN\ldots \ , \quad N_P := NPNPNP\ldots \ . \tag{2.2}$$

Each of the latter two orders (we call them standard*) defines, depending on the parity of d, one of the sign patterns*

$$\sigma_+ := (+,+,-,-,+,+,-,-,\ldots) \quad \text{or} \quad \sigma_- := (+,-,-,+,+,-,-,+,+,\ldots)\,. \tag{2.3}$$

(2) *For each fixed degree d, there are $\binom{d}{[d/2]}$ compatible couples with the order P_N and $\binom{d}{[d/2]}$ with the order N_P, see (2.2). Hence there are $2\binom{d}{[d/2]}-2$ compatible couples in which the order of moduli is rigid (more exactly standard) and which are not realizable, and one has $\lim_{d\to\infty}(2\binom{d}{[d/2]}-2)/\chi(d)=0$, see (1.1) and use Stirling's formula.*

Definition 2.3. *We call superposition of two standard orders of moduli Ω_1 and Ω_2 any order obtained as follows. One inserts the components of Ω_2 at any places between the components of Ω_1 or in front of the first or after the last component of Ω_1 by preserving their relative order. Example: the order*

$$P\bar{N}NP\bar{P}N\bar{N}\bar{P}\bar{N} \quad \text{is superposition of} \quad PNPN \quad \text{and} \quad NPNPN$$

(we overline in this superposition the moduli coming from Ω_2; in this example there is more than one way to attribute the moduli of roots in the superposition as coming from Ω_1 or Ω_2; the superposition of two standard orders is not uniquely defined).

The following proposition explains how one can obtain new examples of non-realizable couples on the basis of standard orders.

Proposition 2.1. *Each superposition of two standard orders is realizable only with sign patterns of the form*

$$(+,+,?,-,?,+,?,-,\ldots)\,, \quad (+,?,-,?,+,?,-,\ldots) \quad \text{or} \quad (+,-,?,+,?,-,?,+,\ldots)$$

which are the "products" of sign patterns $\sigma_+\sigma_+$, $\sigma_+\sigma_-$ and $\sigma_-\sigma_-$.

Proof. Indeed, suppose that in the superposition of standard orders, the roots coming from the order Ω_i are roots of a polynomial T_i, $i=1,2$. Then in the product T_1T_2 every second coefficient, the leading coefficient and the constant term are sums of products of a coefficient of T_1 and a coefficient of T_2 either all with opposite or all with same signs, so the corresponding components of the "products" of sign patterns are well-defined. $\square$

Remark 2.5. *The number of letters N in a standard order is equal to the number of letters P or differs from the latter by 1. Hence in the superposition of two standard orders the modulus of this difference is majorized by 2. Besides, not more than $[d/2]$ of the signs of coefficients are not determined by the order of moduli, so the number of non-realizable couples corresponding to superpositions of standard orders is less than*

$$2\left(\binom{d}{[d/2]}+\binom{d}{[d/2]-1}+\binom{d}{[d/2]-2}\right)\cdot 2^{[d/2]} < 6\binom{d}{[d/2]}\cdot 2^{(d+1)/2}$$

which is $\sim 12\cdot 2^{3d/2}/\sqrt{\pi d}$ (we use Stirling's formula here). At the same time $\chi(d)\sim 2^{2d}/\sqrt{\pi d}$ (see Remark 1.1).

There exist other situations in which the order of moduli defines the signs of part of the coefficients of the polynomial.

Example 2.4. *Consider for $d=8k+2$, $k\in\mathbb{N}^*$, and for $c=2$ the order of moduli*

$$\Omega \;:\; \gamma_1<\cdots<\gamma_{4k}<\alpha_1<\alpha_2<\gamma_{4k+1}<\cdots<\gamma_{8k}\,.$$

It is realizable only with sign patterns having two sign changes. Denote by U_1 and U_2 monic hyperbolic degree $4k+1$ polynomials with roots

$$-\gamma_1\,,\quad -\gamma_2\,,\ldots,\quad -\gamma_{2k}\,,\quad -\gamma_{4k+1}\,,\quad -\gamma_{4k+2}\,,\;\ldots\,,\quad -\gamma_{6k}\,,\quad \alpha_1$$

and

$$-\gamma_{2k+1}\ ,\quad -\gamma_{2k+2}\ ,\ \ldots\ ,\quad -\gamma_{4k}\ ,\quad -\gamma_{6k+1}\ ,\quad -\gamma_{6k+2}\ ,\ \ldots\ ,\quad -\gamma_{8k}\ ,\quad \alpha_2$$

respectively. Hence they define sign patterns of the form Σ_{m_i,n_i}*,* $i = 1, 2$*. According to* [11, Theorem 1]*, if* $n_i < m_i$*, then the polynomial* U_i *has* $\leq 2n_i - 2$ *moduli of negative roots which are* $\leq \alpha_i$*; if* $n_i > m_i$*, then it has* $\leq 2m_i - 2$ *moduli of negative roots which are* $\geq \alpha_i$*. Hence one has* $n_i \geq k+1$ *and* $m_i \geq k+1$*. This implies that the first* $k+1$ *and the last* $k+1$ *coefficients of the product* U_1U_2 *are positive, i. e. the order of moduli* Ω *is not realizable with sign patterns* Σ_{j_1,j_2,j_3} *which do not satisfy the conditions* $j_1 \geq k+1$ *and* $j_3 \geq k+1$*.*

Remark 2.6. *There are* $\binom{d}{2}^2$ *compatible couples with* $c = 2$ *hence less than* $\binom{d}{2}^2$ *non-realizable couples concerned by Example 2.4. Using the involution* i_m *(see Remark 1.2), one can give as many such examples with* $c = d - 2$*. One has* $\lim_{d\to\infty} \binom{d}{2}^2/\chi(d) = 0$*, see (1.1).*

The proposition and theorem that follow describe other situations in which certain compatible couples are not realizable.

Proposition 2.2. *Suppose that* d *is even, that the leading monomial and the constant term are positive (hence* c *is even), that all coefficients of odd powers are negative and that* $c < d$*. Then there is no modulus of a negative root in any of the intervals* $(0, \alpha_1)$*,* (α_2, α_3)*, …,* $(\alpha_{c-2}, \alpha_{c-1})$*,* (α_c, ∞)*.*

Proof. Indeed, for a monic hyperbolic polynomial Q satisfying these conditions one has $Q(t) > 0$, if t belongs to any of the mentioned intervals. As all odd monomials are with negative coefficients, one has also $Q(-t) > Q(t)$ from which the proposition follows. □

Remark 2.7. For d even, the number of sign patterns as defined in Proposition 2.2 is $\leq 2^{d/2}$ (half of the signs of coefficients are fixed), so if d is large, then the number of such non-realizable couples is

$$\leq 2^{d/2}\sum_{c=0}^{d}\binom{d}{c} = 2^{3d/2} < \chi(d) \sim 2^{2d}/\sqrt{\pi d}\,,$$

see Remark 1.1.

Theorem 2.3. *(1) Suppose that*

$$c \leq p \quad \text{and} \quad \alpha_c < \gamma_p, \quad \alpha_{c-1} < \gamma_{p-1}\ ,\ \ldots\ ,\quad \alpha_1 < \gamma_{p-c+1}\ . \tag{2.4}$$

Then $a_{d-1} > 0$*. Hence a couple with* $a_{d-1} < 0$ *and order satisfying conditions (2.4) is not realizable.*

(2) For fixed d*, the number of orders of moduli satisfying conditions (2.4) is*

$$T_d^c := \binom{d}{c} - C_0\binom{d-1}{c-1} - C_1\binom{d-3}{c-2} - C_2\binom{d-5}{c-3} - C_3\binom{d-7}{c-4} - \cdots\ , \tag{2.5}$$

where $C_k := \binom{2k}{k}/(k+1)$ *is the* $k-$*th Catalan number.*

(3) One has

$$T_d^c = \binom{d}{c}\left(1 - \frac{c}{d-c+1}\right) = \binom{d}{c}\frac{d-2c+1}{d-c+1}\ . \tag{2.6}$$

(4) For the number $\nu(d)$ *of non-realizable couples satisfying condition (2.4) and with* $a_{d-1} < 0$ *one has* $\lim_{d\to\infty} \nu(d)/\chi(d) = 0$*, see (1.1).*

Remark 2.8. *The quantity $T_d^c\binom{d-1}{c}$ (resp. $\binom{d}{c}\binom{d-1}{c}$) is the number of couples in which the change-preservation pattern begins with p and the order satisfies condition (2.4) (resp. of all compatible couples in which the change-preservation pattern begins with p). For c fixed, one has $\lim_{d\to\infty} T_d^c/\binom{d}{c} = 1$. Indeed, this is the ratio of two degree c polynomials in d whose leading coefficients equal $1/c!$.*

Proof of Theorem 2.3. Part (1). Indeed, $a_{d-1} = \gamma_1 + \cdots + \gamma_p - \alpha_1 - \cdots - \alpha_c > 0$.

Part (2). The first term in the right-hand side of (2.5) is the number of all orders with c components equal to P. The second term is the number of orders beginning with P; they do not satisfy conditions (2.4). The third (resp. the fourth) term is the number of orders beginning with NPP (resp. with $NPNPP$ or $NNPPP$). The fifth term is the number of orders beginning with $NPNPNPP$, $NNPPNPP$, $NPNNPPP$, $NNPNPPP$ or $NNNPPPP$ etc.

That is, for $k \geq 2$, the kth term is the number of orders among whose first $2k-1$ components there are k letters P and which are not included in one of the previous terms (excluding the initial $\binom{d}{c}$). In an equivalent way, the kth term contains orders among whose $2k-2$ first components there are exactly $k-1$ letters P and for $s \leq 2k-2$, among their s first letters there are not less letters N than letters P. Hence this is the number of lattice paths in the plane with possible steps $(1,1)$ and $(1,-1)$ going from $(0,0)$ to $(2k-2,0)$ which do not descend below the abscissa-axis. The number of such paths is C_{k-1}.

Part (3). Formula (2.6) can be proved by induction on d. For $d=1$ and 2 and for $c \leq d$, it is to be checked directly. Suppose that it is true for $d \leq d_0$. Then for $d = d_0+1$, one applies to any binomial coefficient in the formula the well-known equality $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. Thus

$$\begin{aligned} T_d^c = T_{d-1}^c + T_{d-1}^{c-1} &= \tbinom{d-1}{c}\left(1 - \tfrac{c}{d-c}\right) + \tbinom{d-1}{c-1}\left(1 - \tfrac{c-1}{d-c+1}\right) \\ &= \tbinom{d}{c}\left(1 - \tfrac{c}{d-c+1}\right) , \end{aligned}$$

where the rightmost equality is to be checked straightforwardly.

Part (4). Suppose that $d = 2k$, $k \in \mathbb{N}^*$. Set

$$h_{k,m} := \frac{k(k-1)\cdots(k-m+1)}{(k+1)(k+2)\cdots(k+m)} , \quad \text{so} \quad \binom{2k}{k-m} = \binom{2k}{k} h_{k,m} .$$

For k fixed, the sequence $h_{k,m}$ is decreasing in m; one has $h_{k,0} = 1$. The sum $\sum_{c=0}^{d}\binom{d}{c}^2$ of all compatible couples equals $\tilde{b} := \binom{2k}{k}^2(1 + 2\sum_{m=1}^{k} h_{k,m}^2)$. The number $\nu(d) = \nu(2k)$ is bounded by

$$\sum_{c=0}^{k}\binom{2k}{c}T_{2k}^c = \sum_{m=0}^{k}\binom{2k}{k-m}T_{2k}^{k-m} = \binom{2k}{k}^2 \sum_{m=0}^{k}\frac{2m+1}{k+m+1}h_{k,m}^2$$

(we remind that the orders satisfying condition (2.4) are defined under the assumption that $c \leq p$). Fix $s \in (0,1)$. Then

$$g_1 := \sum_{m=0}^{[sk]}\frac{2m+1}{k+m+1}h_{k,m}^2 \leq \frac{2[sk]+1}{k+[sk]+1}\sum_{m=0}^{[sk]}h_{k,m}^2 .$$

It is clear that $g_1 < \frac{2[sk]+1}{k+[sk]+1}\sum_{m=0}^{k}h_{k,m}^2$, so

$$\binom{2k}{k}^2 g_1 < \frac{2[sk]+1}{k+[sk]+1}\tilde{b} . \tag{2.7}$$

For large values of k and for $m \geq [sk]+1$, the quantity $h_{k,m}$ is majorized by

$$\frac{(k-[sk/2])\cdots(k-m+1)}{(k+[sk/2]+1)\cdots(k+m)} \leq \left(\frac{k-[sk/2]}{k+[sk/2]+1}\right)^{[sk]-[sk/2]} \left(\frac{k-[sk]+1}{k+[sk]}\right)^{m-[sk]-1} .$$

Set $u := \frac{k-[sk/2]}{k+[sk/2]+1}$ and $v := \frac{k-[sk]+1}{k+[sk]}$. Hence

$$\begin{array}{rclcl} g_2 & := & \sum_{m=[sk]+1}^{k} h_{k,m}^2 & < & u^{[sk]-[sk/2]} \sum_{m=[sk]+1}^{\infty} v^{m-[sk]-1} \\ & = & \frac{u^{[sk]-[sk/2]}}{1-v} & = & u^{[sk]-[sk/2]} \frac{k+[sk]}{2[sk]+1} . \end{array}$$

The latter quantity tends to 0 as $k \to \infty$, therefore

$$\lim_{k\to\infty} \binom{2k}{k}^2 g_2/\tilde{b} = 0.$$

As

$$g_3 := \sum_{m=[sk]+1}^{k} \frac{2m+1}{k+m+1} h_{k,m}^2 < g_2,$$

one obtains

$$\lim_{k\to\infty} \binom{2k}{k}^2 g_3/\tilde{b} = 0 . \tag{2.8}$$

One has $\nu(d) \leq \binom{2k}{k}^2 (g_1+g_3)$. The coefficient of $\tilde{b}$ in (2.7) can be made smaller than any positive number by choosing s small enough. Therefore inequality (2.7) and equality (2.8) imply part (4) of Theorem 2.3 for d even.

If $d = 2k+1$, $k \in \mathbb{N}^*$, then one can prove part (4) in much the same way, so we point out only some technical differences. One sets

$$h_{k,m} := \frac{k(k-1)\cdots(k-m+1)}{(k+2)(k+3)\cdots(k+m+1)} , \quad \text{so} \quad \binom{2k+1}{k-m} = \binom{2k+1}{k} h_{k,m} ,$$

and $\tilde{b} = 2\binom{2k+1}{k}^2 (1 + \sum_{m=1}^{k} h_{k,m}^2)$. The definitions of the quantities g_1, g_2 and g_3 are the same, but with respect to the new formula for $h_{k,m}$. One sets $u := \frac{k-[sk/2]}{k+[sk/2]+2}$ and $v := \frac{k-[sk]+1}{k+[sk]+1}$. Inequality (2.7) and equality (2.8) remain the same. □

3. Realizable couples for $d = 3$, 4 and 5

We give the exhaustive answer to Question 1.2 for $d = 3, 4$ and 5; for $d = 1$ and 2, this answer is given by Example 1.1; one finds that $\tilde{r}^*(1) = 1$ and $\tilde{r}^*(2) = 2/3$, see Notation 1.2. It is clear from part (1) of Theorem 1.1 that $\tilde{r}^*(1) < 1$ for $d > 1$. We make use of the involution i_m, see Remark 1.2, to consider only the cases with $a_{d-1} > 0$. For $d = 3$, we give the list of sign patterns and (non)-realizable orders in the following table:

sign pattern	realizable orders	non − realizable orders
$(+,+,+,-)$	PNN	NPN , NNP
$(+,+,-,-)$	PNN , NPN , NNP	
$(+,+,+,+)$	NNN	
$(+,+,-,+)$	PPN	NPP , PNP .

Thus $\tilde{r}^*(3) = 3/5$. The (non)-realizability of these cases can be justified using the results in [11]. For $d = 4$, we list the sign patterns by the value of c:

c	sign pattern	realizable orders	non − realizable orders
0	$(+,+,+,+,+)$	$NNNN$	
1	$(+,+,+,+,-)$	$PNNN$	$NPNN$, $NNPN$, $NNNP$
	$(+,+,+,-,-)$	$PNNN$, $NPNN$, $NNPN$	$NNNP$
	$(+,+,-,-,-)$	$NPNN$, $NNPN$, $NNNP$	$PNNN$
2	$(+,+,-,+,+)$	$NPPN$	$NNPP$, $NPNP$, $PNNP$ $PNPN$, $PPNN$
	$(+,+,-,-,+)$	$PNPN$, $NPPN$, $PPNN$, $PNNP$	$NPNP$, $NNPP$
	$(+,+,+,-,+)$	$PPNN$	$PNPN$, $NPPN$, $NPNP$ $PNNP$, $NNPP$
3	$(+,+,-,+,-)$	$PPPN$	$NPPP$, $PNPP$, $PPNP$

Hence $\tilde{r}^*(4) = 3/7$. The (non)-realizability of the cases can be proved using the results in [11]. The involution i_m transforms the sign pattern with $c = 3$ into $(+,-,-,-,-)$. We illustrate the realizability of the cases with the sign pattern $(+,+,-,-,+)$ by examples:

$$PNPN \quad (x+1.3)(x-1.2)(x+1.1)(x-1) = x^4 + 0.2x^3 - 2.65x^2 - 0.266x + 1.716$$

$$NPPN \quad (x+2)(x-1)(x-0.9)(x+0.8) = x^4 + 0.9x^3 - 2.82x^2 - 0.52x + 1.44$$

$$PPNN \quad (x+2)(x+1.1)(x-1)(x-0.1) = x^4 + 2x^3 - 1.11x^2 - 2.11x + 0.22$$

$$PNNP \quad (x-2)(x+1.9)(x+1)(x-0.8) = x^4 + 0.1x^3 - 4.62x^2 - 0.68x + 3.04 \,.$$

For $d=5$, we show for each sign pattern only the number of realizable and the total number of orders compatible with the sign pattern and in some cases the realizable orders. To justify the tables below, one can use the results in [11] and [13]. There are the following canonical sign patterns:

$$\begin{array}{llllll}
c=0 & (+,+,+,+,+,+) & 1/1 & c=1 & (+,+,+,+,+,-) & 1/5 \\[1em]
c=2 & (+,+,-,+,+,+) & 1/10 & c=3 & (+,+,-,+,-,-) & 1/10 \\
 & (+,+,+,-,+,+) & 1/10 & & (+,+,+,-,+,-) & 1/10 \\
 & (+,+,+,+,-,+) & 1/10 & & & \\[1em]
c=4 & (+,+,-,+,-,+) & 1/5. & & &
\end{array}$$

The remaining sign patterns are:

$$\begin{array}{llcl}
c=1 & (+,+,+,+,-,-) & PNNNN\,, & 3/5 \\
 & & NPNNN\,,\ NNPNN & \\
 & (+,+,+,-,-,-) & & 5/5 \\
 & (+,+,-,-,-,-) & NNPNN\,, & 3/5 \\
 & & NNNPN\,,\ NNNNP & \\[1em]
c=2 & (+,+,-,-,-,+) & PPNNN\,, & 5/10 \\
 & & PNPNN\,,\ PNNPN\,, & \\
 & & PNNNP\,,\ NPPNN & \\
 & (+,+,+,-,-,+) & PPNNN\,,\ PNPNN\,, & 4/10 \\
 & & PNNPN\,,\ NPPNN & \\
 & (+,+,-,-,+,+) & & 10/10 \\[1em]
c=3 & (+,+,-,+,+,-) & & 5/10 \\
 & (+,+,-,-,+,-) & & 4/10.
\end{array}$$

Therefore $\tilde{r}^*(5)=47/126$. The two latter sign patterns (with $c=3$) are obtained from two of the sign patterns with $c=2$ via the involution $i_m i_r$.

The realizability of the sign pattern $(+,+,-,-,+,+)$ with all possible orders results from

$$(x+1)^3(x-1)^2 = x^5 + x^4 - 2x^3 - 2x^2 + x + 1\,.$$

Indeed, by perturbing the triple root at -1 and the double root at 1, one obtains polynomials with the same sign pattern and with any order of the moduli of the roots, see the proof of part (2) of Theorem 1.1.

Remark 3.9. *We obtained the following sequence for the values of the quantity* $\tilde{r}^*(d)$*:* 1, $2/3$, $3/5$, $3/7$, $47/126$, *…. One could conjecture that the sequence is decreasing. For the sequence of the ratios of two consecutive terms, one gets*

$$2/3 = 0.66\ldots\,, \quad 9/10 = 0.9\,, \quad 5/7 = 0.71\ldots\,, \quad 47/54 = 0.87\ldots\,.$$

It seems that the even and the odd terms form two adjacent sequences and that $\lim_{d\to\infty}\tilde{r}^*(d) = 0^+$.

4. Proof of Theorem 1.1

Part (1). As already mentioned, for the orders $PP\ldots P$ and $NN\ldots N$, the only change-preservation patterns compatible with them are $cc\ldots c$ and $pp\ldots p$ respectively and the corresponding couples are realizable.

Suppose that for given $c > 0$ and $p > 0$, the order of moduli Ω is realizable with all compatible change-preservation patterns. Then, in particular, it is realizable with the sign patterns σ' and σ'', where σ' has all its c sign changes at the beginning followed by its p sign preservations and vice-versa for σ''. However, the sign patterns σ' and σ'' are canonical hence realizable only with their respective canonical orders Ω' and Ω'', see Definition 2.2. As $\Omega' \neq \Omega''$, the order Ω is not realizable with both σ' and σ''.

Part (2). For $d \geq 1$, the all-pluses sign pattern is realizable with its only compatible order $N \dots N$. To prove the rest of part (2) for $d \geq 5$, we construct sign patterns with $c = 2$ which are realizable with all compatible orders. Consider the polynomial

$$\begin{aligned}(x+1)^{d-2}(x-1)^2 &= \left(\sum_{k=0}^{d-2}\binom{d-2}{k}x^k\right)(x^2-2x+1) \\ &= \sum_{k=0}^{d} h_k x^k \ , \quad h_k := \binom{d-2}{k} - 2\binom{d-2}{k-1} + \binom{d-2}{k-2} \ .\end{aligned}$$

It has two sign changes (so its sign pattern is of the form Σ_{i_1,i_2,i_3}). To understand in which positions they are, one observes that

$$h_k = \frac{(d-2)!}{k!(d-k)!}(4k^2 - 4dk + d(d-1)) \ ,$$

so $h_k = 0$ if and only if $k = k_\pm := (d \pm \sqrt{d})/2$. If d is not an exact square, then the sign changes occur between the powers $x^{s_\pm}$ and $x^{s_\pm+1}$, where $s_\pm < k_\pm < s_\pm + 1$. If d is an exact square, then the coefficients of $x^{k_\pm}$ are 0.

Suppose that d is not an exact square. One can perturb the roots of the polynomial by keeping the sign pattern the same. If d is an exact square, then one can perturb them so that all coefficients become non-zero. One can choose such a perturbation for any possible order of the moduli of roots which proves part (2). One can observe that as $k_+ - k_- = \sqrt{d}$, for $d \geq 5$, there are at least two consecutive negative coefficients (i. e. $i_2 \geq 2$) and the sign pattern is not canonical.

We prove part (3) of the theorem by induction on d. For $d = 1$, 2 and 3, the claim is to be checked straightforwardly, see Example 1.1 and Section 3. Suppose that $d \geq 4$ and that σ is not canonical. Represent σ in the form $(\sigma_d, \sigma^\dagger, \sigma_0)$, where σ_d and σ_0 are its first and last components. Then at least one of the sign patterns $(\sigma_d, \sigma^\dagger)$ and $(\sigma^\dagger, \sigma_0)$ contains an isolated sign change or an isolated sign preservation. Suppose that this is $(\sigma_d, \sigma^\dagger)$. Then $(\sigma_d, \sigma^\dagger)$ is not canonical and hence is realizable by at least three orders by polynomials P_j. This means that σ is also realizable by at least three orders defined by the roots of the polynomials $P_j(x)(x \pm \varepsilon)$, where $\varepsilon > 0$ is small enough and the sign is $+$ (resp. $-$) if the last two components of σ are equal (resp. are different).

Part (4) is also proved by induction on d. For $d \leq 4$, it is to be checked directly. Suppose that $d \geq 5$. If neither of the sign patterns $(\sigma_d, \sigma^\dagger)$ and $(\sigma^\dagger, \sigma_0)$ contains an isolated sign change or sign preservation, then this is the case of σ as well, so σ is canonical and $k^*(\sigma) = 1$ – a contradiction. Hence at least one of these sign patterns is not canonical. Without loss of generality, we suppose that this is $(\sigma_d, \sigma^\dagger)$ (otherwise we apply the involution i_r). Hence $k^*((\sigma_d, \sigma^\dagger)) \geq 3$, so $k^*((\sigma_d, \sigma^\dagger)) = 3$, otherwise similarly to the proof of part (3) we obtain that $k^*(\sigma) > 3$. Applying if necessary the involution i_m, we assume that $(\sigma_d, \sigma^\dagger) = \Sigma_{2,d-2}$ or $\Sigma_{d-2,2}$. In the first case, one has $\sigma = \Sigma_{2,d-1}$. Indeed, if $\sigma = \Sigma_{2,d-2,1}$, then $k^*(\sigma) > 3$, see [11, Theorems 3 and 4]. In the second case, either $\sigma = \Sigma_{d-2,3}$ and $k^*(\sigma) = 5$ (see [11, Theorem 1]) or $\sigma = \Sigma_{d-2,2,1}$ and $k^*(\sigma) = 4$ (see [11, Theorems 3 and 4]).

Part (5). For d even, the order $\Omega := PNN \dots N$ is realizable exactly with the sign patterns $\Sigma_{m,n}$, $m + n = d + 1$, $n < m$, see [11, Theorem 1], so $\ell_*(\Omega) = d/2$.

5. Proof of theorem 1.2

Proof of part (1). A) For a vector-row v of length $2d$, we denote by v_ℓ the vector-row obtained from v by shifting v by ℓ positions to the right (the rightmost ℓ positions are then lost and the leftmost ℓ positions are filled with zeros). We represent R as determinant of the Sylvester $2d \times 2d$-matrix of the polynomials $Q(x)$ and $(-1)^d Q(-x)$ whose first and $(d+1)$st row equal respectively

$$u := (\,1 \quad a_{d-1} \quad a_{d-2} \quad a_{d-3} \quad a_{d-4} \quad \ldots \quad a_1 \quad a_0 \quad 0 \quad \ldots \quad 0\,)$$

and

$$w := (\,1 \quad -a_{d-1} \quad a_{d-2} \quad -a_{d-3} \quad a_{d-4} \quad \ldots \quad (-1)^{d-1}a_1 \quad (-1)^d a_0 \quad 0 \quad \ldots \quad 0\,)\,;$$

its second and $(d+2)$nd rows equal u_1 and w_1, its third and $(d+3)$rd rows equal u_2 and w_2 etc. For $d=2$ and $d=3$, we obtain the determinants

$$\begin{vmatrix} 1 & a_1 & a_0 & 0 \\ 0 & 1 & a_1 & a_0 \\ 1 & -a_1 & a_0 & 0 \\ 0 & 1 & -a_1 & a_0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 1 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 1 & a_2 & a_1 & a_0 \\ 1 & -a_2 & a_1 & -a_0 & 0 & 0 \\ 0 & 1 & -a_2 & a_1 & -a_0 & 0 \\ 0 & 0 & 1 & -a_2 & a_1 & -a_0 \end{vmatrix}\,.$$

B) For $j = 1, \ldots, d$, we add the $(j+d)$th row to the jth row. Hence the first row of the determinant is now

$$g := (\,2 \quad 0 \quad 2a_{d-2} \quad 0 \quad 2a_{d-4} \quad \ldots \quad 2a_{d-2[d/2]} \quad 0 \quad 0 \quad \ldots \quad 0\,)$$

and the next $d-1$ rows equal g_j, $j = 1, \ldots, d-1$. After this one subtracts the kth row multiplied by $1/2$ from the $(d+k)$th one, $k = 1, \ldots, d$. Hence, the $(d+1)$st row equals

$$h := (\,0 \quad -a_{d-1} \quad 0 \quad -a_{d-3} \quad 0 \quad \ldots \quad -a_{d-2[(d+1)/2]+1} \quad 0 \quad 0 \quad \ldots \quad 0\,)$$

and the next $d-1$ rows are of the form h_j, $j = 1, \ldots, d-1$. For $d=2$ and $d=3$, this gives

$$\begin{vmatrix} 2 & 0 & 2a_0 & 0 \\ 0 & 2 & 0 & a_0 \\ 0 & -a_1 & 0 & 0 \\ 0 & 0 & -a_1 & 0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 2 & 0 & 2a_1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2a_1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2a_1 & 0 \\ 0 & -a_2 & 0 & -a_0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_0 & 0 \\ 0 & 0 & 0 & -a_2 & 0 & -a_0 \end{vmatrix}\,.$$

C) We permute the rows of the determinant (which does not change the determinant up to a sign). In the first $d-[d/2]$ positions we place the first, third, fifth etc. rows, in the next $[d/2]$ positions the $(d+2)$nd, $(d+4)$th, $(d+6)$th etc. rows, in the next $[d/2]$ positions the second, fourth, sixth etc. rows and in the last $d-[d/2]$ positions the $(d+1)$st, $(d+3)$rd, $(d+5)$th etc. rows. After this permutation the first d rows have non-zero entries only in the odd and the last d rows have non-zero entries only in the even columns.

Then we permute the columns of the determinant placing the odd columns in the first d positions and the even columns in the last d positions by preserving the relative order of the even and odd columns. For $d=2$ and $d=3$, the result is

$$\begin{vmatrix} 2 & 2a_0 & 0 & 0 \\ 0 & -a_1 & 0 & 0 \\ 0 & 0 & 2 & 2a_0 \\ 0 & 0 & -a_1 & 0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 2 & 2a_1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2a_1 & 0 & 0 & 0 \\ 0 & -a_2 & -a_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2a_1 & 0 \\ 0 & 0 & 0 & -a_2 & -a_0 & 0 \\ 0 & 0 & 0 & 0 & -a_2 & -a_0 \end{vmatrix}.$$

For any $d \geq 2$, the determinant is now block-diagonal, with two diagonal blocks $d \times d$. For $d=4$, these blocks are

$$\begin{vmatrix} 2 & 2a_2 & 2a_0 & 0 \\ 0 & 2 & 2a_2 & 2a_0 \\ 0 & -a_3 & -a_1 & 0 \\ 0 & 0 & -a_3 & -a_1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 2 & 2a_2 & 2a_0 & 0 \\ 0 & 2 & 2a_2 & 2a_0 \\ -a_3 & -a_1 & 0 & 0 \\ 0 & -a_3 & -a_1 & 0 \end{vmatrix}.$$

The first and the $(d+1)$st rows equal respectively

$$\tilde{g} := (\, 2 \quad 2a_{d-2} \quad 2a_{d-4} \quad \ldots \quad 2a_{d-2[d/2]} \quad 0 \quad 0 \quad \ldots \quad 0 \,)$$

and $\tilde{g}_d$. The first $d-[d/2]$ rows equal $\tilde{g}$, $\tilde{g}_1$, $\tilde{g}_2$, …, $\tilde{g}_{d-[d/2]-1}$ while the rows with indices $d+1$, $d+2$, …, $d+[d/2]$ are $\tilde{g}_d$, $\tilde{g}_{d+1}$, …, $\tilde{g}_{d+[d/2]-1}$. The $(d-[d/2]+1)$st row equals

$$\tilde{h} := (\, 0 \quad -a_{d-1} \quad -a_{d-3} \quad -a_{d-5} \quad \ldots \quad -a_{d-2[(d+1)/2]+1} \quad 0 \quad 0 \quad \ldots \quad 0 \,)\,.$$

The next $[d/2]-1$ rows are $\tilde{h}_j$, $j=1, \ldots, [d/2]-1$. The last $d-[d/2]$ rows equal $\tilde{h}_k$, $k=d-1$, …, $2d-[d/2]-2$.

The total number of transpositions of rows and columns is even, so the sign of the determinant does not change.

D) One develops the determinant thus obtained w.r.t. its first and then w.r.t. its last column. For d even (resp. for d odd), this yields $-4a_0\Delta$ (resp. $-2a_0\Delta$), where the $(2d-2)\times(2d-2)$-determinant Δ is block-diagonal, with two diagonal blocks $(d-1)\times(d-1)$ each of which is the Sylvester matrix of the polynomials $2Q^1$ and $-Q^2$. This implies part (1) of the theorem. □

Proof of part (2). One can assign quasi-homogeneous weights to the variables a_j as follows: 0 to a_{d-1}, 1 to a_{d-2} and a_{d-3}, 2 to a_{d-4} and a_{d-5}, 3 to a_{d-6} and a_{d-7} etc., in accordance with the fact that a_{d-2}, a_{d-4}, … and a_{d-3}/a_{d-1}, a_{d-5}/a_{d-1}, … are up to a sign elementary symmetric polynomials of the roots of Q^1 and Q^2. Hence R_0 is a quasi-homogeneous polynomial of weight $d_0 := [(d-1)/2][d/2]$. For d even (resp. for d odd), it contains monomials $\alpha a_0^{[(d-1)/2]}a_{d-1}^{[d/2]}$ and $\beta a_1^{[d/2]}$, $\alpha \neq 0 \neq \beta$ (resp. $\gamma a_1^{[(d-1)/2]}a_{d-1}^{[d/2]}$ and $\delta a_0^{[d/2]}$, $\gamma \neq 0 \neq \delta$), all other monomials containing factors a_0^k and a_1^s only with $k < [(d-1)/2]$ and $s < [d/2]$ (resp. with $k < [d/2]$ and $s < [(d-1)/2]$). Hence R_0 cannot be the product of two quasi-homogeneous polynomials of weights b_1 and b_2, $0 < b_1, b_2 < d_0$. □

References

[1] F. Cajori: *A history of the arithmetical methods of approximation to the roots of numerical equations of one unknown quantity*. Colo. Coll. Publ. Sci. Ser., **12** (7) (1910), 171–215 .

[2] D. R. Curtiss: *Recent extensions of Descartes' rule of signs*, Ann. of Math., **19** (4) (1918), 251–278.

[3] J.-P. de Gua de Malves: *Démonstrations de la Règle de Descartes*, Pour connoître le nombre des Racines positives & négatives dans les Équations qui n'ont point de Racines imaginaires, Memoires de Mathématique et de Physique tirés des registres de l'Académie Royale des Sciences, (1741), 72–96.

[4] The Geometry of René Descartes with a facsimile of the first edition, translated by D. E. Smith and M.L. Latham, New York, Dover Publications, 1954.

[5] J. Forsgård, V. P. Kostov and B. Shapiro: *Could René Descartes have known this?*, Exp. Math., **24** (4) (2015), 438–448.
[6] J. Forsgård, D. Novikov and B. Shapiro: *A tropical analog of Descartes' rule of signs*, Int. Math. Res. Not. IMRN, **2017** (12), 3726–3750.
[7] J. Fourier: Sur l'usage du théorème de Descartes dans la recherche des limites des racines. Bulletin des sciences par la Société philomatique de Paris, (1820) 156–165, 181–187; œuvres 2, 291–309, Gauthier-Villars, 1890.
[8] Y. Gati, V. P. Kostov and M. C. Tarchi: *Sign patterns and rigid moduli orders*, Grad. J. Math., **6** (1) (2021), 60–72.
[9] C. F. Gauss: *Beweis eines algebraischen Lehrsatzes*. J. Reine Angew. Math., **3** (1828), 1-4; Werke 3, 67–70, Göttingen, 1866.
[10] J. L. W. Jensen: *Recherches sur la théorie des équations*, Acta Math., **36** (1913), 181–195 .
[11] V. P. Kostov: *Descartes' rule of signs and moduli of roots*, Publ. Math. Debrecen, **96** (1-2) (2020) 161–184,
[12] V. P. Kostov: *Hyperbolic polynomials and canonical sign patterns*, Serdica Math. J., **46** (2020) 135–150.
[13] V. P. Kostov: *Which Sign Patterns are Canonical*, Results Math., **77** (6) (2022), 235.
[14] V. P. Kostov: *Hyperbolic polynomials and rigid moduli orders*, Publ. Math. Debrecen, **100** (1-2) (2022), 119–128,
[15] V. P. Kostov: *The disconnectedness of certain sets defined after uni-variate polynomials*, Constr. Math. Anal., **5** (3) (2022), 119–133.
[16] E. Laguerre: *Sur la théorie des équations numériques*, Journal de Mathématiques pures et appliquées, s. 3, t. 9, 99–146 (1883); œuvres 1, Paris, 1898, Chelsea, New-York, 1972, pp. 3–47.
[17] B. E. Meserve: *Fundamental Concepts of Algebra*, New York, Dover Publications, 1982.

VLADIMIR PETROV KOSTOV
UNIVERSITÉ CÔTE D'AZUR
DÉPARTEMENT DE MATHÉMATIQUES
CNRS, LJAD, FRANCE
ORCID: 0000-0001-5836-2678
E-mail address: vladimir.kostov@unice.fr