A NOTE ON THE SOLVABILITY OF A FINITE GROUP IN WHICH EVERY NON-NILPOTENT MAXIMAL SUBGROUP IS NORMAL

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Abstract. We provide a new and simple proof to show that a finite group in which every non-nilpotent maximal subgroup is normal is solvable.

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1. Introduction

In this paper all groups are assumed to be finite. It is known that every maximal subgroup of a group $G$ is normal if and only if $G$ is a nilpotent group. As a generalization, Li and Shi [1] gave a proof to show that the following result holds.

Theorem 1.1. [1, Theorem 1.1] A group $G$ with all non-nilpotent maximal subgroups being normal is solvable.

Moreover, based on the solvability of the group $G$ in [1, Theorem 1.1], Shi [3, Theorem 5] proved that such a group $G$ has a Sylow tower.

In this paper, our main goal is to provide a new and simpler proof of [1, Theorem 1.1], see Section 2.

2. New proof of [1, Theorem 1.1]

Proof. We first claim that $G$ has a normal subgroup of prime-power order.

Suppose not. We divide the following discussions into three cases.

Case 1: Assume that every maximal subgroup of $G$ is nilpotent. It follows that $G$ is either a nilpotent group or a minimal non-nilpotent group. Then one can easily get that $G$ has a normal Sylow subgroup that has prime-power order by the structure of minimal non-nilpotent group [2, Theorem 9.1.9], a contradiction.

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Case 2: Assume that every maximal subgroup of $G$ is non-nilpotent. It follows that every maximal subgroup of $G$ is normal by the hypothesis and then $G$ is nilpotent, this contradicts that every maximal subgroup of $G$ is non-nilpotent.

Case 3: Assume that $G$ not only has nilpotent maximal subgroups but also has non-nilpotent maximal subgroups. Since $G$ has no normal subgroup of prime-power order, every Sylow $p$-subgroup $P$ of $G$ is not normal in $G$ for any prime divisor $p$ of $|G|$, that is $N_G(P) < G$. Then there exists a maximal subgroup $M$ of $G$ such that $N_G(P) \leq M$. Note that every non-nilpotent maximal subgroup of $G$ is normal and $P$ is not normal in $G$, one has that $M$ is nilpotent by the Frattini-argument. Therefore, every Sylow subgroup of $G$ is contained in some nilpotent maximal subgroup of $G$.

For any nilpotent maximal subgroup $M$ of $G$, if there exists a prime divisor $q$ of $|M|$ such that the Sylow $q$-subgroup $Q_M$ of $M$ is not a Sylow $q$-subgroup of $G$, then $N_G(Q_M) > M$ as $M$ being nilpotent. It follows that $Q_M$ is a normal subgroup of $G$ of prime-power order since $M$ is maximal in $G$, a contradiction.

Next assume that every Sylow subgroup of $G$ is also a Sylow subgroup of $G$ for any nilpotent maximal subgroup $M$ of $G$.

For the case when $G$ has exactly one nilpotent maximal subgroup $M$, then $M$ is normal in $G$, which implies that $G$ has a normal Sylow subgroup, a contradiction.

For another case when $G$ has at least two nilpotent maximal subgroups. Let $M_1$ and $M_2$ be any two distinct nilpotent maximal subgroups of $G$.

(i) Suppose $([M_1], [M_2]) = 1$. Let $N$ be a non-nilpotent maximal subgroup of $G$, then $G = M_1N = M_2N$. One has $|G| = \frac{|M_1||N|}{|M_1 \cap N|} = \frac{|M_2||N|}{|M_2 \cap N|}$ and then $|M_1| = \frac{|M_2|}{|M_2 \cap N|}$. Note that $(\frac{|M_1|}{|M_1 \cap N|}, \frac{|M_2|}{|M_2 \cap N|}) = 1$ by the hypothesis. It follows that $|M_1| = |M_1 \cap N|$ and then $M_1 \leq N$. One has $M_1 = N$ since $M_1$ is maximal in $G$, a contradiction.

(ii) Suppose $([M_1], [M_2]) > 1$ and $|M_1| \neq |M_2|$. Then there exists a prime $r$ such that $r \mid ([M_1], [M_2])$. Let $R_1$ be a Sylow $r$-subgroup of $M_1$ and $R_2$ be a Sylow $r$-subgroup of $M_2$. Since both $R_1$ and $R_2$ are also Sylow $r$-subgroups of $G$, there exists an $x \in G$ such that $R_2 = R_1^x$. That is $R_1x \in \text{Syl}_r(M_2)$. It follows that $R_1 \in \text{Syl}_r(M_2^{x^{-1}})$. Since $|M_1| \neq |M_2|$, one has $M_1 \neq M_2^{x^{-1}}$. Then $N_G(R_1) \geq \langle M_1, M_2^{x^{-1}} \rangle > M_1$. Thus $N_G(R_1) = G$ since $M_1$ is maximal in $G$, which implies that $R_1$ is a normal Sylow subgroup of $G$, a contradiction.

(iii) Suppose that all nilpotent maximal subgroups of $G$ have the same order. Let $M$ be any nilpotent maximal subgroup of $G$. Since every Sylow subgroup of $G$ is contained in some nilpotent maximal subgroup of $G$ and all nilpotent maximal subgroups of $G$ have the same order, one has $|G| = |M|$, a contradiction.
All above arguments imply that our assumption is not true. Hence $G$ has a normal subgroup of prime-power order.

In the following let $G_1$ be a normal subgroup of $G$ of prime-power order. Consider the quotient group $G/G_1$. It is clear that every non-nilpotent maximal subgroup of $G/G_1$ is also normal, arguing as above, one has that $G/G_1$ has a normal subgroup $G_2/G_1$ of prime-power order. We go on considering the quotient group $G/G_2$, one by one, we can obtain a normal subgroups series: $1 = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_i \triangleleft \cdots \triangleleft G_{s-1} \triangleleft G_s = G$, where $s > 1$ and every quotient group $G_i/G_{i-1}$ has prime-power order for each $1 \leq i \leq s$. Therefore, one has that $G$ is solvable. □

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References


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