Hacettepe Journal of Mathematics & Statistics
Hacet. J. Math. Stat.
Volume 53 (4) (2024), 952 – 962
DOI : 10.15672/hujms.1254445

Research Article

Spectral properties and inverse nodal problems for singular diffusion equation

Rauf Amirov*, Sevim Durak

Sivas Cumhuriyet University, Faculty of Science, Department of Mathematics, 58140, Türkiye

Abstract

In this study, some properties for the pencils of singular Sturm-Liouville operators are investigated. Firstly, the behaviors of eigenvalues were learned, then the solutions of the inverse problem were given to determine the potential function and parameters of the boundary condition with the help of a dense set of nodal points and lastly we obtain a constructive solution to the inverse problems of this class.

Mathematics Subject Classification (2020). 34A55, 34B24, 34L05

Keywords. singular diffusion equation, inverse spectral problems, inverse nodal problems

1. Introduction

Solvable models of quantum mechanics are investigated in detail in the paper [1]. As can be seen, these models are generally expressed with Hamilton operators or Schrödinger operators with singular coefficients. Many of the problems expressed by these models are related to the solution of spectral inverse problems for differential operators with singular coefficients. However, many problems in mathematical physics are reduced to the study of differential operators whose coefficients are generalized functions.

For example, the stationary vibrations of a spring-tied homogeneous wire fixed at both ends, density $R^{'}(x) = a\delta(x-x_0)$ ($\delta(x)$-Dirac function) and stiffness $R(x)$ at point x_0, whose domain set is

$$D(L_o) = \left\{ y(x) \in W_2^2[0,1] : y^{'}(x_0+0) - y^{'}(x_0-0) = ay(x_0), x_0 \in (0,1); y(0) = 0 = y(1) \right\}$$

and is expressed by the differential operator given as $L_o = -\dfrac{d^2}{dx^2}$ in Hilbert space $L_2[0,1]$. There is detailed information about the correct (regular) definition of such operators and the examination of their spectral properties in [2,7,11] studies.

We consider the following quadratic pencils of Sturm-Liouville equation of the form

$$\ell y := -y^{''} + [\lambda p(x) + q(x)] y = \lambda^2 y, \qquad x \in [0,\pi] \setminus \{a\}, \tag{1.1}$$

with the boundary conditions

*Corresponding Author.
Email addresses: emirov@cumhuriyet.edu.tr (R. Amirov), sdurak@cumhuriyet.edu.tr (S. Durak)
Received: 21.02.2023; Accepted: 19.08.2023

$$U(y) := y'(0) - hy(0) = 0, \tag{1.2}$$

$$V(y) := y'(\pi) + Hy(\pi) = 0, \tag{1.3}$$

where $q(x)$ is a real function belonging to the space $L_2[0,\pi]$, λ is a spectral parameter, $p(x) = \beta\delta(x-a)$, h, a, H and β are real numbers.

Definition 1.1. Any function $y(x) \in W_2^2(\Omega)$ $(\Omega = [0,\pi] \setminus \{a\})$, satisfying the Sturm-Liouville equation

$$-y'' + q(x)y(x) = \lambda^2 y, \tag{1.4}$$

and the discontinuity condition at the point a:

$$y'(a+0) - y'(a-0) = \lambda\beta y(a), \tag{1.5}$$

is called the solution of the equation (1.1).

Next, suppose that for all functions $y(x) \in W_2^2(\Omega), y(x) \neq 0$, satisfying conditions (1.2),(1.3) and (1.5), we have

$$h|y(0)|^2 + H|y(\pi)|^2 + \int_0^\pi \left\{ \left| y'(x) \right|^2 + q(x)|y(x)|^2 \right\} dx > 0. \tag{1.6}$$

Here we denote by $W_2^n(\Omega)$ the space of functions $f(x)$,$x \in \Omega$, such that the derivatives $f^{(m)}(x)$, $(m = \overline{1, n-1})$ are absolute continuous and $f^{(n)}(x) \in L_2(\Omega)$.

Quadratic pencils of Sturm-Liouville equations with singular coefficient appear frequently in various models of classical and quantum mechanics.

In studies [13–15], the spectral properties of the operator produced by the regular differential equation given with non-separated boundary conditions containing the spectral parameter were examined and the uniqueness theorems related to the solution of the spectral inverse problem were proved. In studies [10], [16–19], the spectral properties of the operator produced by the Schrödinger equation with the singular coefficient given with the boundary conditions depending on the spectral parameter were examined and the solution of the inverse spectral problems according to different spectral data was given.

2. Preliminaries

Let $y(x,\lambda)$ and $z(x,\lambda)$ be continuously differentiable solutions on $(0,a)\cup(a,\pi)$ of equation (1.4), satisfying the discontinuity condition (1.5), then

$$\langle y, z\rangle_{x=a-0} = \langle y, z\rangle_{x=a+0}, \tag{2.1}$$

i.e. the function $\langle y, z\rangle$ is continuous on $(0,\pi)$.

Let $\varphi(x,\lambda)$ be solution of equation (1.4), satisfying the initial conditions

$$\varphi(0,\lambda) = 1, \varphi'(0,\lambda) = h \tag{2.2}$$

and the discontinuity condition (1.5).

The characteristic function of the problem (1.1)-(1.3) is in the form

$$\Delta(\lambda) = \varphi'(\pi,\lambda) + H\varphi(\pi,\lambda)$$

with the function $\varphi(x,\lambda)$ being the solution of equation (1.1) satisfying the initial conditions (2.2).

It is also clear that this is an entire function [4], so this problem has a countable number of eigenvalues. We can also prove the following propositions from the methods used in [6], [8].

In addition, using the methods used in papers [5], [7], the following propositions are proved:

Lemma 2.1. *The eigenvalues of the problem (1.1)-(1.3) are real and not equal to zero.*

Lemma 2.2. *The eigenvalues of problem (1.1) are simple.*

Let $\Delta_0(\lambda)$ be the characteristic function of the problem corresponding to the case is $q(x) \equiv 0$ problem (1.1)-(1.3). In this case, it becomes

$$\Delta_0(\lambda) = \varphi_0^{'}(\pi, \lambda) + H\varphi_0(\pi, \lambda), \tag{2.3}$$

where $\varphi_0(x, \lambda)$ is the solution of the equation (1.4), satisfying initial conditions (1.2) and discontinuity condition (1.5).

Lemma 2.3. *Let $G_\delta = \{\lambda : |\lambda - \lambda_n^0| \geq \delta, n = 1, 2, ...\}$ be a small enough number $\delta < \frac{r}{2}$. The zeros of the $\Delta_0(\lambda)$ function λ_n^0 are discrete, so*

$$\inf_{n \neq k} \left|\lambda_n^0 - \lambda_k^0\right| = r > 0.$$

Lemma 2.4. *There is a constant $C_\delta > 0$ so that the inequality*

$$|\Delta_0(\lambda)| \geq C_\delta |\lambda| e^{|\mathrm{Im}\,\lambda|\pi}, \lambda \in G_\delta, \tag{2.4}$$

is satisfied.

Theorem 2.5. *When $\lambda_n, n = 1, 2, ...$ eigenvalues of problem (1.1)-(1.3) are $n \to \infty$,*

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + o\left(\frac{1}{\lambda_n^0}\right), \tag{2.5}$$

has behavior, where,

$$d_n = \frac{1}{\overset{\bullet}{\Delta}(\lambda_n^0)} \left\{(\omega_0(\pi) + H)\sin\lambda_n^0\pi + \left(\frac{H}{2}\beta - \omega_1(\pi)\right)\cos\lambda_n^0\pi + \right.$$
$$\left. + \left(\omega_2(\pi) - \frac{H}{2}\beta\right)\cos\lambda_n^0(2a - \pi) + \omega_3(\pi)\sin\lambda_n^0(2a - \pi)\right\}$$

is the bounded sequence. Where $\overset{\bullet}{\Delta}(\lambda_n^0) = \left[\frac{d}{d\lambda}\Delta_o(\lambda)\right]_{\lambda=\lambda_n^0}$.

Proof. It is clear from the definition given above that the problem (1.1)-(1.3) is equivalent to the problem (1.4)-(1.5), (1.2)-(1.3), that is, each solution of the problem (1.1)-(1.3) is equivalent to the solution of the problem (1.4) satisfying the (1.2),(1.3) boundary and (1.5) discontinuity conditions. Let us denote the problem of seeking the solution of (1.4) equation satisfying (1.2)-(1.3) boundary conditions and (1.5) discontinuity condition with L. By applying the method in the paper [9], we obtain the solution of the problem L that satisfies the initial conditions for (2.2), while $|\lambda| \to \infty$, according to the x variable,

$$\varphi(x, \lambda) = \cos\lambda x + \left(h + \frac{1}{2}\int_0^x q(t)dt\right)\frac{\sin\lambda x}{\lambda} + o\left(\frac{\exp(|\tau|\, x)}{|\lambda|}\right), \tag{2.6}$$

$$\varphi'(x, \lambda) = -\lambda\sin\lambda x + \left(h + \frac{1}{2}\int_0^x q(t)dt\right)\cos\lambda x + o(\exp(|\tau|\, x)), \tag{2.7}$$

in the case of $x < a$ and

$$\varphi(x,\lambda) = \cos\lambda x + \frac{1}{2}\beta\left(\sin\lambda x - \sin\lambda(2a-x)\right) + \omega_0(x)\frac{\sin\lambda x}{\lambda} + \omega_1(x)\frac{\cos\lambda x}{\lambda} \tag{2.8}$$
$$+\left(1-2\beta^2\right)\int_0^a q(t)dt\frac{\sin\lambda(2a-x)}{4\lambda} + \omega_2(x)\frac{\cos\lambda(2a-x)}{2\lambda}$$
$$+ o\left(\frac{\exp\left(|\tau| x\right)}{|\lambda|}\right),$$

$$\varphi'(x,\lambda) = \lambda\left[-\sin\lambda x + \frac{1}{2}\beta\left(\cos\lambda x + \cos\lambda(2a-x)\right)\right] + \omega_0(x)\cos\lambda x - \omega_1(x)\sin\lambda x \tag{2.9}$$
$$+\omega_2(x)\sin\lambda(2a-x) - \frac{1}{4}\left(1-2\beta^2\right)\int_0^a q(t)dt\cos\lambda(2a-x)$$
$$+ o\left(\exp\left(|\tau| x\right)\right)$$

in the case of $x > a$ are valid. Here

$$\omega_0(x) = h - \frac{1}{4}\beta\int_0^a q(t)dt + \frac{1}{2}\int_0^x q(t)dt, \omega_1(x) = -\frac{1}{2}\beta\left(h - \frac{1}{2}\int_0^a q(t)dt + \int_0^x q(t)dt\right)$$
$$\omega_2(x) = \frac{1}{2}\beta\left(h - \frac{3}{2}\int_0^a q(t)dt + \int_0^x q(t)dt\right), \omega_3(x) = \frac{1}{4}(1-\beta^2)\int_0^a q(t)dt.$$

In this case,

$$\Delta(\lambda) = \lambda\left[-\sin\lambda\pi + \frac{1}{2}\beta\left(\cos\lambda\pi + \cos\lambda(2a-\pi)\right)\right] + \left(H + \omega_0(\pi)\right)\cos\lambda\pi \tag{2.10}$$
$$+\left(\frac{H}{2}\beta - \omega_1(\pi)\right)\sin\lambda\pi + \left(-\frac{H}{2}\beta + \omega_2(\pi)\right)\sin\lambda(2a-\pi) - \omega_3(\pi)\cos\lambda(2a-\pi)$$
$$+ o\left(\exp\left(|\tau|\pi\right)\right)$$

is for $|\lambda| \to \infty$.

Let

$$\Delta_0(\lambda) = \lambda\left[-\sin\lambda\pi + \frac{1}{2}\beta\left(\cos\lambda\pi + \cos\lambda(2a-\pi)\right)\right] \tag{2.11}$$

be a function. Using [6], for the roots of the equation $\Delta_0(\lambda) = 0$,

$$\lambda_n^0 = n + h_n, \ \sup_n |h_n| = h < +\infty \tag{2.12}$$

we obtain the following equality.

If we use the method given in the paper [6] for the characteristic equation $\Delta(\lambda) = 0$, it is obtained from (2.10) that

$$\lambda_n = \lambda_n^0 + o(1) \tag{2.13}$$

according to Rouche's theorem.

Denote $G_n := \left\{\lambda : |\lambda| = |\lambda_n^0| + \delta/2\right\}$. On the other hand [8], since

$$\Delta(\lambda) - \Delta_0(\lambda) = O\left(\exp\left(|\mathrm{Im}\,\lambda|\pi\right)\right), |\lambda| \to \infty,$$

for sufficiently large values of hand $\lambda \in G_n$, we get

$$|\Delta(\lambda) - \Delta_0(\lambda)| < \frac{1}{2} C_\delta \exp(|\mathrm{Im}\,\lambda| \pi).$$

Thus, for $\lambda \in G_n$,

$$|\Delta_0(\lambda)| \geq C_\delta |\lambda| \exp(|\mathrm{Im}\,\lambda| \pi) > \frac{1}{2} C_\delta |\lambda| \exp(|\mathrm{Im}\,\lambda| \pi) > |\Delta(\lambda) - \Delta_0(\lambda)|$$

such that n is sufficiently large natural number. It follows from that for sufficiently large values n,functions $\Delta_0(\lambda)$ and $\Delta_0(\lambda) + (\Delta(\lambda) - \Delta_0(\lambda)) = \Delta(\lambda)$ have the same number of zeros counting multiplicities inside contour G_n according to Rouches theorem. So, they have the $(n+1)$ number of zeros $\lambda_0, \lambda_1, ..., \lambda_n$. Analogously, it is shown by Rouche's teorem that for sufficiently large values of n, function $\Delta(\lambda)$ has a unique of zero λ_n inside each circle $C(\delta) = \{\lambda : |\lambda - \lambda_n^0| \leq \delta\}$. Since δ is orbitrary sufficiently small number, we must have

$$\lambda_n = \lambda_n^0 + \varepsilon_n, \quad \varepsilon_n = o(1), \quad n \to \infty. \tag{2.14}$$

Since function $\Delta_0(\lambda)$ is type of "sine" [6], p.119, the number $\gamma_\delta > 0$ exsists such that for all n, $\left|\dot{\Delta}(\lambda_n^0)\right| \geq \gamma_\delta > 0$. Since λ_n are zeros of $\Delta(\lambda)$, from (2.10) we get

$$\varepsilon_n = -\frac{1}{\lambda_n^0 \dot{\Delta}(\lambda_n^0)} \left\{ (\omega_0(\pi) + H) \sin \lambda_n^0 \pi + \left(\frac{H}{2}\beta - \omega_1(\pi) \right) \cos \lambda_n^0 \pi + \right. \tag{2.15}$$
$$\left. + \left(\omega_2(\pi) - \frac{H}{2}\beta \right) \cos \lambda_n^0 (2a - \pi) + \omega_3(\pi) \sin \lambda_n^0 (2a - \pi) \right\} + o\left(\frac{1}{\lambda_n^0} \right).$$

Substituting (2.15) into (2.14), we get (2.5). □

3. Inverse Nodal Problems

In this section, the solution of the nodal inverse problem for the diffusion operator with $p(x) = \beta\delta(x - a)$-Dirac delta potential and any of the set of nodal points dense in the interval $(0, \pi)$ of the constants β, h, H and $q(x)$ function, an algorithm for determining with the help of subsequence will be given. Such problems have been studied in studies of [3, 12, 20–22] for the regular diffusion operator.

In the [20] investigate inverse nodal problems for energy-dependant p-Laplacian equations and of the study applies the Tikhonov regularization method to reconstruct potential functions by only using zeros of one eigenfunction and show that the space of the p-Laplacian operator is homeomorphic to the partition set of the space of nodal sequences.

Now, we examine the case of $a = \frac{\pi}{2}$ and $H = +\infty$ for the sake of simplicity. In this case, the (1.2), (1.3) boundary value conditions are written as $y'(0) - hy(0) = 0$ and $y(\pi) = 0$, respectively. Then (1.5) a discontinuity condition, $y'\left(\frac{\pi}{2} + 0\right) - y'\left(\frac{\pi}{2} - 0\right) = \lambda\beta y\left(\frac{\pi}{2}\right)$.

In this case, since $\Delta(\lambda) = \varphi(\pi, \lambda)$, from the expression (2.8),

$$\Delta(\lambda) = \cos \lambda\pi + \frac{1}{2}\beta \sin \lambda\pi + \omega_0(\pi) \frac{\sin \lambda\pi}{\lambda} + \omega_1(\pi) \frac{\cos \lambda\pi}{\lambda}$$
$$+ \frac{\omega_2(\pi)}{\lambda} + o\left(\frac{1}{|\lambda|} \exp(|\tau| \pi) \right),$$

is obtained.

If we take

$$\Delta_0(\lambda) = \cos \lambda\pi + \frac{1}{2}\beta \sin \lambda\pi,$$

we get $\lambda_n^0 = n - \frac{\alpha}{\pi}$, $n \in \mathbb{Z}$, $\alpha = \arctan\left(\frac{2}{\beta}\right)$ from the equation $\Delta_0(\lambda) = 0$. Therefore, according to Theorem 2.5, the behavior of

$$\lambda_n = \lambda_n^0 + \frac{c_n}{\lambda_n^0} + o\left(\frac{1}{n}\right) \tag{3.1}$$

is obtained when $n \to \infty$ for the roots of the equation $\Delta(\lambda) = 0$, that is, for the eigenvalues of the problem (1.4), (1.2), (1.3) and (1.5). Where

$$c_n = \frac{(-1)^n}{\overset{\bullet}{\Delta}(\lambda_n^0)} \left\{\omega_1(\pi)\cos\alpha - \omega_0(\pi)\sin\alpha + (-1)^n \omega_2(\pi)\right\}$$

$$= \frac{1}{\pi\sqrt{1+\left(\frac{1}{2}\beta\right)^2}} \left\{\omega_1(\pi)\frac{\beta}{2\sqrt{1+\left(\frac{1}{2}\beta\right)^2}} - \frac{\omega_0(\pi)}{\sqrt{1+\left(\frac{1}{2}\beta\right)^2}} + (-1)^n \omega_2(\pi)\right\}.$$

The eigenfunctions of the boundary value problem (1.1)- (1.3) or (1.4), (1.2), (1.3) and (1.5) have the form $y_n(x) = \varphi(x, \lambda_n)$. We note that $y_n(x)$ are real-valued functions. Substituting (3.1) into (2.6)-(2.9) we obtain the following asimptotic formulae for $n \to \infty$ uniformly in x:

$$y_n(x) = \cos\lambda_n^0 x + \left\{-c_n x + h + \frac{1}{2}\int_0^x q(t)dt\right\} \frac{\sin\lambda_n^0 x}{\lambda_n^0} + o\left(\frac{1}{n}\right),\ x < \frac{\pi}{2} \tag{3.2}$$

$$\begin{aligned} y_n(x) = {} & \left(1 + (-1)^n \frac{\beta}{2\sqrt{1+\left(\frac{1}{2}\beta\right)^2}}\right)\cos\lambda_n^0 x + \frac{1}{2}\beta\left(1 + (-1)^n \frac{\beta}{2\sqrt{1+\left(\frac{1}{2}\beta\right)^2}}\right)\sin\lambda_n^0 x \\ & + \left[\omega_0(x) - x c_n - \frac{(-1)^n}{\sqrt{1+\left(\frac{1}{2}\beta\right)^2}}\left(\omega_2(x) - \frac{\pi - x}{2}\beta c_n\right) - (-1)^n \frac{\beta\omega_3(x)}{2\sqrt{1+\left(\frac{1}{2}\beta\right)^2}}\right]\frac{\sin\lambda_n^0 x}{\lambda_n^0} \\ & + \left[\omega_1(x) + \frac{x}{2}\beta c_n + \frac{(-1)^n \beta}{2\sqrt{1+\left(\frac{1}{2}\beta\right)^2}}\left(\omega_2(x) - \frac{\pi - x}{2}\beta c_n\right) - \frac{(-1)^n \omega_3(x)}{\sqrt{1+\left(\frac{1}{2}\beta\right)^2}}\right]\frac{\cos\lambda_n^0 x}{\lambda_n^0} \\ & + o\left(\frac{1}{n}\right),\ x > \frac{\pi}{2}. \end{aligned} \tag{3.3}$$

From oscillation theorem it is easy to see the eigenfunction $y_n(x)$ has exactly n (simple) zeros inside in the interval $(0, \pi)$, namely:

$$0 < x_n^1 < x_n^2 < ... < x_n^n < \pi.$$

The set is $X := \{x_n^j\}_{n\geq 1}$, $j = \overline{1, n}$, called the set of nodal points of the problem (1.1)-(1.3) or (1.4), (1.2), (1.3) and (1.5). Denote $X^{(t)} := \left\{x_{2m-t}^j\right\}, t = 0, 1$. Clearly, $X^{(0)} \cup X^{(1)} = X^{(t)}$ and the set $X^{(t)}$ is dense in $(0, \pi)$.

Inverse problem: Given a set $X = \{x_n^j : n \in \mathbb{N}, j = \overline{1, n}\}$ nodal points or subset

$$X_s = \left\{ x_{n_k}^j : n_k \in \mathbb{N}, j = \overline{1, n_k}, k \in \mathbb{N} \right\}$$

of a set X, where X_s is dens in $(0, \pi)$, find the data h, β and potential $q(x)$.

Taking (3.2) and (3.3) into account, we obtain the following asimptotic formulae for nodal points as $n \to \infty$ uniformly in $j \in \mathbb{Z}$:

$$\begin{gathered} x_n^j = \frac{\left(j - \frac{1}{2}\right)\pi}{n} + \frac{\alpha}{\pi n} \frac{\left(j - \frac{1}{2}\right)\pi}{n} \\ + \left\{ -c_0 x_n^j + h + \frac{1}{2} \int_0^{x_n^j} q(t)dt \right\} \frac{1}{n^2} + o\left(\frac{1}{n^2}\right), x_n^j \in \left(0, \frac{\pi}{2}\right), n = 2m, \end{gathered} \tag{3.4}$$

$$\begin{gathered} x_n^j = \frac{\left(j - \frac{1}{2}\right)\pi}{n} + \frac{\alpha}{\pi n} \frac{\left(j - \frac{1}{2}\right)\pi}{n} \\ + \left\{ -c_1 x_n^j + h + \frac{1}{2} \int_0^{x_n^j} q(t)dt \right\} \frac{1}{n^2} + o\left(\frac{1}{n^2}\right), x_n^j \in \left(0, \frac{\pi}{2}\right), n = 2m-1, \end{gathered} \tag{3.5}$$

$$\begin{gathered} x_n^j = \frac{\left(j - \frac{1}{2}\right)\pi}{n} + \frac{\alpha\left(j - \frac{1}{2}\right)\pi}{\pi n^2} + \frac{1}{n}\arctan\left(\frac{1}{2}\beta\right) + \frac{\alpha \arctan\left(\frac{1}{2}\beta\right)}{\pi n^2} \\ - \frac{\frac{1}{2}\beta B^0\left(x_n^j\right) - A^0\left(x_n^j\right)}{n^2} + o\left(\frac{1}{n^2}\right), x_n^j \in \left(\frac{\pi}{2}, \pi\right), n = 2m, \end{gathered} \tag{3.6}$$

$$\begin{gathered} x_n^j = \frac{\left(j - \frac{1}{2}\right)\pi}{n} + \frac{\alpha\left(j - \frac{1}{2}\right)\pi}{\pi n^2} + \frac{1}{n}\arctan\left(\frac{1}{2}\beta\right) + \frac{\alpha \arctan\left(\frac{1}{2}\beta\right)}{\pi n^2} \\ - \frac{\frac{1}{2}\beta B^1\left(x_n^j\right) - A^1\left(x_n^j\right)}{n^2} + o\left(\frac{1}{n^2}\right), x_n^j \in \left(\frac{\pi}{2}, \pi\right), n = 2m-1, \end{gathered} \tag{3.7}$$

where,

$$A^{(t)}(x) = \frac{1}{A^t} \left\{ \omega_0(x) - x c_t - (-1)^t \sin\alpha \left(\omega_2(x) - \frac{\pi - x}{2} \beta c_t \right) - (-1)^t \omega_3 \cos\alpha \right\}, \tag{3.8}$$

$$B^{(t)}(x) = \frac{1}{A^t} \left\{ \omega_1(x) - \frac{x}{2} \beta c_t + (-1)^t \cos\alpha \left(\omega_2(x) - \frac{\pi - x}{2} \beta c_t \right) - (-1)^t \omega_3 \sin\alpha \right\} \tag{3.9}$$

$$c_t = \frac{1}{\pi\sqrt{1 + \left(\frac{1}{2}\beta\right)^2}} \left\{ \frac{\beta\omega_1(\pi) - 2\omega_0(\pi)}{2\sqrt{1 + \left(\frac{1}{2}\beta\right)^2}} + (-1)^t \omega_2(\pi) \right\}, \tag{3.10}$$

$$A^{(t)} = 1 + (-1)^t \cos\alpha = 1 + \frac{(-1)^t \beta}{2\sqrt{1 + \left(\frac{1}{2}\beta\right)^2}}, t = 0, 1. \tag{3.11}$$

The equality (3.4)-(3.7) for the sufficilutly large gives

$$x_n^{j+1} - x_n^j = \frac{\pi}{n} + o\left(\frac{1}{n}\right)$$

uniformly with respect to j. Cousequently, for sufficiently large n, we have $x_n^j < x_n^{j+1}$.

On the other hand, from equality (3.4)-(3.7) it follows that the set $\{x_n^j\}$ is dense in $[0,\pi]$. Therefore, for all $x \in [0,\pi]$, there exists $\{x_n^j\}$ such that $\lim\limits_{n\to\infty} x_n^j = x$. Moreover, note that

$$\lim_{n\to\infty} \frac{\left(j-\frac{1}{2}\right)\pi}{n} = x.$$

Theorem 3.1. *Fix $t = 0,1$ and $x \in [0,\pi]$. Suppose that $\{x_n^j\} \in X^{(t)}$, be chosen such that $\lim\limits_{n\to\infty} x_n^j = x$. Then the following finite limits exist and the corresponding equalities hold for $x < \frac{\pi}{2}$*

$$\lim_{n\to\infty}\left(nx_n^j - \left(j-\frac{1}{2}\right)\pi\right) \overset{def}{=} f_1(x), \tag{3.12}$$

$$\lim_{n\to\infty}\left[nx_n^j - \left(j-\frac{1}{2}\right)\pi - \frac{\alpha}{\pi}\frac{\left(j-\frac{1}{2}\right)\pi}{n}\right] \overset{def}{=} g_1^t(x), \tag{3.13}$$

for $x > \frac{\pi}{2}$

$$\lim_{n\to\infty}\left(nx_n^j - \left(j-\frac{1}{2}\right)\pi\right) \overset{def}{=} f_2(x), \tag{3.14}$$

$$\lim_{n\to\infty}\left[nx_n^j - \left(j-\frac{1}{2}\right)\pi - \frac{\alpha\left(j-\frac{1}{2}\right)\pi}{\pi n} - \arctan\left(\frac{1}{2}\beta\right)\right] n \overset{def}{=} g_2^t(x), \tag{3.15}$$

and

$$f_1(x) = \frac{\alpha}{\pi}x, \quad x \in \left[0, \frac{\pi}{2}\right), \tag{3.16}$$

$$g_1^t(x) = -c_t x + h + \frac{1}{2}\int_0^x q(t)dt, \ \ x \in \left[0, \frac{\pi}{2}\right), \tag{3.17}$$

$$f_2(x) = \frac{\alpha}{\pi}x + \arctan\left(\frac{1}{2}\beta\right), \ \ x \in \left(\frac{\pi}{2}, \pi\right], \tag{3.18}$$

$$g_2^t(x) = \frac{\alpha}{\pi}\arctan\left(\frac{1}{2}\beta\right) + A^{(t)}(x) - \frac{1}{2}\beta B^t(x), x \in \left(\frac{\pi}{2}, \pi\right]. \tag{3.19}$$

Proof. If we use the asymptotical formulaes (3.4)-(3.7), we get that:

for $x_n^j \in \left(0, \frac{\pi}{2}\right)$

$$nx_n^j - \left(j-\frac{1}{2}\right)\pi = \frac{\alpha}{\pi}\frac{\left(j-\frac{1}{2}\right)\pi}{n} + \left\{-c_t x_n^j + h + \frac{1}{2}\int_0^{x_n^j} q(t)dt\right\}\frac{1}{n} + o\left(\frac{1}{n}\right), \tag{3.20}$$

$$\left[nx_n^j - \left(j-\frac{1}{2}\right)\pi - \frac{\alpha}{\pi}\frac{\left(j-\frac{1}{2}\right)\pi}{n}\right] n = -c_t x_n^j + h + \frac{1}{2}\int_0^{x_n^j} q(t)dt + o(1), \tag{3.21}$$

for $x_n^j \in \left(\frac{\pi}{2}, \pi\right)$

$$nx_n^j - \left(j - \frac{1}{2}\right)\pi = \frac{\alpha}{\pi}\frac{\left(j-\frac{1}{2}\right)\pi}{n} + \arctan\left(\frac{1}{2}\beta\right) + \frac{\alpha}{\pi n}\arctan\left(\frac{1}{2}\beta\right) + \frac{A^{(t)}\left(x_n^j\right) - \frac{1}{2}\beta B^{(t)}\left(x_n^j\right)}{n} + o\left(\frac{1}{n}\right), \tag{3.22}$$

$$\left[nx_n^j - \left(j - \frac{1}{2}\right)\pi - \frac{\alpha}{\pi}\frac{\left(j-\frac{1}{2}\right)\pi}{n} - \arctan\left(\frac{1}{2}\beta\right)\right] n = \frac{\alpha}{\pi}\arctan\left(\frac{1}{2}\beta\right) + A^{(t)}\left(x_n^j\right) - \frac{1}{2}\beta B^{(t)}\left(x_n^j\right) + o(1). \tag{3.23}$$

Since, $\lim_{n\to\infty} x_n^j = x$, $\lim_{n\to\infty}\frac{\left(j-\frac{1}{2}\right)\pi}{n} = x$ and

$$\lim_{n\to\infty} A^{(t)}(x_n^j) = A^{(t)}(x), \quad \lim_{n\to\infty} B^{(t)}(x_n^j) = B^{(t)}(x)$$

from this and (3.20)-(3.23), we conclude that as $n \to \infty$ the limits of left hand side of (3.20)-(3.23) holds. Theorem 3.1 is proved. □

Remark 3.2. We get from Theorem 3.1 that the regularity orders of the functions $g^t(x)$ and $\int_0^x q(t)dt$ are the same.

We now state a uniqueness theorem and present a constructive prosedure for solving inverse nodal problem.

Theorem 3.3. *Fix $t = 0, 1$. Let $X_s \subset X^{(t)}$ be a subset of nodes which is dense $(0, \pi)$. Then, the specification of X_s uniquely determines the potential $q(x) - \langle q\rangle$ a.e. on $(0, \pi)$ and the coefficient h of the boundary condition and coefficient β. The potential $q(x) - \langle q\rangle$ and the numbers h and β, can be constructed via the following algoritm:*

1. *For each $x \in [0, \pi]$, we choose a sequence $\{x_n^j\} \subset X_0$ such that $\lim_{n\to\infty} x_n^j = x$.*
2. *From (3.13), we find the function $g_1^t(x)$ and calculate value for $g_1^t(x)$ at $x = 0$, i.e.*

$$h = g_1^t(0) \tag{3.24}$$

3. *From (3.12), we find the function $f_1(x)$ and calculate value for $f_1(x)$ at $x = 1$, i.e.*

$$\beta = \frac{2}{\tan\left(\pi f_1(1)\right)} \tag{3.25}$$

4. *The function $q(x) - \langle q^t\rangle$ can be determined as*

$$q(x) - \left\langle q^t\right\rangle = 2\left(g_1^t(x)\right)' + \frac{g_1^t(0)}{\pi\left[1 + 2\left(\frac{1}{2}\beta\right)^2\right]}\left[(-1)^t\beta - \sqrt{1 + \left(\frac{1}{2}\beta\right)^2}\right], x \in \left[0, \frac{\pi}{2}\right], \tag{3.26}$$

where

$$\langle q^t \rangle = -\frac{2}{\pi\left[1+2\left(\frac{1}{2}\beta\right)^2\right]}\left\{\frac{1}{2}\beta\left[(\beta-1)-3(-1)^t\sqrt{1+\left(\frac{1}{2}\beta\right)^2}\right]\int_0^a q(t)dt+\right.$$
$$\left.\left[\frac{1}{2}+\left(\frac{1}{2}\beta\right)^2-(-1)^t\beta\sqrt{1+\left(\frac{1}{2}\beta\right)^2}\right]\int_0^\pi q(t)dt\right\}.$$

$$q(x)-\langle q^t\rangle = \frac{\left(2\sqrt{1+\left(\frac{1}{2}\beta\right)^2}+(-1)^t\beta\right)\left(g_2^t(x)\right)'}{\sqrt{1+\left(\frac{1}{2}\beta\right)^2}\left(1+2\left(\frac{1}{2}\beta\right)^2\right)-(-1)^t\beta\left(1+\left(\frac{1}{2}\beta\right)^2\right)}+$$
$$\frac{g_1^t(0)}{\pi\left[1+2\left(\frac{1}{2}\beta\right)^2\right]}\left[(-1)^t\beta-\sqrt{1+\left(\frac{1}{2}\beta\right)^2}\right], x\in\left[\frac{\pi}{2},\pi\right], \tag{3.27}$$

where

$$\left\langle q^t\right\rangle = -\frac{2}{\pi\left[1+2\left(\frac{1}{2}\beta\right)^2\right]}\frac{2\left[1+\left(\frac{1}{2}\beta\right)^2\right]+(-1)^t\beta\sqrt{1+\left(\frac{1}{2}\beta\right)^2}}{\left[1+2\left(\frac{1}{2}\beta\right)^2-(-1)^t\beta\sqrt{1+\left(\frac{1}{2}\beta\right)^2}\right]}.$$
$$\left\{\frac{1}{2}\beta\left[(\beta-1)-3(-1)^t\sqrt{1+\left(\frac{1}{2}\beta\right)^2}\right]\int_0^a q(t)dt+\right.$$
$$\left.\left[\frac{1}{2}+\left(\frac{1}{2}\beta\right)^2-(-1)^t\beta\sqrt{1+\left(\frac{1}{2}\beta\right)^2}\right]\int_0^\pi q(t)dt\right\}.$$

Proof. Formulas (3.24), (3.25) and (3.27) can be derived from, (3.16), (3.17) and (3.19) step by step. We obtain the following reconstruction prosedure:

i) Taking value for $g_1^t(x)$ at $x=0$, then it yields $h=g_1^t(0)$.

ii) Taking value for $f_1(x)$ at $x=1$, then it yield $\beta=\dfrac{2}{\tan(\pi f_1(1))}$.

iii) After hand β are reconstructed, on take derivatives of the functions $g_i^t(x)$, $(i=1,2)$ we have (3.27) and (3.26). □

References

[1] S. Albeverio, F. Gesztesy, R. Hoegh-Kron and H. Holden, *Solvable models in quantum mechanics*, Springer, New York-Berlin, 1988.

[2] R.Kh. Amirov and I.M. Guseinov, *Boundary value problems for a class of Sturm-Liouville operator with nonintegrable potential*, Dif. Eq. **38**(8), 1195-1197, 2002.

[3] S.A. Buterin and S.T. Chung, *Inverse nodal problem for differential pencils*, Appl. Math. Letters **22**, 1240-1247, 2009.

[4] I.M. Guseinov and L.I. Mammadova, *Properties of the eigenvalues of the Sturm-Liouville operator with discontinuity conditions inside the interval*, Pross. Baku State University, Phys-Math. Sci. Series **3**, 2011.

[5] I.M. Guseinov and L.I. Mammadova, *Reconstruction of the diffusion equation with singular coefficients for two spectra*, Doklady Math. **90**(1), 401-404, 2014.

[6] B.Y. Levin, *Lectures on entire functions*, Transl. Math. Monographs **150**, Amer. Math. Soc, Providence RI, 1996.

[7] M.Dzh. Manafov, *Inverse spectral and inverse nodal problems for Sturm-Liouville equations with point δ and δ' - interactions*, Proc. of the Institute of Math. and Mech. **45**(2), National Acad. Sci. Azerbaijan, 286-294, 2019.

[8] V.A. Marchenko, *Sturm–Liouville operators and their applications*, Naukova Dumka, Kiev, 1977. English transl., Birkhäuser, Basel, 1986.

[9] A.A. Nabiev and R.Kh. Amirov, *Integral representations for the solutions of the generalized Schroedinger equation in a finite interval*, Adv. Pure Math. **5**(13), 777-795, 2015.

[10] N. Pronska, *Reconstruction of energy-dependent Sturm-Liouville equations from two spectra*, Integral Equations Operator Theory **76**, 403-419, 2013.

[11] A.M. Savchuk, *On the eigenvalues and eingenfunctions of the Sturm-Liouville Operator with singular potential*, Math. Notes **69**(2), 245-252, 2001.

[12] C.F. Yang, *An inverse problem for a differential pencil using nodal points as data*, Israel J. Math **204**, 431-446, 2014.

[13] Ch.G. Ibadzadeh, L.I. Mammadova and I.M. Nabiev, *Inverse problem of spectral analysis for diffusion operator with nonseparated boundary conditions and spectral parameter in boundary condition*, Azerbaijan J. Math. **9**(1), 171-189, 2019.

[14] I.M. Nabiev, *Reconstruction of the differential operator with spectral parameter in the boundary condition*, Mediterr. J. Math. **19**(3), 1-14, 2022.

[15] L.I. Mammadova, I.M. Nabiev and Ch.H. Rzayeva, *Uniqueness of the solution of the inverse problem for differential operator with semiseparated boundary conditions*, Baku Math. J. **1**(1), 47-52, 2022.

[16] N.J. Guliyev, *Inverse square singularities and eigenparameter-dependent boundary conditions are two sides of the same coin*, The Quarterly J. Math. 74(3), 889-910, 2023.

[17] N.J. Guliyev, *Schrödinger operators with distributional potentials and boundary conditions dependent on the eigenvalue parameter*, J. Math. Phys. **60**(6), 063501, 23 pp, 2019.

[18] N.J. Guliyev, *Essentially isospectral transformations and their applications*, Ann. Mat. Pura Appl. **199**(4), 1621-1648, 2020.

[19] N.J. Guliyev, *On two-spectra inverse problems*, Proc. Amer. Math. Soc. **148**(10), 4491-4502, 2020.

[20] Y.H. Cheng, *Reconstruction and stability of inverse nodal problems for energydependent p-Laplacian equation*, J. Math. Ann. Appl. **491**(2), 124388, 2020.

[21] C. F. Yang, *Direct and inverse nodal problem for differential pencil with coupled boundary conditions*, Inverse Prob. Sci. Eng. **21**(4), 562-584, 2013.

[22] M. Dzh. Manafov and A. Kablan, *Inverse spectral and inverse nodal problems for energydependent Sturm-Liouville equations with δ-interaction*, Elec. J. Dif. Eq. **2015**(26), 1-10, 2015.

.