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BERNSTEIN POLİNOMLARI İÇİN LIPSCHITZ SABİTLERİ

Mine AKTAŞ*

Özet

$f \in C[0,1]$ $\alpha, \beta \in IR^+$, $\alpha < \beta$ olmak üzere Bernstein polinomunun Stancu tipindeki genelleşmesi olan

$$B_n^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

polinomu ve iki değişkenli Bernstein polinomu olan

$$B_{n,r}(f;x_1,x_2) = \sum_{j=0}^{n}\sum_{q=0}^{r}\binom{n}{j}\binom{r}{q}(1-x_1)^{n-j}(1-x_2)^{r-q}\, x_1^j x_2^q f\left(\frac{j}{n},\frac{q}{r}\right)$$

polinomunun Lipschitz sınıfına ait olduğu bulunur.

Anahtar Kelimeler : Lipschtz, sabit, Bernstein, polinom

LIPSCHITZ CONSTANT FOR THE BERNSTEIN POLYNOMIALS

Abstract

In this paper, we first find the polynomial which belongs to Lipschitz's class.

$$B_n^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

The above polynomial is a Stancu type generalization of Bernstein polynomial where

$$f \in C[0,1],\ \alpha,\beta \in R^+,\ \alpha < \beta.$$

We also find the below polynomial which belongs to Lipschitz's class, that is a Bernstein polynomial with two variable.

$$B_{n,r}(f;x_1,x_2) = \sum_{j=0}^{n}\sum_{q=0}^{r}\binom{n}{j}\binom{r}{q}(1-x_1)^{n-j}(1-x_2)^{r-q}\, x_1^j x_2^q f\left(\frac{j}{n},\frac{q}{r}\right)$$

Key Words: : Lipschtz, constant, Bernstein, polynomial

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1. GIRIŞ

Bernstein polinomları ve onların çeşitli genelleşmelerine ait bir çok çalışmalar mevcuttur. (Bloom and Elliot, 1981 ve Stancu, 1967). Bloom, Elliot (1981) ve Brown, Elliot, Paget (1987) makalelerinde Lipschitz sınıfında olan fonksiyonların Bernstein polinomları için Lipschitz sabiti bulunmuştur.

Stancu (1969) de ise Bernstein polinomlarının düğüm noktaları α,β sabit sayılarına bağlı bir genelleşmesi verilmiştir. Bu makalede, Stancu (1969) da tanımlanmış polinomların Lipschitz sınıfına ait olduğu gösterilecektir ve ayrıca Brown, Elliot, Paget (1987) nin sonuçlarının iki değişkenli halde de geçerli olduğu gösterilecektir.

Belirlidir ki f, $[0,1]$ aralığında sürekli bir f fonksiyonunun Bernstein polinomu

$$B_n(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} \tag{1}$$

ile tanımlıdır ve bu fonksiyon konveks ise $B_n(f)$ dizisi monoton azalandır, yani

$n=2,3,4,\ldots$ ve $\forall x\in[0,1]$ için

$$B_{n-1}(f;x)\geq B_n(f;x)\geq f(x) \tag{2}$$

dir (Davis 1963). Burada f fonksiyonunun konveksliği onun $[0,1]$ aralığındaki tüm ikinci ve üçüncü bölünmüş farklarının pozitif olması anlamındadır (Stancu (1967)

Bunun bir sonucu olarak $0<\mu\leq 1$ olmak üzere $(-x^{\mu})$ konveks olduğundan

$n=1,2,3,\ldots$ için

$$B_n(x^{\mu};h)\leq h^{\mu},\quad 0\leq h\leq 1 \tag{3}$$

elde edilir.

Tanım 1. $x_1,x_2,y_1,y_2\in[0,1]$ olmak üzere iki değişkenli f fonksiyonu için

$$|f(x_1,x_2)-f(y_1,y_2)|\leq A_1|x_1-y_1|^{\mu}+A_2|x_2-y_2|^{\mu} \tag{4}$$

olacak şekilde $A_1,A_2\geq 0$ sabitleri varsa $f\in Lip_{A_1,A_2}\mu$ dir. Burada A_1,A_2, f nin Lipschitz sabiti olarak bilinir (Brown, Elliot, Paget, 1981).

2. Bu çalışmamızda $f\in C[0,1]$, $\alpha,\beta\in IR^+$, $\alpha<\beta$ olmak üzere Bernstein polinomunun Stancu tipindeki genelleşmesi olan (Stancu,1969))

$$B_n^{\alpha,\beta}(f;x)=\sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)\binom{n}{k}x^k(1-x)^{n-k}$$

polinomu için Lipschitz sınıfına ait olma problemini inceleyeceğiz. Daha sonra iki değişkenli Bernstein polinomu olan

$$B_{n,s}(f;x_1,x_2)=\sum_{j=0}^{n}\sum_{t=0}^{s}\binom{n}{j}\binom{s}{t}(1-x_1)^{n-j}(1-x_2)^{s-t}\,x_1^j x_2^t f\left(\frac{j}{n},\frac{t}{s}\right)$$

aynı problemin çözümünü bulacağız.

Teorem 1. $f\in Lip_A\mu$ ise $B_n^{\alpha,\beta}(f;x)\in Lip_A\mu$ dir.

İspat. $x_1,x_2\in[0,1]$ ve $x_1<x_2$ olsun. Bu durumda

$$B_r^{\alpha,\beta}(f;x_2)=\sum_{j=0}^{n}\binom{n}{j}(1-x_2)^{n-j}f\left(\frac{j+\alpha}{n+\beta}\right)(x_1+(x_2-x_1))^j$$

$$=\sum_{j=0}^{n}\binom{n}{j}(1-x_2)^{n-j}f\left(\frac{j+\alpha}{n+\beta}\right)\left\{\sum_{k=0}^{j}\binom{j}{k}x_1^k(x_2-x_1)^{j-k}\right\}$$

$$=\sum_{j=0}^{n}\sum_{k=0}^{j}\frac{n!x_1^k(x_2-x_1)^{j-k}(1-x_2)^{n-j}}{k!(j-k)!(n-j)!}f\left(\frac{j+\alpha}{n+\beta}\right)$$

dır. $k+\ell=j$ alırsak,

$$B_n^{\alpha,\beta}(f;x_2)=\sum_{k=0}^{n}\sum_{\ell=0}^{n-k}\frac{n!x_1^k(x_2-x_1)^{\ell}(1-x_2)^{n-k-\ell}}{k!\ell!(n-k-\ell)!}f\left(\frac{k+\ell+\alpha}{n+\beta}\right) \tag{5}$$

ifadesi elde edilir. Benzer işlemleri $B_n^{\alpha,\beta}(f;x_1)$ için yaparsak

$$B_n^{\alpha,\beta}(f;x_1)=\sum_{k=0}^{n}\binom{n}{k}x_1^k f\left(\frac{k+\alpha}{n+\beta}\right)((x_2-x_1)+(1-x_2))^{n-k}$$

$$=\sum_{k=0}^{n}\binom{n}{k}x_1^k f\left(\frac{k-\alpha}{n+\beta}\right)\left\{\sum_{\ell=0}^{n-k}\binom{n-k}{\ell}(x_2-x_1)^{\ell}(1-x_2)^{n-k-\ell}\right\}$$

$$=\sum_{k=0}^{n}\sum_{\ell=0}^{n-k}\frac{n!x_1^k(x_2-x_1)^{\ell}(1-x_2)^{n-k-\ell}}{k!\ell!(n-k-\ell)!}f\left(\frac{k+\alpha}{n+\beta}\right) \tag{6}$$

olduğunu görürüz. (5) den (6) yı çıkartırsak

$$\left|B_n^{\alpha,\beta}(f;x_2)-B_n^{\alpha,\beta}(f;x_1)\right|=\left|\sum_{k=0}^{n}\sum_{\ell=0}^{n-k}\frac{n!x_1^k(x_2-x_1)^{\ell}(1-x_2)^{n-k-\ell}}{k!\ell!(n-k-\ell)!}\right.$$

$$\left.\left\{f\left(\frac{k+\ell+\alpha}{n+\beta}\right)-f\left(\frac{k+\alpha}{n+\beta}\right)\right\}\right|$$

bulunur. (4) den

$$\left|B_n^{\alpha,\beta}(f;x_2)-B_n^{\alpha,\beta}(f;x_1)\right| \le A\sum_{k=0}^{n}\sum_{\ell=0}^{n-k}\frac{n!\,x_1^k(x_2-x_1)^{\ell}(1-x_2)^{n-k-\ell}}{k!\,\ell!\,(n-k-\ell)!}\left(\frac{\ell}{n+\beta}\right)^{\mu}$$

$$=A\sum_{\ell=0}^{n}\frac{(x_2-x_1)^{\ell}n!}{\ell!(n-\ell)!}\left(\frac{\ell}{n+\beta}\right)^{\mu}\left\{\sum_{k=0}^{n-\ell}\binom{n-\ell}{k}x_1^k(1-x_2)^{n-\ell-k}\right\}$$

$$=A\sum_{\ell=0}^{n}\binom{n}{\ell}(x_2-x_1)^{\ell}\left(\frac{\ell}{n+\beta}\right)^{\mu}(x_1+1-x_2)^{n-\ell}$$

$$\le A\sum_{\ell=0}^{n}\binom{n}{\ell}(x_2-x_1)^{\ell}\left(\frac{\ell+\alpha}{n+\beta}\right)^{\mu}(x_1+1-x_2)^{n-\ell}$$

$$=AB_n^{\alpha,\beta}(x^{\mu};x_2-x_1)$$

buluruz. (3) den

$$\left|B_n^{\alpha,\beta}(f;x_2)-B_r^{\alpha,\beta}(f;x_1)\right| \le A(x_2-x_1)^{\mu}$$

çıkar. Bu ise $B_n^{\alpha,\beta}\in \mathrm{Lip}_A\mu$ olduğunu ispatlar.

Teorem 2. $n,s>1$ doğal sayılar olmak üzere

$$B_{n,r}(f;x_1,x_2)=\sum_{j=0}^{n}\sum_{q=0}^{r}\binom{n}{j}\binom{r}{q}(1-x_1)^{n-j}(1-x_2)^{r-q}\,x_1^j x_2^q f\left(\frac{j}{n},\frac{q}{r}\right)$$

olsun. Bu durumda $f\in \mathrm{Lip}_{A_1,A_2}\mu$ ise $B_{r,r}(f)\in \mathrm{Lip}_A\mu$ dır.

İspat. $x_1,x_2,y_1,y_2\in[0,1]$ olsun. O zaman

$$B_{n,r}(f;x_1,x_2)=\sum_{j=0}^{n}\sum_{q=0}^{r}\binom{n}{j}\binom{r}{q}(1-x_1)^{n-j}(1-x_2)^{q-r}f\left(\frac{j}{n},\frac{q}{r}\right)$$

$$(y_1+(x_1-y_1))^j(y_2+(x_2-y_2))^q$$

$$=\sum_{j=0}^{n}\sum_{q=0}^{r}\binom{n}{j}\binom{r}{q}(1-x_1)^{n-j}(1-x_2)^{r-q}f\left(\frac{j}{n},\frac{q}{r}\right)$$

$$\left\{\sum_{k=0}^{j}\binom{j}{k}y_1^{\,k}(x_1-y_1)^{j-k}\right\}\left\{\sum_{d=0}^{q}\binom{q}{d}y_2^{\,d}(x_2-y_2)^{q-d}\right\}$$

$$=\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}(1-x_1)^{n-j}y_1^{\,k}(x_1-y_1)^{j-k}$$

$$\sum_{q=0}^{r}\sum_{d=0}^{q}\binom{r}{q}\binom{q}{d}(1-x_2)^{r-q}y_2^{\,d}(x_2-y_2)^{q-d}f\left(\frac{j}{n},\frac{q}{r}\right)$$

ifadesi elde edilir. $k+\ell=j$ ve $d+p=q$ alarak toplamı yeniden düzenlersek,

$$B_{n,r}(f;x_1,x_2)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{n!}{k!l!(n-k-l)!}y_1^{\,k},(x_1-y_1)^l(1-x_1)^{n-k-l}$$

$$=\sum_{d=0}^{r}\sum_{p=0}^{r-d}\frac{r!}{d!p!(s-d-p)!}y_2^{\,d},(x_2-y_2)^p(1-x_2)^{r-d-p}\,f\left(\frac{k+l}{n},\frac{d+p}{r}\right) \quad (7)$$

bulunur. Benzer işlemleri $B_{n,r}(f,y_1,y_2)$ için yapalım.

$$B_{n,r}(f,y_1,y_2)=\sum_{k=0}^{n}\sum_{d=0}^{r}\binom{n}{k}\binom{r}{d}y_1^{\,k}y_2^{\,d}f\left(\frac{k}{n},\frac{d}{r}\right)$$

$$((x_1-y_1)+(1-x_1))^{n-k}((x_2-y_2)+(1-x_2))^{r-d}$$

$$=\sum_{k=0}^{n}\sum_{d=0}^{r}\binom{n}{k}\binom{r}{d}y_1^{\,k}y_2^{\,d}f\left(\frac{k}{n},\frac{d}{r}\right)\left\{\sum_{l=0}^{n-k}\binom{n-k}{l}(x_1-y_1)^l(1-x_1)^{n-k-l}\right\}$$

$$\left\{\sum_{p=0}^{r-d}\binom{r-d}{p}(x_2-y_2)^p(1-x_2)^{r-d-p}\right\}$$

$$B_{n,r}(f;y_1,y_2)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{n!}{k!l!(n-k-l)!}y_1^k(x_1-y_1)^l(1-x_1)^{n-k-l}$$

$$\sum_{d=0}^{r}\sum_{p=0}^{r-d}\frac{r!}{d!p!(r-d-p)!}y_2^d(x_2-y_2)^p(1-x_2)^{r-d-p}f\left(\frac{k}{n},\frac{d}{r}\right) \quad (8)$$

(7) den (8)'i çıkartırsak

$$\left|B_{n,r}(f;x_1,x_2)-B_{n,r}(f;y_1,y_2)\right| \le \sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{d=0}^{r}\sum_{p=0}^{r-d}\frac{n!}{k!l!(n-k-l)!}\frac{r!}{d!p!(r-d-p)!}$$

$$y_1^k(x_1-y_1)^l(1-x_1)^{n-k-l}y_2^d(x_2-y_2)^l(1-x_2)^{r-d-p}\left(f\left(\frac{k+l}{n},\frac{d+p}{r}\right)-f\left(\frac{k}{n},\frac{d}{r}\right)\right)$$

bulunur. (4) den

$$\left|B_{n,r}(f;x_1,x_2)-B_{n,r}(f;y_1,y_2)\right| \le \sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{d=0}^{r}\sum_{p=0}^{r-d}\frac{n!}{k!l!(n-k-l)!}\frac{r!}{d!p!(r-d-p)!}$$

$$y_1^k(x_1-y_1)^{\ell}(1-x_1)^{n-k-\ell}\,y_2^d(x_2-y_2)^{p}(1-x_2)^{r-d-p}\left[A_1\left(\frac{\ell}{n}\right)^{\mu}+A_2\left(\frac{p}{r}\right)^{\mu}\right]$$

elde edilir.

Böylece

$$\left|B_{n,r}(f;x_1,x_2)-B_{n,r}(f;y_1,y_2)\right|\le$$

$$A_1\left\{\sum_{\ell=0}^{n}\binom{n}{\ell}(x_1-y_1)^{\ell}\left[\sum_{k=0}^{n-\ell}\binom{n-\ell}{k}y_1^k(1-x_1)^{n-\ell-k}\right]\right.$$

$$\left.\sum_{p=0}^{r}\binom{r}{p}(x_2-y_2)^{p}\left[\sum_{d=0}^{r-p}\binom{r-d}{p}y_2^d(1-x_2)^{r-d-p}\right]\left(\frac{\ell}{n}\right)^{\mu}\right\}$$

$$+A_2\left\{\sum_{\ell=0}^{n}\binom{n}{\ell}(x_1-y_1)^{\ell}\left[\sum_{k=0}^{n-\ell}\binom{n-\ell}{k}y_1^k(1-x_1)^{n-\ell-k}\right]\right.$$

$$\left.\sum_{p=0}^{r}\binom{r}{p}(x_2-y_2)^{p}\left[\sum_{d=0}^{r-p}\binom{r-d}{p}y_2^d(1-x_2)^{r-d-p}\right]\left(\frac{p}{r}\right)^{\mu}\right\}$$

$$=A_1\left\{\sum_{\ell=0}^{n}\binom{n}{\ell}(x_1-y_1)^{\ell}(y_1+1-x_1)^{n-\ell}\right.$$

$$\sum_{p=0}^{r}\binom{r}{p}(x_2-y_2)^{p}(y_2+1-x_2)^{r-p}\left(\frac{\ell}{n}\right)^{\mu}$$

$$+A_2\sum_{\ell=0}^{n}\binom{n}{\ell}(x_1-y_1)^{\ell}(y_1+1-x_1)^{n-\ell}$$

$$\sum_{p=0}^{r}\binom{r}{p}(x_2-y_2)^{p}(y_2+1-x_2)^{r-p}\left(\frac{p}{r}\right)^{\mu}$$

elde edilir. Buradan

$$\left|B_{n,r}(f;x_1,x_2)-B_{n,r}(f;y_1,y_2)\right|\le A_1B_r(1,x_2-y_2)B_n(x^{\mu};x_1-y_1)+$$

$$+A_2B_n(1,x_1-y_1)B_r(x^{\mu};x_2-y_2)$$

çıkar. (3) den ve

$$B_r(1,x_2-y_2)=B_n(x^{\mu};x_1-y_1)=1$$

olduğundan

$$\left|B_{n,r}(f;x_1,x_2)-B_{n,r}(f;y_1,y_2)\right|\le A_1(x_1-y_1)^{\mu}+A_2(x_2-y_2)^{\mu}$$

bulunur.

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