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Inverse Problems for Dirac Operator with Boundary Conditions Involving A Herglotz-Nevanlinna function

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Abstract. In this paper, we deal with the inverse problems for Dirac operator with rationally eigenvalue dependent boundary condition and linearly eigenvalue dependent jump conditions. We prove that when $Q(x)$ is known on $\left(\frac{1}{2},1\right)$ then only one spectrum excluding a finite number of eigenvalues is sufficient to determine $Q(x)$ on the interval $(0,1)$ and the other coefficients of the problem. Moreover, it is shown that $Q(x)$ is uniquely determined by the classical spectral data, i.e., eigenvalues and normalising numbers.

Keywords. Inverse problem, impulsive differential equation, boundary value problem

Herglotz-Nevanlinna Fonksiyonu İçeren Sınır Koşullarına Sahip Dirac Operatörü için Ters Problemler

Özet. Bu makalede, sınır koşulları spektral parametreye rasyonel, süreksizlik koşulları ise lineer şekilde bağlı olan Dirac operatörü için ters problem ele alınmıştır. (½,1) aralığında Q(x) potansiyel fonksiyonu biliniyorken, tek spektrumun sonlu sayıda özdeğerlerin dışında (0,1) aralığında Q(x) potansiyel fonksiyonunu ve problemin diğer katsayılarını tek olarak belirlediği ispatlanmaktadır. Ayrıca, Q(x) fonksiyonunun klasik spektral veriler, yani özdeğerler ve normalleştirici sayılar yardımıyla tek olarak belirlendiği gösterilmektedir.

Anahtar kelimeler: Ters problem, süreksiz Dirac operatör, sınır-değer problem

1. INTRODUCTION

We consider the boundary value problem $L = L(Q, f, \alpha, \beta, \gamma)$ generated by the system of Dirac differential equation

$$\ell y := By' + Q(x)y = \lambda y,\ x \in I := \left(0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right) \tag{1}$$

with $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}$, $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$ subject to the boundary conditions

$$U(y) := y_1(0) = 0 \tag{2}$$

$$V(y) := y_2(1) + f(\lambda) y_1(1) = 0 \tag{3}$$

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and the jump conditions

$$\begin{cases} y_1(\frac{1}{2}+0)=\alpha y_1(\frac{1}{2}-0) \\ y_2(\frac{1}{2}+0)=\alpha^{-1}y_2(\frac{1}{2}-0)+(\beta\lambda+\gamma)y_1(\frac{1}{2}-0). \end{cases} \tag{4}$$

Here $p(x)$ and $q(x)$ are real valued functions in $L_2(0,1)$; λ is a spectral parameter; α , β and γ are real numbers, $\alpha>0$, $\beta>0$; $f(\lambda)$ is a rational function of Herglotz–Nevanlinna type such that

$$f(\lambda)=a\lambda+b-\sum_{k=1}^{N}\frac{f_k}{\lambda-g_k}$$

where $a,$ $b,$ $f_k,$ g_k are real numbers, $a>0,$ $f_k>0,$ $g_1<g_2<...<g_N$. We shall note that if $f(\lambda)=\infty$ then the condition (3) will be as the condition $y_1(1)=0$.

Inverse problems for Dirac operator have been rather well studied and continued acceleratingly(see [14], [15], [18], [24] and references therein)

When the spectrum of a differential operator and the potential on a half of the interval are known, one can recover the differential operator on whole interval. Such problems are known as half-inverse problems. The first study on the half-inverse problem for Sturm-Liouville operator was made by Hochstadt and Lieberman in 1978 [21]. They proved that if the potential function $q(x)$ of Sturm-Liouville equation is given over the interval $\left(\frac{1}{2},1\right)$, then one spectrum is enough to determine $q(x)$ on the interval $\left(0,\frac{1}{2}\right)$. After that in 1984, Hald [20] showed that if the potential is known over half the interval and if one boundary condition is given, then the eigenvalues uniquely determine the potential and the other boundary condition. In 1994, T. N. Arutyunyan [3] proved that the potential $Q(x)$ is uniquely determined by the eigenvalues and normalising coefficients. Later, in [16] and in [25], it is provided some results in inverse spectral analysis with partial data on the potential.

Spectral problems involving eigenvalue dependent boundary conditions arise in various problems of mathematics as well as in applications. Firstly, in 1973, Walter [36] was interested in an expansion theorem for the this kind of eigenvalue problem. In 1977, Fulton [13] also examined the Sturm-Liouville eigenvalue problem. Inverse problems for some classes of differential operators linearly eigenvalue dependent are analysed in diverse papers (see [1, 2, 9, 17, 22]). More general boundary conditions are observed in [4-6, 10, 12, 26, 29, 30, 34,35]. On the other hand, when $f(\lambda)$ is a rational function of Herglotz–Nevanlinna type, direct and inverse spectral problems for Sturm-Liouville operator were investigated in [7, 8, 27].

Recently, also some new uniqueness results in inverse spectral analysis with partial information on the potential for Sturm-Liouville and Dirac operator have been given (see , [28, 31-33]).

In the present paper, our research is connected to a Dirac equation with rationally eigenvalue dependent boundary conditions and linearly eigenvalue dependent jump conditions. We prove that the potential in

$(0,1)$ and the remaining coefficients of the boundary value problem can be uniquely determined by the potential in $(\frac{1}{2},1)$ and one spectrum excluding finite eigenvalues. We also show that $Q(x)$ is uniquely determined by the sequences of eigenvalues and normalising numbers.

2. PRELIMINARIES

Firstly, we define an operator such that (1)-(4) can be regarded as an eigenvalue problem of it. See [8] and [27] for a similar operator associated with the Sturm- Liouville problem.

Consider the space $H = L_2(0,1) \oplus L_2(0,1) \oplus \mathbb{C}^{N+2}$ with the inner product defined by

$$\langle Y, Z \rangle := \int_0^1 \left[y_1(x)\overline{z}_1(x) + y_2(x)\overline{z}_2(x) \right] dx + \sum_{k=1}^{N} \frac{Y_k \overline{Z}_k}{f_k} + \frac{Y_{N+1}\overline{Z}_{N+1}}{a} + \frac{\alpha Y_{N+2}\overline{Z}_{N+2}}{\beta} \tag{5}$$

where $Y = \left(y_1(x), y_2(x), Y_1, Y_2, \ldots Y_{N+2}\right)$, $Z = \left(z_1(x), z_2(x), Z_1, Z_2, \ldots Z_{N+2}\right)$. Define the operator T with the domain

$D(T) = \{Y \in H:$ $y_1(x)$ and $y_2(x)$ are absolutely continuous in I, $\ell y \in L_2(0,1)$, $y_1(0) = 0$, $y_1(\frac{1}{2}+0) - \alpha y_1(\frac{1}{2}-0) = 0$, $Y_{N+1} = -a y_1(1)$, $Y_{N+2} = \beta y_1(\frac{1}{2}-0)\}$ such that

$$T(Y) = Z = \left(z_1(x), z_2(x), Z_1, Z_2, \ldots Z_{N+2}\right),$$

where $z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} = \ell y(x)$,

$$Z_i = g_i Y_i - f_i y_1(1), \text{ for } i = \overline{1, N}$$

$$Z_{N+1} = y_2(1) + b y_1(1) + \sum_{k=1}^{N} Y_k$$

and

$$Z_{N+2} = y_2(\frac{1}{2}+0) - \alpha^{-1} y_1(\frac{1}{2}-0) - \gamma y_1(\frac{1}{2}-0)$$

Theorem 1 *Eigenvalues of the operator* T *coincide with eigenvalues of the problem* L.

Proof. The following equalities obtained from $TY = \lambda Y$ are enough to see validity of this theorem:

$$g_1 Y_1 - b_1 y_1(1) = \lambda Y_1$$

$$g_2 Y_2 - b_2 y_1(1) = \lambda Y_2$$

$$\ldots$$

$$g_N Y_1 - b_N y_1(1) = \lambda Y_N$$

$$y_2(1) + b y_1(1) + \sum_{k=1}^{N} Y_k = -a\lambda y_1(1)$$

$$y_2(\frac{1}{2}+0) - \alpha^{-1} y_2(\frac{1}{2}-0) - \gamma y_1(\frac{1}{2}-0) = \lambda\beta y_1(\frac{1}{2}-0).$$

Let the function $\varphi(x,\lambda) = (\varphi_1(x,\lambda)\ \ \varphi_2(x,\lambda))^T$ be the solution of (1) under the initial conditions

$$\varphi(0,\lambda) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{6}$$

and under the jump conditions (4). $\varphi(x,\lambda)$ satisfies the following integral equations:

For $x < \frac{1}{2}$

$$\varphi_1(x,\lambda) = \sin \lambda x + \int_0^x \sin \lambda(x-t)\Phi^+(t,\lambda)dt + \int_0^x \cos\lambda(x-t)\Phi^-(t,\lambda)dt$$

$$\varphi_2(x,\lambda) = -\cos\lambda x - \int_0^x \cos\lambda(x-t)\Phi^+(t,\lambda)dt + \int_0^x \sin \lambda(x-t)\Phi^-(t,\lambda)dt$$

and for $x > \frac{1}{2}$

$$\varphi_1(x,\lambda) = \alpha^+ \sin \lambda x + \alpha^- \sin \lambda(1-x) +$$

$$+ \int_0^{1/2} \left(\alpha^+ \sin \lambda(x-t) + \alpha^- \sin \lambda(1-x-t)\right)\Phi^+(t,\lambda)dt$$

$$+ \int_0^{1/2} \left(\alpha^+ \cos\lambda(x-t) + \alpha^- \cos\lambda(1-x-t)\right)\Phi^-(t,\lambda)dt$$

$$+\int_{1/2}^{x}\left[\sin\lambda(x-t)\Phi^{+}(t,\lambda)+\cos\lambda(x-t)\Phi^{-}(t,\lambda)\right]dt$$

$$+\frac{(\beta\lambda+\gamma)}{2}\left\{\cos\lambda x-\cos\lambda(1-x)+\int_{0}^{1/2}[\cos\lambda(x-t)-\cos\lambda(1-x-t)]\Phi^{+}(t,\lambda)dt\right.$$

$$\left.-\int_{0}^{1/2}[\sin\lambda(x-t)-\sin\lambda(1-x-t)]\Phi^{-}(t,\lambda)dt\right\}$$

$$\varphi_2(x,\lambda)=-\alpha^{+}\cos\lambda x+\alpha^{-}\cos\lambda(1-x)$$

$$-\int_{0}^{1/2}\left(\alpha^{+}\cos\lambda(x-t)-\alpha^{-}\cos\lambda(1-x-t)\right)\Phi^{+}(t,\lambda)dt$$

$$+\int_{0}^{1/2}\left(\alpha^{+}\sin\lambda(x-t)-\alpha^{-}\sin\lambda(1-x-t)\right)\Phi^{-}(t,\lambda)dt$$

$$-\int_{1/2}^{x}\left[\cos\lambda(x-t)\Phi^{+}(t,\lambda)+\sin\lambda(x-t)\Phi^{-}(t,\lambda)\right]dt$$

$$+\frac{(\beta\lambda+\gamma)}{2}\left\{\sin\lambda x+\sin\lambda(1-x)+\int_{0}^{1/2}[\sin\lambda(x-t)+\sin\lambda(1-x-t)]\Phi^{+}(t,\lambda)dt\right.$$

$$\left.+\int_{0}^{1/2}[\cos\lambda(x-t)+\cos\lambda(1-x-t)]\Phi^{-}(t,\lambda)dt\right\}$$

where $\Phi^{+}(t,\lambda)=\varphi_1(t,\lambda)p(t)+\varphi_2(t,\lambda)q(t),\quad \Phi^{-}(t,\lambda)=\varphi_1(t,\lambda)q(t)-\varphi_2(t,\lambda)p(t)$ and $\alpha^{\pm}=\frac{1}{2}\left(\alpha\pm\frac{1}{\alpha}\right)$.

Moreover, it is shown in [14] that, $\varphi(x,\lambda)$ has a representation as follows:

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+\int_0^x K(x,t)\varphi_0(t,\lambda)dt,\ \text{for}\, x\in\left(0,\frac{1}{2}\right) \tag{7}$$

where $\varphi_0(x,\lambda)=\left(\sin\lambda x\quad -\cos\lambda x\right)^{T},\quad K(x,t)=\left(K_{ij}(x,t)\right)_{i,j=1,2}$

$K_{ij}(x,.)\in L_2(0,1)$ for each fixed x.

The following asymptotic relations can be proved easily;

$$\varphi_1(x,\lambda)=\begin{cases}\sin\lambda x+o(\exp\tau x), & x<\dfrac{1}{2}\\ \dfrac{\beta\lambda}{2}\left[\cos\lambda x-\cos\lambda(1-x)\right]+o(\lambda\exp\tau x), & x>\dfrac{1}{2}\end{cases}$$

$$\varphi_2(x,\lambda)=\begin{cases}-\cos\lambda x+o(\exp\tau x), & x<\dfrac{1}{2}\\ \dfrac{\beta\lambda}{2}\left[\sin\lambda x+\sin\lambda(1-x)\right]+o(\lambda\exp\tau x), & x>\dfrac{1}{2}\end{cases}$$

where $\tau=|Im\lambda|$.

It is clear that $f(\lambda)$ can be written as follows:

$$f(\lambda)=\frac{a(\lambda)}{b(\lambda)}$$

where

$$a(\lambda)=(a\lambda+b)\prod_{k=1}^{N}(\lambda-g_k)-\sum_{k=1}^{N}f_k\prod_{\substack{j=1\\ j\neq k}}^{N}(\lambda-g_j),$$

$$b(\lambda)=\prod_{k=1}^{N}(\lambda-g_k).$$

Let the function $\psi(x,\lambda)=(\psi_1(x,\lambda)\ \psi_2(x,\lambda))^T$ be the solution of (1) under the initial conditions

$$\psi(1,\lambda)=\begin{pmatrix}b(\lambda)\\ -a(\lambda)\end{pmatrix} \tag{8}$$

and under the jump conditions (4).

Consider the function

$$\begin{aligned}\Delta(\lambda)&:=W[\psi,\varphi]=\varphi_2(x,\lambda)\psi_1(x,\lambda)-\varphi_1(x,\lambda)\psi_2(x,\lambda) \qquad (9)\\ &=a(\lambda)\varphi_1(1,\lambda)+b(\lambda)\varphi_2(1,\lambda)=-\psi_1(0,\lambda).\end{aligned}$$

It is obvious that $\Delta(\lambda)$ is entire function and the zeros, namely $\{\lambda_n\}_{n\in Z}$ of $\Delta(\lambda)$ are eigenvalues of the problem (1)-(4). Additionally, the equality $\psi(x,\lambda_n)=\chi_n\varphi(x,\lambda)$ holds for all x and λ, where $\chi_n=\psi_2(0,\lambda_n)=\dfrac{b(\lambda_n)}{\varphi_1(1,\lambda_n)}$.

3. UNIQUENESS THEOREMS

3.1 According to the mixed given data

The first main result of this work is a generalized of Hochstadt and Lieberman theorem [21]. We prove that when $Q(x)$ is known on $\left(\frac{1}{2},1\right)$ then only one spectrum excluding a finite number of eigenvalues is sufficient to determine $Q(x)$ on the interval $(0,1)$ and the coefficients α , β and γ. Together with L, we consider the problem $\tilde{L}$ of the same form but with a different coefficients $\tilde{Q}(x)=\begin{pmatrix}\tilde{p}(x) & q(x)\\ q(x) & -\tilde{p}(x)\end{pmatrix}$, $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$. It is assumed in what follows that if a certain symbol s denotes an object related to L, then the corresponding symbol $\tilde{s}$ with tilde denote the analogous object related to $\tilde{L}$. Let us denote by $\varphi(x,\lambda_n)$, the eigenfunction which corresponds to λ_n.

Let Z_0 be any subset with $N-1$ elements of Z. Denote $\sigma:=\{\lambda_n\}_{n\in Z\setminus Z_0}$ and consider the following representation:

$$\Delta(\lambda)=R(\lambda)\prod_{n\in Z_0}(\lambda-\lambda_n)$$

where $R(\lambda)=C\prod_{\lambda_n\in\sigma}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right)$, C is a constant which depends only on λ_n (the case $\Delta(0)=0$ requires minor modifications).

Theorem 2 *If* $\sigma=\tilde{\sigma}$, $f(\lambda)\equiv\tilde{f}(\lambda)$ *and* $Q(x)=\tilde{Q}(x)$ *on* $\left(\frac{1}{2},1\right)$ *then* $Q(x)=\tilde{Q}(x)$ *almost everywhere on* $(0,1)$, $\alpha=\tilde{\alpha}$ *and* $\beta=\tilde{\beta}$ *and* $\gamma=\tilde{\gamma}$.

Before the proof of theorem, we need to prove the following lemma.

Lemma 1 *i) The eigenvalues* λ_n *are real numbers.*

ii) $\left.\frac{d\Delta(\lambda)}{d\lambda}\right|_{\lambda=\lambda_n}\neq 0$, i.e., $\{\lambda_n\}_{n\in Z}$ *are simple zeros of* $\Delta(\lambda)$.

Proof. i) According to the Theorem 1, it is enough to show that eigenvalues of T are real. For Y in $D(T)$, we calculate that

$$\langle TY,Y\rangle=\int_0^1 \ell y(x)\overline{y}(x)dx+\sum_{k=1}^{N}\frac{TY_k\overline{Y}_k}{f_k}+\frac{TY_{N+1}\overline{Y}_{N+1}}{a}+\frac{\alpha TY_{N+2}\overline{Y}_{N+2}}{\beta}$$

$$=\int_0^1 p(x)\left(|y_1(x)|^2-|y_2(x)|^2\right)dx+\int_0^1 2Re\left[y_2(x)\overline{y}_1(x)\right]q(x)dx$$

$$+\overline{y}_1\left(\frac{1}{2}-0\right)y_2\left(\frac{1}{2}-0\right)-\overline{y}_1(0)y_2(0)-\overline{y}_1\left(\frac{1}{2}+0\right)y_2\left(\frac{1}{2}+0\right)$$

$$+\overline{y}_1(1)y_2(1)-\int_0^1 2Re\left[y_2(x)\overline{y}_1(x)\right]dx+\sum_{k=1}^{N}\frac{\left[g_kY_k-f_ky_1(1)\right]\overline{Y}_k}{f_k}$$

$$-\left[y_2(1)+by_1(1)+\sum_{k=1}^{N}Y_k\right]\overline{y}_1(1)$$

$$+\alpha\left[y_2(\frac{1}{2}+0)-\alpha^{-1}y_2(\frac{1}{2}-0)-\gamma y_1(\frac{1}{2}-0)\right]\overline{y}_1(\frac{1}{2}-0).$$

By using the structure of T, we obtain

$$\langle TY,Y\rangle=\int_0^1 p(x)\left(|y_1(x)|^2-|y_2(x)|^2\right)dx+\int_0^1 2Re\left[y_2(x)\overline{y}_1(x)\right]q(x)dx$$

$$-\int_0^1 2Re\left[y_2(x)\overline{y}_1(x)\right]dx+\sum_{k=1}^{N}\frac{g_k}{f_k}|Y_k|^2-\sum_{k=1}^{N}2Re\left[y_1(1)\overline{Y}_k\right]$$

$$-b|y_1(1)|^2-\alpha\gamma\left|y_1(\frac{1}{2}-0)\right|^2.$$

We conclude that $\langle TY,Y\rangle$ are real for each Y in $D(T)$. Thus, all eigenvalues of the operator T (or problem L) are real numbers.

ii) Assume $\lambda_n\neq g_k$. If $\lambda_n=g_k$, the proof is similar.

Write the equation (1) for $\varphi(x,\lambda_n)$ and $\psi(x,\lambda)$ respectively.

$$\begin{cases}\varphi_2'(x,\lambda_n)+p(x)\varphi_1(x,\lambda_n)+q(x)\varphi_2(x,\lambda_n)=\lambda_n\varphi_1(x,\lambda_n),\\-\varphi_1'(x,\lambda_n)+q(x)\varphi_1(x,\lambda_n)-p(x)\varphi_2(x,\lambda_n)=\lambda_n\varphi_2(x,\lambda_n)\end{cases}$$

$$\begin{cases} \psi_2'(x,\lambda) + p(x)\psi_1(x,\lambda) + q(x)\psi_2(x,\lambda) = \lambda\psi_1(x,\lambda), \\ -\psi_1'(x,\lambda) + q(x)\psi_1(x,\lambda) - p(x)\psi_2(x,\lambda) = \lambda\psi_2(x,\lambda). \end{cases}$$

After some operations, we obtain

$$\varphi_1(x,\lambda_n)\psi_2(x,\lambda) - \varphi_2(x,\lambda_n)\psi_1(x,\lambda)\left(\Big|_0^{\frac{1}{2}} + \Big|_{\frac{1}{2}}^{1} \right)$$

$$= (\lambda - \lambda_n)\int_0^1 [\psi_1(x,\lambda)\varphi_1(x,\lambda_n) + \psi_2(x,\lambda)\varphi_2(x,\lambda_n)]dx.$$

Take into account of boundary and discontinuity conditions to get

$$\psi_1(0,\lambda) - a(\lambda)\varphi_1(1,\lambda_n) - b(\lambda)\varphi_2(1,\lambda_n)$$

$$-\varphi_1(\tfrac{1}{2}+0,\lambda_n)\psi_2(\tfrac{1}{2}+0,\lambda) + \varphi_2(\tfrac{1}{2}+0,\lambda_n)\psi_1(\tfrac{1}{2}+0,\lambda)$$

$$+\varphi_1(\tfrac{1}{2}-0,\lambda_n)\psi_2(\tfrac{1}{2}-0,\lambda) - \varphi_2(\tfrac{1}{2}-0,\lambda_n)\psi_1(\tfrac{1}{2}-0,\lambda)$$

$$= (\lambda - \lambda_n)\int_0^1 [\psi_1(x,\lambda)\varphi_1(x,\lambda_n) + \psi_2(x,\lambda)\varphi_2(x,\lambda_n)]dx.$$

Using (4) and (9) we get

$$-\Delta(\lambda) - b(\lambda)\varphi_1(1,\lambda_n)(f(\lambda) - f(\lambda_n)) - \alpha\beta(\lambda-\lambda_n)\varphi_1(\frac{1}{2}-0,\lambda_n)\psi_1(\tfrac{1}{2}-0,\lambda)$$

$$= (\lambda - \lambda_n)\int_0^1 [\psi_1(x,\lambda)\varphi_1(x,\lambda_n) + \psi_2(x,\lambda)\varphi_2(x,\lambda_n)]dx.$$

If we divide both side of this equality by $(\lambda - \lambda_n)$ and take limit for $\lambda \to \lambda_n$,

$$\chi_n\left[\int_0^1 [\varphi_1^2(x,\lambda_n) + \varphi_2^2(x,\lambda_n)]dx + \varphi_1^2(1,\lambda_n)f'(\lambda_n) + \alpha\beta\varphi_1^2(\frac{1}{2}-0,\lambda_n)\right] = -\Delta'(\lambda_n).$$

Since $\alpha > 0$, $\beta > 0$ and $f'(\lambda_n) > 0$ for all λ_n, $\Delta'(\lambda_n) \neq 0$.

Let us write the equation (1) for φ and $\tilde{\varphi}$,

$$B\varphi'(x,\lambda) + Q(x)\varphi(x,\lambda) = \lambda\varphi(x,\lambda)$$

$$B\tilde{\varphi}'(x,\lambda)+\tilde{Q}(x)\tilde{\varphi}(x,\lambda)=\lambda\tilde{\varphi}(x,\lambda)$$

Multiply these equalities by $\tilde{\varphi}^T(x,\lambda)$ and $\varphi^T(x,\lambda)$ respectively from left hand side and subtract then we get

$$\begin{aligned}&\frac{d}{dx}\{\varphi_1(x,\lambda)\tilde{\varphi}_2(x,\lambda)-\tilde{\varphi}_1(x,\lambda)\varphi_2(x,\lambda)\}\\&=\left[Q(x)-\tilde{Q}(x)\right]\varphi(x,\lambda)\tilde{\varphi}(x,\lambda).\end{aligned}\tag{10}$$

After integrating this equality on $(0,1)$, taking into account the hypothesis

$Q(x)=\tilde{Q}(x)$ on $\left(\frac{1}{2},1\right)$, we find

$$\begin{aligned}&\{\varphi_1(x,\lambda)\tilde{\varphi}_2(x,\lambda)-\tilde{\varphi}_1(x,\lambda)\varphi_2(x,\lambda)\}\left(\Big|_0^{1/2}+\Big|_{1/2}^1\right)\\&=\int_0^{\frac{1}{2}}[p(x)-\tilde{p}(x)]\langle\tilde{\varphi}(x,\lambda)^T J,\varphi(x,\lambda)\rangle dx\end{aligned}\tag{11}$$

where $\langle .,. \rangle$ denotes the classical inner product of $\mathbb{C}^2$ and $J:=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Denote

$$H(\lambda):=\varphi_2(1,\lambda)\tilde{\varphi}_1(1,\lambda)-\varphi_1(1,\lambda)\tilde{\varphi}_2(1,\lambda).\tag{12}$$

One can calculate from (4) and (11) that

$$\begin{aligned}&\varphi_2(1,\lambda)\tilde{\varphi}_1(1,\lambda)-\varphi_1(1,\lambda)\tilde{\varphi}_2(1,\lambda)=\int_0^{\frac{1}{2}}[p(x)-\tilde{p}(x)]\langle\tilde{\varphi}(x,\lambda)^T J,\varphi(x,\lambda)\rangle dx\\&+\left(1-\frac{\alpha}{\tilde{\alpha}}\right)\varphi_1(\frac{1}{2}-0,\lambda)\tilde{\varphi}_2(\frac{1}{2}-0,\lambda)-\left(1-\frac{\tilde{\alpha}}{\alpha}\right)\varphi_2(\frac{1}{2}-0,\lambda)\tilde{\varphi}_1(\frac{1}{2}-0,\lambda)\\&+\left[\lambda\left(\tilde{\alpha}\beta-\alpha\tilde{\beta}\right)+\tilde{\alpha}\gamma-\alpha\tilde{\gamma}\right]\varphi_1(\frac{1}{2}-0,\lambda)\tilde{\varphi}_1(\frac{1}{2}-0,\lambda).\end{aligned}\tag{13}$$

It is obtained from $f(\lambda)\equiv\tilde{f}(\lambda)$ that $\varphi_2(1,\lambda_n)\tilde{\varphi}_1(1,\lambda_n)-\varphi_1(1,\lambda_n)\tilde{\varphi}_2(1,\lambda_n)=0$. Therefore, $H(\lambda_n)=0$ for all $\lambda_n\in\sigma$.

Now, define $F(\lambda):=\dfrac{H(\lambda)}{R(\lambda)}$ which is an entire function on λ. From asymptotic relations of $\varphi_1(x,\lambda)$ and $\varphi_2(x,\lambda)$, it is valid that $F(\lambda)=O\left(\dfrac{1}{|\lambda|}\right)$. Therefore, from Liouville's Theorem,

$F(\lambda)\equiv 0$ and so $H(\lambda)\equiv 0.$ One can calculate that $\int_0^{\frac{1}{2}}[p(x)-\tilde{p}(x)]\langle\tilde{\varphi}(x,\lambda)^T J,\varphi(x,\lambda)\rangle dx=\frac{1}{2}\int_0^{\frac{1}{2}}[p(x)-\tilde{p}(x)]dx+o(1),$ for $\lambda\to\infty,$ $\lambda\in\mathrm{R}$. Therefore it can be written the following equality from (13),

$$\frac{1}{2}\left[\lambda\left(\tilde{\alpha}\beta-\alpha\tilde{\beta}\right)+\tilde{\alpha}\gamma-\alpha\tilde{\gamma}\right](1-\cos\lambda+o(1))-\frac{1}{2}\left(\frac{\tilde{\alpha}}{\alpha}-\frac{\alpha}{\tilde{\alpha}}\right)[\sin\lambda+o(1)]$$

$$+\frac{1}{2}\int_0^{\frac{1}{2}}[p(x)-\tilde{p}(x)]dx=o(1)\ \text{for}\ \lambda\to\infty,\ \lambda\in\mathrm{R}.$$

We obtain from the last relation that, $\frac{\tilde{\alpha}}{\alpha}=\frac{\alpha}{\tilde{\alpha}}=\frac{\beta}{\tilde{\beta}}=\frac{\gamma}{\tilde{\gamma}}$ and so, $\alpha=\tilde{\alpha},$ $\gamma=\tilde{\gamma},$ $\beta=\tilde{\beta}$, $\int_0^{\frac{1}{2}}[p(x)-\tilde{p}(x)]dx=0.$ As a consequent,

$\int_0^{\frac{1}{2}}[p(x)-\tilde{p}(x)]\langle\tilde{\varphi}(x,\lambda)^T J,\varphi(x,\lambda)\rangle dx=0$ holds on the whole λ -plane.

On the other hand, the following equality is valid,

$$\langle\tilde{\varphi}(x,\lambda)^T J,\varphi(x,\lambda)\rangle=-\cos 2\lambda x+\int_0^x K_1(x,t)\cos 2\lambda t dt+\int_0^x K_2(x,t)\sin 2\lambda t dt$$

where $K_i(x,t),$ $i=1,2,$ depend only on x and $t.$ It follows from (13) that

$$\int_0^{\frac{1}{2}}P(x)\left\{\cos 2\lambda x-\int_0^x K_1(x,t)\cos 2\lambda t dt-\int_0^x K_2(x,t)\sin 2\lambda t dt\right\}dx=0 \tag{14}$$

for all λ , where $P(x):=[p(x)-\tilde{p}(x)]$. This can be rewritten as

$$\int_0^{\frac{1}{2}}\cos 2\lambda\tau\left[P(\tau)+\int_\tau^{\frac{1}{2}}P(x)K_1(x,t)dx\right]dt$$

$$+\int_0^{\frac{1}{2}}\sin 2\lambda t\int_\tau^{\frac{1}{2}}P(x)K_2(x,t)dxdt=0 \tag{15}$$

Therefore, we obtain from the completeness of the vector functions $(\cos 2\lambda\tau \quad \sin 2\lambda t)^T$ in $L_2(0,\frac{1}{2})\oplus L_2(0,\frac{1}{2})$ that $P(x)=0$, i.e. $p(x)=\tilde{p}(x)$ for $x\in(0,\frac{1}{2})$. This completes the proof.

3.2 According to the classical spectral data

In [23], it is proven that the coefficients of the problem L are uniquely determined by the Weyl function. We aim to prove uniqueness of the coefficients according to the eigenvalues and normalising numbers, namely spectral data. Consider the problem $\tilde{L}$ with the coefficient $\tilde{Q}(x)=\begin{pmatrix}\tilde{p}(x) & \tilde{q}(x)\\ \tilde{q}(x) & -\tilde{p}(x)\end{pmatrix}$.

For an element $Y=\left(y_1(x),y_2(x),Y_1,Y_2,...Y_{N+2}\right)$ in H, the norm of Y is defined by $\|Y\|^2:=\langle Y,Y\rangle$. From (5), we get

$$\|Y\|^2=\int_0^1\left[|y_1(x)|^2+|y_2(x)|^2\right]dx+\sum_{k=1}^{N}\frac{|Y_k|^2}{f_k}+\frac{|Y_{N+1}|^2}{a}+\frac{\alpha|Y_{N+2}|^2}{\beta} \tag{16}$$

Let λ_n be an eigenvalue of T (or the problem L) and $Y(n)$ eigenvector for λ_n. Then the numbers $\rho_n:=\|Y(n)\|^2$ are called as normalizing numbers.

Lemma 2 *The equality*

$$\rho_n=\int_0^1\left[\varphi_1^2(x,\lambda_n)+\varphi_2^2(x,\lambda_n)\right]dx$$

$$+f^{'}(\lambda_n)\varphi_1^2(1,\lambda_n)+\alpha\beta\varphi_1^2(d_1-0,\lambda_n)$$

is valid.

Proof. Let $\lambda_n\neq g_k$. Since $\lambda_n=g_k$ is equivalent to $\varphi_1(1,g_k)=0$, this case requires minor modification in the following proof.

Using the structure of T and the inner product in (5), a direct calculation yields

$$\|Y(n)\|^2=\int_0^1\left[\varphi_1^2(x,\lambda_n)+\varphi_2^2(x,\lambda_n)\right]dx$$

$$+\sum_{k=1}^{N}\frac{|Y_k|^2}{f_k}+\frac{|Y_{N+1}|^2}{a}+\frac{\alpha|Y_{N+2}|^2}{\beta}$$

$$= \int_0^1 \left[\varphi_1^2(x,\lambda_n) + \varphi_2^2(x,\lambda_n)\right] dx$$

$$+ \varphi_1^2(1,\lambda_n) \sum_{k=1}^{N} \frac{f_k}{[g_k - \lambda_n]^2} + a\varphi_1^2(1,\lambda_n)$$

$$= \int_0^1 \left[\varphi_1^2(x,\lambda_n) + \varphi_2^2(x,\lambda_n)\right] dx$$

$$+ \varphi_1^2(1,\lambda_n) \left\{ a + \sum_{k=1}^{N} \frac{f_k}{[g_k - \lambda_n]^2} \right\} + \alpha\beta\varphi_1^2(\frac{1}{2} - 0, \lambda_n)$$

$$= \int_0^1 \left[\varphi_1^2(x,\lambda_n) + \varphi_2^2(x,\lambda_n)\right] dx + f'(\lambda_n)\varphi_1^2(1,\lambda_n) + \alpha\beta\varphi_1^2(\frac{1}{2} - 0, \lambda_n)$$

Take into account Lemma 1 and Lemma 2 to get the following relation

$$\Delta'(\lambda_n) = \rho_n \chi_n \tag{17}$$

The Weyl function is defined as follows;

$$m(\lambda) = \frac{\psi_2(0,\lambda)}{\Delta(\lambda)}. \tag{18}$$

Theorem 3 *[23] If* $m(\lambda) = \tilde{m}(\lambda)$ *then* $L = \tilde{L}$; *i.e. the Weyl function* $m(\lambda)$ *determines uniquely the problem* L .

Theorem 4 *If* $\{\lambda_n, \rho_n\} = \{\tilde{\lambda}_n, \tilde{\rho}_n\}$ *then* $L = \tilde{L}$; *i.e. the spectral data* $\{\lambda_n, \rho_n\}$ *determine uniquely the problem* L .

Proof. Since $\lambda_n = \tilde{\lambda}_n$, $\Delta(\lambda) = \tilde{\Delta}(\lambda)$. Therefore, from (17) $\chi_n = \tilde{\chi}_n$ so $\psi_2(0,\lambda_n) = \tilde{\psi}_2(0,\lambda_n)$. Hence the function defined as

$$G(\lambda) := \frac{\psi_2(0,\lambda) - \tilde{\psi}_2(0,\lambda)}{\Delta(\lambda)}$$

is an entire on λ. Moreover, one can obtained that $G(\lambda) = o(1)$ for $|\lambda| \to \infty$. $G(\lambda) \equiv 0$ and so $\psi_2(0,\lambda) \equiv \tilde{\psi}_2(0,\lambda)$. From (18) we have $m(\lambda) = \tilde{m}(\lambda)$. Consequently, from Theorem 3, $L = \tilde{L}$.

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