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Acceleratly Expanding Cosmologies in $f(R, \Phi, X)$ Theory

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Abstract: In this study, beginning and today expansion of universe are viewed in $f(R, \Phi, X)$ gravity. Field equations and their solutions of Friedmann-Lemaître-Robertson-Walker cosmologies with perfect fluid are obtained by considering $f(R, \Phi, X) = f_0 R + f_1 X^q - V(\Phi)$ model. Validity of both $f(R, \Phi, X)$ gravity and $f(R, \Phi, X) = f_0 R + f_1 X^q - V(\Phi)$ model for non-static space-time geometries is discussed by making use of the obtained matter dynamics results such as pressure and energy density. It is seen that in all obtained solutions by taking into account early and late period expansion, f function is a constant. This indicates that $f(R, \Phi, X)$ function is a first-order dependent function of Ricci scalar. When $f(R, \Phi, X) = f_0 R + f_1 X^q - V(\Phi)$ model is considered together, it is understood that the obtained solutions could be reduced to $\Lambda - CDM$ model for $f(R)$ gravity in limits of $\Phi \to 0$ and $X \to 0$. The fact that the obtained results agree with expected situations supports. So, $f(R, \Phi, X)$ theory is a consistent theory of gravity.

$f(R, \Phi, X)$ Teori'de İvmeli Genişleyen Kozmolojiler

Anahtar Kelimeler
$f(R, \Phi, X)$ Gravite,
İdeal akışkan,
Friedmann- Lemaître-Robertson-Walker Uzay-Zamanı

Öz: Bu çalışmada, evrenin başlangıç ve günümüz genişlemesi, $f(R, \Phi, X)$ gravite çerçevesinde gözden geçirilmiştir. İdeal akışkanlı Friedmann-Lemaître-Robertson-Walker uzay-zamanı için alan denklemleri ve çözümleri $f(R, \Phi, X) = f_0 R + f_1 X^q - V(\Phi)$ modeli dikkate alınarak elde edilmiştir. Statik olmayan uzay-zaman geometrileri için $f(R, \Phi, X)$ gravite ve $f(R, \Phi, X) = f_0 R + f_1 X^q - V(\Phi)$ modelinin geçerliliği basınç ve enerji yoğunluğu gibi elde edilen madde dinamikleri kullanılarak tartışılmıştır. Elde edilen sonuçlardan erken ve geç dönem genişlemeleri için f fonksiyonunun sabit değer aldığı görülmüştür. Bu durum $f(R, \Phi, X)$ fonksiyonunun Ricci skalerine birinci dereceden bağlı bir fonksiyon olduğunu göstermektedir. $f(R, \Phi, X) = f_0 R + f_1 X^q - V(\Phi)$ modeli ile birlikte düşünüldüğünde $\Phi \to 0$ ve $X \to 0$ limitinde elde edilen çözümlerin $f(R)$ gravitenin $\Lambda - CDM$ modeline indirgeneceği anlaşılmaktadır. Öyleki elde edilen çözümlerin beklenen durum ile uyuşum içinde olması teorinin tutarlı bir gravitasyon teorisi olduğunu desteklemektedir.

1. Introduction

Einstein's General Relativity has lost its popularity due to problems such as cosmological constant problem and recent findings about last period expansion of the universe. Recently, search for new gravitational theories which provide solutions to these problems has attracted attention in studies on cosmology and gravitation. The universe has entered a period of accelerating expansion. Observational data consistently confirm the findings [1-3]. Although standard cosmology is successful in explaining many observed features of universe, it cannot explain the periods from beginning to end of universe in one go. This gave us idea that General Relativity could be modified to explain some dominant periods of universe [4]. As a result, many modified gravitational theories put forward. Brans-Dicke theory [5], $f(R)$ gravity [6], $f(R, T)$ gravity [7], $f(R, \Phi)$ gravity [8], k-essence gravity [9] and $f(R, \Phi, X)$ gravity [10-12] could be listed among these theories.

There may be different approaches to thinking about an acceptable theory of gravity. One of them is to differentiate variables of Lagrange functions used to relate matter and geometry. Recently, many theories have been proposed using this method. Theories with f function are in this class. $f(R)$ gravity, considered

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among dark energy-based theories, was firstly proposed by Buchdahl in 1970 with addition of a function dependent on Ricci scalar [7]. The theory is a family of theories integrating more than one $f(R)$ functions with differing function properties. The case where $f(R)$ function is constant, corresponds to Einstein's General Relativity. Also, it is base member of the family [7]. While early expansion can be explained by a repulsive force, scalar field is usually taken into account for transition in the radiation dominant period [8]. Later, by adding scalar field potential, $f(R,\Phi)$ gravity was put forward by Stabile and Capozziello [8]. A more generalized variation of these theories, $f(R,\Phi,X)$ gravity containing kinetic term was given by Tsujikawa in 2007 [11]. In order to realize solutions in these theories, degrees of derivative of equation of motion are important [13]. It is also necessary to explore alternatives that also provide solar system observational tests, rather than just considering dark energy behavior [14-16]. A gravitational effect that differs from General Relativity means that gravitational contribution of matter also differs. Therefore, in general, modified gravitational theories offer different observational results compared to models in General Relativity. So, $f(R)$ theory [17-19], scalar tensor theories [19-21], a general Lagrangian derived by assuming that mass of Φ field about Hubble constant H_0. This is controlled by light mass approach. In 2015, Bahamonde *et al.* examined recent cosmic acceleration-specific models and applications of a wide variety of dark energy and some gravitational models in $f(R,\Phi)$ gravity [12]. They also discussed solutions of some classes of $f(R,\Phi,X)$ gravity under spherically symmetric space-time [12]. Also, Malik *et al.* examined by using field equations of $f(R,\Phi,X)$ gravity [22]. They obtained six different solutions of some realistic regions in cylindrically symmetric space-time.

In this study, we aim to investigate the validity of some models proposed for $f(R,\Phi,X)$ gravity. For the purpose, behavior of matter in the perfect fluid form filling the Friedmann-Lemaître-Robertson-Walker (FLRW) type space-time is investigated by considering the $f(R,\Phi,X)=f_0R+f_1X^q-V(\Phi)$. The substance behavior obtained by taking into account the limit value conditions and limits are compared with its widely accepted nature in the literature. The study is planned as follows: Derivation of field equations for $f(R,\Phi,X)$ gravity is presented. Field equations and their solutions for FLRW space-time with perfect fluid in the case of $f(R,\Phi,X)=f_0R+f_1X^q-V(\Phi)$ are researched in Section 2 & 3. Then, Section 4 includes discussion of the current situation.

2. Material and Method

It is known that a class of alternative gravitational theories propose modifications to action function which derives Lagrange-Euler equations, one of the most important equations revealing relations between cosmic matters, fields and geometries. So, Lagrangian function L_m of geometrical part is not directly dependent on Ricci scalar as used in Einstein's General Relativity, but is considered as a function dependent on other factors thought to affect space-time geometry. By considering the success of $f(R)$ gravity, scalar field attempts taken into account in explaining transition from early period expansion to radiation dominant period expansion, and kinetic term proposed as a solution to dynamic dark energy problem, a powerful gravity that can respond to all of these problems at the same time can be researched. In this context, the action function proposed in $f(R,\Phi,X)$ gravity is considered as follows [10-12]:

$$S=\frac{1}{16\pi G}\int\sqrt{-g}d^4xf(R,\Phi,X)+S_m \tag{1}$$

where R, Φ and X indicate Ricci scalar, scalar field potential and kinetic term, respectively [11-12]. Kinetic term is defined as

$$X(\Phi)=-\frac{1}{2}[\partial^\alpha\Phi\partial_\alpha\Phi]\varepsilon \tag{2}$$

here ε is a parameter which can be chosen as $\varepsilon=1$ [11-12]. From variation of Eq.(1), field equation of $f(R,\Phi,X)$ gravity is defined as

$$\begin{aligned}FG_{ik}-\frac{1}{2}(f-RF)g_{ik}-\nabla_i\nabla_kF+g_{ik}\nabla_\alpha\nabla^\alpha F\\-\frac{\varepsilon}{2}H(\nabla_i\Phi)(\nabla_k\Phi)=T_{ik}\end{aligned} \tag{3}$$

here $F\equiv\frac{df}{dR}$ and $H\equiv\frac{df}{dX}$ [11-12]. Behavior of spineless particles located at source of quantum scalar or pseudoscalar fields requires considering Klein-Gordon Equation together in scalar field gravitational theories. Due to nature of the scalar field and principles of variation, the Klein-Gordon equation must also be provided in scalar field tensor theories. The equation for $f(R,\Phi,X)$ gravity is defined as

$$\nabla_i(H\nabla^i\Phi)+\varepsilon N=0 \tag{4}$$

here $N\equiv\frac{df}{d\Phi}$ [11-12]. It is known that homogenous, isotropic and expanding model of space-time geometry is defined as

$$\begin{aligned}ds^2=-dt^2+a^2(t)[\frac{dr^2}{1-kr^2}\\+r^2(d\theta^2+sin^2\theta d\varphi^2)]\end{aligned} \tag{5}$$

here $a(t)$ is scale factor. k is a parameter such as $k=-1,0,+1$. Also, matter additive which depends on energy density $\rho(t)$ and pressure $p(t)$ is

$$T_{ik}=(p+\rho)u_iu_k+pg_{ik}. \tag{6}$$

From Eqs.(1)-(6) one can obtain field equations for FLRW space-time with perfect fluid in $f(R,\Phi,X)$ gravity as follows:

$$\varepsilon H\Phi'^2 a^2 + 2\rho a^2 + 2F'' a^2 + 4Faa'' - 2F'aa' - 4Fa'^2 - 4Fk + 2w\rho a^2 = 0, \quad (7)$$

$$-3\varepsilon H\Phi'^2 a^2 - 6\rho a^2 - 6F'' a^2 - 12Faa'' + 6F'aa' + 12Fa'^2 + 12Fk - 6w\rho a^2 = 0 \quad (8)$$

where prime represents partial derivative according to time. Also, according to Equation of State (EoS), $p = w\rho$. From Eqs.(4)-(5), the Klein-Gordon equation is obtained for $f(R,\Phi,X)$ gravity:

$$-H'\Phi' + H\left(-\Phi'' - \frac{3\Phi' a'}{a}\right) + \varepsilon N = 0. \quad (9)$$

Bahamonde *et al*. proposed several different usable models, supported by the solutions by using Noether symmetry [12]. From the results, they proposed some class of $f(R,\Phi,X)$ gravity that includes scalar field, kinetic terms and Ricci scalar contributions. It is helpful to discuss the power series and exponential expansion phases in accordance with the nature of the universe by considering $f(R,\Phi,X) = f_0 R + f_1 X^q - V(\Phi)$ model. Bahamonde *et al*. discussed $V(\Phi)$ potentials for non-trivial states of Noether symmetry in the model [12]. Therefore, it can be said that it is a suitable model for discussing different cosmologies according to the $V(\Phi)$ choices.

3. Results

In this section, within the framework of considered viable model, early expansion is analyzed for $f(R,\Phi,X)$ gravity. For early expansion period, $a(t)$ scale factor is directly proportional to the positive powers of time $a(t) \propto t^n$. While $n = \frac{2}{3}$ indicates dust dominant expansion period and $n = \frac{1}{2}$ indicates radiation dominant expansion period. Let us now reconstruct the field equations in Eqs.(7)-(9) of $f(R,\Phi,X)$ gravity taking into account power low scale factor.

$$\varepsilon H\Phi'^2 t^{2n+2} + 2\rho t^{2n+2} + 2F'' t^{2n+2} - 4Fnt^{2n} - 2F'nt^{2n+1} - 4Fkt^2 + 2w\rho t^{2n+2} = 0, \quad (10)$$

$$-3\varepsilon H\Phi'^2 t^{2n} - 6\rho t^{2n} - 6F'' t^{2n} + 12Fnt^{2n-2} + 6F'nt^{2n-1} + 12Fk - 6w\rho t^{2n} = 0, \quad (11)$$

$$-H'\Phi' + H(-\Phi'' - 3\Phi' n t^{-1}) + \varepsilon N = 0. \quad (12)$$

From Eqs.(10)-(12), the solutions for the FRLW universe with perfect fluid within $f(R,\Phi,X)$ gravity, which can correspond to early universe, are suggested as follows:

$$H = \frac{2^{1-C_3} C_2 C_3 \varepsilon (\alpha\beta)^{2C_3-3}}{2C_3\beta - 2C_3 - 2\beta + 3n + 3} t^{4-3\beta+2C_3\beta-2C_3} + (\alpha\beta)^{-1} C_4 t^{1-3n-\beta}, \quad (13)$$

$$\rho = \frac{p}{w} = \frac{1}{2(w+1)} \left(-\frac{2^{1-C_3} C_2 C_3 \varepsilon^2 (\alpha\beta)^{2C_3-1}}{2C_3\beta - 2C_3 - 2\beta + 3n + 3} t^{2\beta C_3 - \beta - 2C_3 + 2} - \alpha\beta C_4 \varepsilon t^{-3n+\beta-1} + 4nC_1 t^{-2} + 4kC_1 t^{-2n} \right), \quad (14)$$

$$\Phi = \alpha t^{\beta}, \quad (15)$$

$$F = C_1, \quad (16)$$

$$N = C_2 C_3 X^{C_3 - 1} \quad (17)$$

where C_i represents constants.

As seen from Eq.(16), f function is obtained as a constant. This indicates that $f(R,\Phi,X)$ is a first-order dependent function of Ricci scalar. When the $f(R,\Phi,X) = f_0 R + f_1 X^q - V(\Phi)$ model is considered together, it is understood that the obtained solution could be reduced to $\Lambda - CDM$ model for $f(R)$ gravity in limits of $\Phi \to 0$ and $X \to 0$.

On the other hand, it is seen that dynamic components of ρ and p show different distributions according to the definition ranges of $\alpha, \beta, C_i, k, n, \varepsilon$ constants. From Eq.(14), existence of singularity is clear in the case of $w = -1$. Dark energy is thought to be carried in equivalent amount from early stages. According to $f(R,\Phi,X)$ gravity, considering $f(R,\Phi,X) = f_0 R + f_1 X^q - V(\Phi)$ model, an exotic energy without $w = -1$ can only exist in early stages. In addition, the change of energy density over time is given in Fig.(1)-(3). As can be seen from the figures, the energy density decreases with time. As can be seen from Fig.(1)-(3), the decrease is faster when $n = \frac{1}{2}$ than when $n = \frac{2}{3}$.

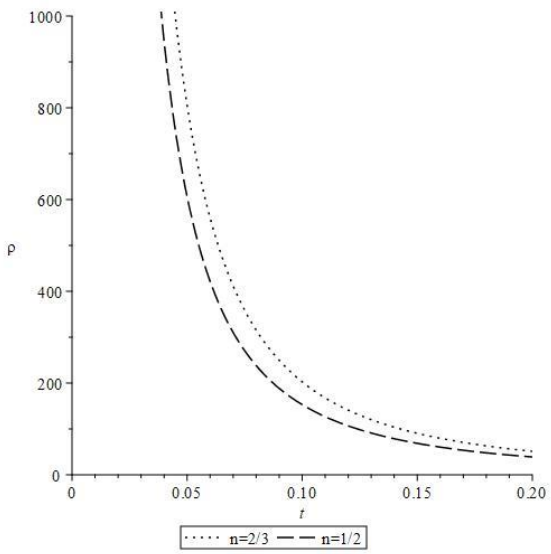

Figure 1. Change of the obtained energy density in early universe ($k = C_4 = 0,\ \varepsilon = \alpha = C_1 = C_2 = C_3 = 1,\ \beta = -1, w = -\frac{1}{3}$).

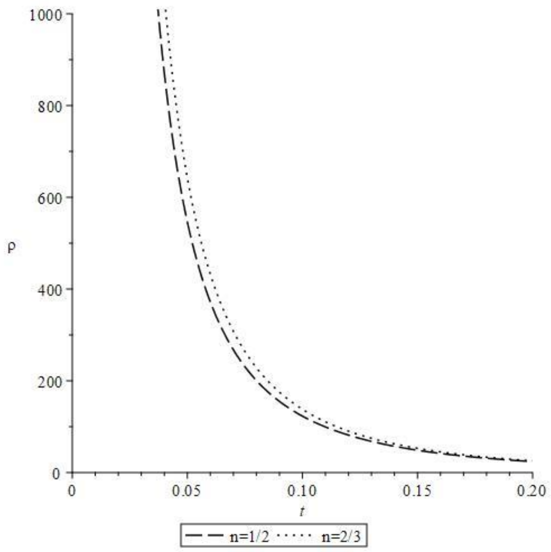

Figure 2. Change of the obtained energy density in early universe ($k = -1,\ \varepsilon = \alpha = C_1 = C_2 = C_3 = 1, C_4 = 0,\ \beta = -1, w = -\frac{1}{3}$).

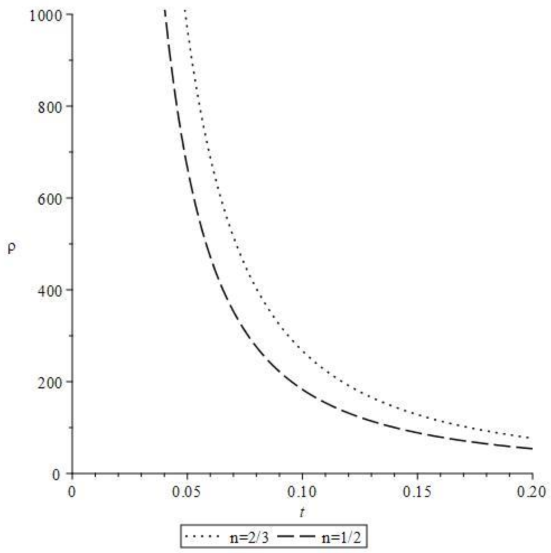

Figure 3. Change of the obtained energy density in early universe ($k = 1,\ \varepsilon = \alpha = C_1 = C_2 = C_3 = 1, C_4 = 0,\ \beta = -1, w = -\frac{1}{3}$).

Observational data show in which present universe is expanding at an accelerating rate. In FLRW cosmologies, this is possible with exponential scale factor $a(t) = e^{H_0 t}$, in which H_0 represents Hubble constant. At this point, let's consider the exponential scale factor and the $f(R, \Phi, X) = f_0 R + f_1 X^q - V(\Phi)$ model in order to interpret last period expansion of universe for $f(R, \Phi, X)$ gravity. From Eqs.(7)-(9), we get

$$e^{2H_0 t}\left[\varepsilon H \Phi'^2 + 2\rho(w+1) + 2F'' - 2F' H_0\right] = 4Fk, \quad (18)$$

$$-3\varepsilon H \Phi'^2 - 6\rho - 6F'' + 6F' H_0 + 12Fke^{-2H_0 t} - 6w\rho = 0, \quad (19)$$

$$-H'\Phi' + H(-\Phi'' - 3\Phi' H_0) + \varepsilon N = 0. \quad (20)$$

From Eqs.(18)-(20), a solution for the FLRW universe with perfect fluid within $f(R, \Phi, X)$ gravity, which can correspond to late time universe, is suggested as follows:

$$H(t) = \frac{H_1(t)}{\Phi' e^{3H_0 t}}, \quad (21)$$

$$\rho = \frac{p}{w} = -\frac{e^{-2H_0 t}}{2(w+1)}(\epsilon\alpha\beta t^{\beta-1} e^{-H_0 t} H_1(t) - 4C_1 k, \quad (22)$$

$$\Phi = \alpha t^{\beta}, \quad (23)$$

$$F = C_1, \quad (24)$$

$$N = C_2 C_3 X^{C_3 - 1}, \quad (25)$$

where

$$H_1(t) = \int \epsilon C_2 C_3 e^{3H_0 t}\left(\frac{\alpha\beta t^{\beta-1}}{\sqrt{2}}\right)^{2C_3 - 2} dt + C_5. \quad (26)$$

In this case, while the functions given in Eqs.(23)-(26) have the same values as in the early universe, dynamical components of perfect fluid show different distributions of them. As can be seen from Eq.(22), there is $w = -1$ singularity in this case, as well. In addition, for the finite pressure and density components, H_0 parameter must also take a non-zero value. Variations of energy density with respect to fixed choices are given in Fig.(4). As can be seen from Eq.(22), positive or negative w constant does not change the energy density as expected, while pressure is affected. Also, values of k constant of FLRW cosmologies affect the increasing or decreasing nature of energy density as can be seen from Fig.(4).

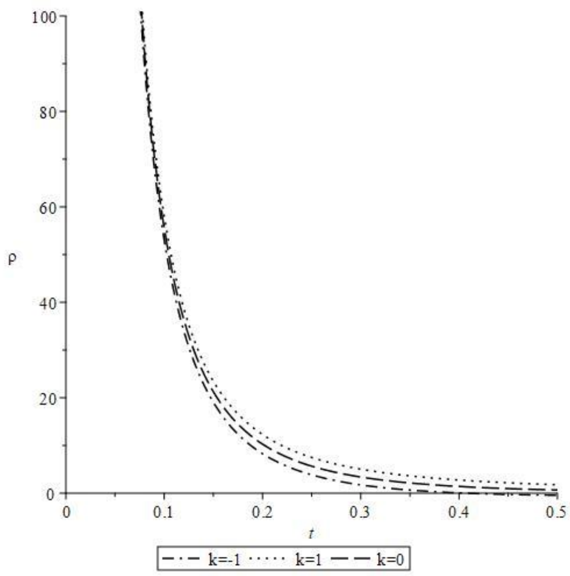

Figure 4. Change of the obtained energy density in late-time expansion ($\varepsilon = \alpha = C_1 = C_3 = 1,\quad C_2 = C_5 = 0,\ \beta = -1,\ w = -\frac{1}{3},\ H_0 = 1$).

4. Conclusion

In this study, early and late period expansions of universe are examined by considering dynamic components of the dominant matter of the universe within $f(R,\Phi,X)$ gravity, which is proposed as an alternative to Einstein's General Relativity and later $f(R)$, $f(R,T)$, $f(R,\Phi)$, k-essence and other theories. Field equations and their solutions are sought in a remarkable model of $f(R,\Phi,X)$ gravity for universe expansions of dust dominated, radiation dominated and dark energy dominated stages. By taking the $f(R,\Phi,X)=f_0R+f_1X^q-V(\Phi)$ model into account for each cases, the field equations and Klein-Gordon equation for FLRW space-time with perfect fluid were obtained. It is seen that the $f(R,\Phi,X)$ gravity allows for solutions of the obtained field equations.
The scale factor with a power low approach is taken into account for early stages. As can be seen from Eq.(14), $w=-1$ singularity attracts attention for early period. According to $f(R,\Phi,X)$ gravity, a dark energy existence may be possible in early stages of universe, but dark energy candidate must be negative pressure and positive density ($p=w\rho$, $w<0$) without $w=-1$. In Fig.(1)-(3), the variation of the energy density of the cosmic matter form filling the early universe with time is given for some fixed choices. In Fig.(1)-(3), it is seen that obtained energy densities of dust and radiation decrease according to $f(R,\Phi,X)$ gravity. So, it is seen from Fig.(1)-(3) that obtained energy density for scale factor indicating the radiation period decreases faster than scale factor indicating the dust substance period. In addition, according to $f(R,\Phi,X)$ gravity, the decreasing trend of obtained energy density for all values k parameter which has a decisive role in the universe model, does not change.

The scale factor with exponential form approach is taken into account for late-time stages. There is $w=-1$ singularity in this stage, too. Also, as expected, it is seen from Eq.(22) that energy density is positive. The variation of obtained energy density for late-time stage with time is given in the Fig.(4). From the Fig.(4), it is clear that this component tends to decrease. It is seen that k parameter does not have an effect on decrease trend of the density, but affects rate of decrease in the late period results as it is in the early period results.

From Eqs.(16) and (24), it is seen that in both of solutions obtained by taking into account early and late period expansion, F function is a constant. This indicates that $f(R,\Phi,X)$ function is a first-order dependent function of Ricci scalar. When the $f(R,\Phi,X)=f_0R+f_1X^q-V(\Phi)$ model is considered together, it is understood that obtained solution could be reduced to $\Lambda-CDM$ model for $f(R)$ gravity in limits of $\Phi\to 0$ and $X\to 0$. The fact that the results obtained are in agreement with the expected situations supports that the theory is a consistent theory of gravity and that this situation needs to be investigated intensively.

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Declaration of Ethical Code

In this study, we undertake that all the rules required to be followed within the scope of the "Higher Education Institutions Scientific Research and Publication Ethics Directive" are complied with, and that none of the actions stated under the heading "Actions Against Scientific Research and Publication Ethics" are not carried out.

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