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Research Article

Finite commutative rings whose line graphs of comaximal graphs have genus at most two

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Abstract

Let R be a ring with identity. The comaximal graph of R, denoted by $\Gamma(R)$, is a simple graph with vertex set R and two different vertices a and b are adjacent if and only if $aR + bR = R$. Let $\Gamma_2(R)$ be a subgraph of $\Gamma(R)$ induced by $R\backslash\{U(R) \cup J(R)\}$. In this paper, we investigate the genus of the line graph $L(\Gamma(R))$ of $\Gamma(R)$ and the line graph $L(\Gamma_2(R))$ of $\Gamma_2(R)$. All finite commutative rings whose genus of $L(\Gamma(R))$ and $L(\Gamma_2(R))$ are 0, 1, 2 are completely characterized, respectively.

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Keywords. finite commutative ring, comaximal graph, line graph, genus, induced subgraph

1. Introduction

The cross research on rings and graphs has attracted lots of attention by many mathematicians. The definition of graph on a ring one based on the special elements of the ring, for example, zero-divisor graphs[6], unit graphs[3] and total graphs[1] of rings; another one based on the ideals of the ring, for example, comaximal ideal graph[4, 5, 20] and zero-divisor graphs with respect to ideals[10] of rings. The comaximal graphs of rings based on both elements and ideals of rings seems to be more interesting. Let R be a ring with identity. The comaximal graph of R, denoted by $\Gamma(R)$, is a simple graph whose vertices are elements of the ring R, and two different vertices a and b are adjacent if and only if $aR + bR = R$. In 1995, Sharma and Bhatwadekar[16] firstly gave the definition and studied its basic properties. Maimani et al.[11] conducted research on the connectedness and diameter of comaximal graphs of rings in 2008. Meanwhile, Wang[20] determined the finite commutative rings whose comaximal graphs having genus 0 and 1, respectively. In 2011, Moconja and Petrovi[12] investigated the center, radius and girth of the comaximal graphs of commutative semilocal rings. In 2020, Sinha and Rao[17] discussed the planarity of line graphs of comaximal graphs. More recently, a book [2] on graphs over rings summaries the research on graph structures over rings. This book gives an overview of

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research on graphs associated with commutative rings. Many other papers are devoted to the comaximal graphs of rings, see [13, 15, 19, 21, 22].

In this paper, all graphs are finite and simple. Let G be a graph with vertex set V and edge set E. The degree of a vertex of v, denoted by $deg(v)$, is the number of vertices incident with v. $\Delta(G)$ is the maximal degree of G among all vertices in G. For two graphs H and G, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then we call H a subgraph of G. Let S be a subset of the vertex set of a graph G. Then the subgraph of G induced by S is a graph whose vertex set is S and two vertices u and v are adjacent if and only if they are adjacent in G. K_n stands for the complete graph on n vertices and $K_{m,n}$ is the complete bipartite graph on size m and n.

This paper concerns with the genus of the line graph of the comaximal graph of a finite commutative ring. Also, in this paper, surfaces are compact 2-manifolds without boundary. An orientatable surface $\mathbb{S}_g$ is said to be of genus g if it is topologically homeomorphic to a sphere with g handles. A graph that can be drawn without crossing on a compact surface of genus g, but not on one of genus $g-1$, is called a graph of genus g. In general, determining the genus of a graph is not an easy task. It is shown by Thomassen in [18] that the graph genus problem is indeed NP-complete. The genus of a graph G is denoted by $\gamma(G)$. It is clear that $\gamma(H) \leq \gamma(G)$ for any subgraph H of G. A graph is said to be embeddable on a surface if it can be drawn on that surface in such a way that no two edges cross. Such a drawing is referred to as an embedding. Note that the nonorientable genus of a planar graph is zero.

The line graph of a graph G, denoted by $L(G)$, is a graph whose vertex set is the edge set of G and two vertices are adjacent in $L(G)$ if and only if they share a common vertex in G. In 1978, Bénard[7] studied the genus of line graphs for some class of graphs and determined the lower bound of genus of line graph of a complete graph. In 2010, Chiang-Hsieh et al. [8] studied the genus of line graph of zero divisor graphs of rings and classified all finite commutative rings with genus at most two. In 2014, Eric et al. [9], studied the girth and clique number of line graphs of total graphs of rings and determined when it is Eulerian.

There is a lower bound for the genus of a connected simple graph.

Lemma 1.1. ([23, Corollaries 6.14 and 6.15]). *Suppose that a simple graph G is connected with $p(\geq 3)$ vertices and q edges. Then $\gamma(G) \geq \frac{q}{6} - \frac{p}{2} + 1$. Furthermore, if G has no triangles, then $\gamma(G) \geq \frac{q}{4} - \frac{p}{2} + 1$.*

For the genus of the line graph of a graph, we have

Lemma 1.2 ([14]). *Let G be a non-empty graph. Its line graph $L(G)$ is planar if and only if the following conditions hold:*
(1) *G is planar;*
(2) *$\Delta(G) \leq 4$;*
(3) *If $deg(v) = 4$, then v is a cut vertex of G.*

Based on the research of Bénard[7], Chiang-Hsieh and Lee et al. [8] obtained the formulae of genus of line graph of a complete graph or a complete bipartite graph, see the following lemma.

Lemma 1.3. (1) $\gamma(L(K_n)) \geq \lceil \frac{(n+1)(n-3)(n-4)}{12} \rceil$, *the equality hold if and only if* $n \equiv 0, 3, 4, 7 (mod\ 12)$;
(2) $\gamma(L(K_{1,n})) = \lceil \frac{(n-3)(n-4)}{12} \rceil$;
(3) $\gamma(L(K_{2,n})) \geq \lceil \frac{(n-2)(n-3)}{6} \rceil$, *when* $n \neq 5, 9 (mod\ 12)$ *the equality hold;*
(4) $\gamma(L(K_{3,3})) = 1$, $\gamma(L(K_{3,4})) = 2$, $\gamma(L(K_{4,4})) \geq 3$.

Throughout, rings are associative with identity. We use $J(R)$ and $U(R)$ to denote the Jacobson radical and the group of units of a ring R respectively. We write $\overline{R} = R/J(R)$ and $\bar{a} = a + J(R) \in \overline{R}$ for $a \in R$. As usual, we write $\mathbb{Z}_n$ for the ring of integers modulo n and by $\mathbb{F}_p$ the field of p elements. The cardinal of a set X is denoted $|X|$. $\lceil x \rceil$ is the least positive integer greater than or equal to x.

2. The genus of $L(\Gamma(R))$

In this section, we shall classify all finite commutative rings whose genus of line graph of comaximal graph is 0, 1, and 2, respectively. As we all known, for any finite commutative ring R, R can be uniquely expressed as the direct product of several finite local rings, that is, $R \cong R_1 \times R_2 \times \cdots \times R_s$, where $R_1, R_2, \ldots, R_s$ are finite commutative local rings. According to the value of s, we can complete the task. First, we give a lemma.

Lemma 2.1. *Let $R \cong R_1 \times R_2 \times \cdots \times R_s$. If $s \geq 3$, then $\gamma(L(\Gamma(R))) \geq 3$.*

Proof. If $|R| \geq 10$, then $deg(1) \geq 10$ in $\Gamma(R)$, which deduces that $L(\Gamma(R))$ contains a subgraph K_9. So we have $\gamma(L(\Gamma(R))) \geq \gamma(K_9) = 3$. So, we just need to consider the case of $s = 3$ as $|R| \geq 16$ when $s \geq 4$.

When $s = 3$. Agian by the fact that $|R| \geq 10$ implies $\gamma(L(\Gamma(R))) \geq \gamma(K_9) = 3$, we only need to consider the case that $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $u_1 = (1,1,1),\ u_2 = (0,1,1),\ u_3 = (1,0,1),\ u_4 = (1,1,0),\ u_5 = (1,0,0),\ u_6 = (0,1,0),\ u_7 = (0,0,1),\ u_8 = (0,0,0)$. Then we know that $|V(\Gamma(R))| = 8$, $E(\Gamma(R)) = \{u_1u_i | 2 \leq i \leq 8\} \cup \{u_2u_3, u_2u_4, u_3u_4, u_2u_5, u_3u_6, u_4u_7\}$, that is, $|E(\Gamma(R))| = 13$. In $L(\Gamma(R))$, we mark the point v_{ij} corresponding to the edge $u_i - u_j (i \neq j)$ in $\Gamma(R)$, (where $v_{ij} = v_{ji}$), and two distinct vertices v_{ij} and v_{kl} are adjacent if and only if $\{i,j\} \cap \{k,l\} \neq \emptyset$. Thus, it is easy to check that $|V(L(\Gamma(R)))| = 13$, $|E(L(\Gamma(R)))| = 42$. By Lemma 1.1, we have $\gamma(L(\Gamma(R))) \geq \lceil \frac{42}{6} - \frac{13}{2} + 1 \rceil = 2$. Now we show that $\gamma(L(\Gamma(R))) \neq 2$ in the following.

Assume on the contrary that $\gamma(L(\Gamma(R))) = 2$. Fix a presentation of $L(\Gamma(R))$ on the surface of $\mathbb{S}_2$. Then, by Euler formulae $v - e + f = 2 - 2\gamma$, there are 27 faces in the presentation, say $\{F_1, F_2, \cdots, F_{27}\}$. Now, by deleting $\{v_{24}, v_{34}\}$ and the edges incident with $\{v_{24}, v_{34}\}$, we obtain a presentation of a subgraph of $L(\Gamma(R))$, denote it by G. Observe that the presentation of $L(\Gamma(R))$ can be recovered from the presentation of G by inserting $\{v_{24}, v_{34}\}$ and the edges incident with $\{v_{24}, v_{34}\}$. Observe also that the presentation of G has 23 faces, say $\{F_1', F_2', \cdots, F_{23}'\}$. Let $S_{F_i'}$ be the length of face F_i'; then $\sum_{i=1}^{23} S_{F_i'} = 72$. Suppose that $S_{F_1'} \leq S_{F_2'} \leq \cdots \leq S_{F_{23}'}$. Then the following hold:

(1) $S_{F_i'} = 3$, where $1 \leq i \leq 20$;

(2) $S_{F_{21}'} \leq 4$;

(3) $S_{F_{23}'} \leq 6$.

Notice that we need to insert the vertices $\{v_{24}, v_{34}\}$ and the edges into the same face of the presentation of G. Since $deg(v_{24}) = deg(v_{34}) = 6$, there is at least one face F_i' such that $S_{F_i'} \geq 7$, a contradiction. So, we conclude that $\gamma(L(\Gamma(R))) \geq 3$. □

With the help of previous lemma, we are able to find all finite commutative rings R whose $L(\Gamma(R))$ has genus 0, 1, and 2, respectively.

Theorem 2.2. *Let R be a finite commutative ring. Then:*

(1) $\gamma(L(\Gamma(R))) = 0$ *if and only if R is isomorphic to one of the following rings:* $\mathbb{Z}_2,\ \mathbb{Z}_3,\ \mathbb{Z}_4,\ \frac{\mathbb{Z}_2[x]}{(x^2)}, \mathbb{F}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_2$;

(2) $\gamma(L(\Gamma(R))) \neq 1$;

(3) $\gamma(L(\Gamma(R))) = 2$ *if and only if R is isomorphic to one of the following rings:* $\mathbb{Z}_5,\ \mathbb{Z}_2 \times \mathbb{Z}_3$.

Proof. We may write $R \cong R_1 \times R_2 \times \cdots \times R_s$ with R_i a finite local commutative ring. By Lemma 2.1, $\gamma(L(\Gamma(R))) \geq 3$ when $s \geq 3$.

Assume that $s = 2$, that is, $R \cong R_1 \times R_2$. If $|R| \geq 10$, then the degree of the unit 1 is great than 8 and thus there is a complete graph K_9 in $L(\Gamma(R))$. In this case, $\gamma(L(\Gamma(R))) \geq 3$. So $\gamma(L(\Gamma(R))) \leq 2$ implies that $|R| \leq 9$. We need only consider the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times F_4$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Since $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ have only four edges, $L(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))$ have only four vertices, it is apparently planar. If $R = \mathbb{Z}_2 \times \mathbb{F}_4$, then $\Gamma(R)$ contains 8 vertices and 21 edges. So $L(\Gamma(R))$ contains twenty-one vertices. It is not difficult to check that $L(\Gamma(R))$ has 99 edges. Then by Lemma 1.1, $\gamma(L(\Gamma(R))) \geq \lceil \frac{99}{6} - \frac{21}{2} + 1 \rceil = 7$. Since $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$ and $\Gamma(\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)})$ are isomorphic, we only need to consider the case that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, then $\Gamma(R)$ contains 8 vertices and 17 edges. It is easy to check that $L(\Gamma(R))$ contains 17 vertices and 68 edges. According to Lemma 1.1, $\gamma(L(\Gamma(R))) \geq \lceil \frac{68}{6} - \frac{17}{2} + 1 \rceil = 3$. Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $\Gamma(R)$ contains 9 vertices and 30 edges. By direct verification, $L(\Gamma(R))$ contains 30 vertices and 178 edges. By the Lemma 1.1, $\gamma(L(\Gamma(R))) \geq \lceil \frac{178}{6} - \frac{30}{2} + 1 \rceil = 15$. If $R = \mathbb{Z}_2 \times \mathbb{Z}_3$, then $\Gamma(R)$ contains 6 vertices and 11 edges, then $L(\Gamma(R))$ contains 11 vertices and 35 edges, which are given by Lemma 1.1, $\gamma(L(\Gamma(R)) \geq \lceil \frac{35}{6} - \frac{11}{2} + 1 \rceil = 2$. In $L(\Gamma(R))$, let the vertex v_i correspond to the edge e_i in $\Gamma(R)$ (as shown in the figure 1). Then $L(\Gamma(R))$ can be embedded in the topological plane $\mathbb{S}_2$ (as shown in the figure 2), so $\gamma(L(\Gamma(R))) = 2$.

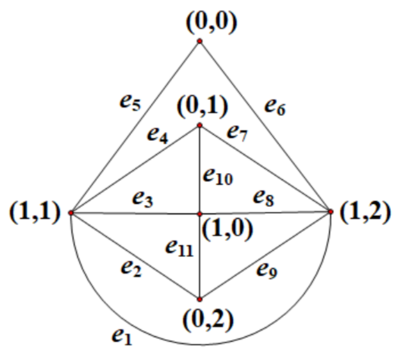

Figure 1. $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3)$

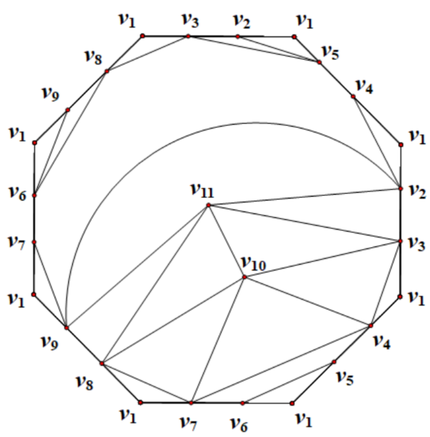

Figure 2. The embedding of $L(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3))$ in $\mathbb{S}_2$

Assume now that $s = 1$. R is a finite commutative local ring. From $\gamma(L(\Gamma(R))) \leq 2$, we have $|R| \leq 9$. Then R may be isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_3$, F_4, $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_5$, $\mathbb{Z}_7$, F_8, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{(x^3)}$, $\frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$, F_9, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{(x^2)}$.

When $R = \mathbb{Z}_2$ or $\mathbb{Z}_3$, the graph $\Gamma(R)$ has at most 4 edges, so $L(\Gamma(R))$ has at most 4 vertices, which is obviously a planar graph. $\Gamma(F_4)$ is a complete graph with 4 vertices, by Lemma 1.3(1), $\gamma(L(\Gamma(F_4))) = \gamma(L(K_4)) = 0$. Since $\Gamma(\mathbb{Z}_4)$ and $\Gamma(\frac{\mathbb{Z}_2[x]}{(x^2)})$ are isomorphic, so we only need to consider $\Gamma(\mathbb{Z}_4)$. By Lemma 1.2, we know $L(\Gamma(\mathbb{Z}_4))$ ($L(\Gamma(\frac{\mathbb{Z}_2[x]}{(x^2)}))$) is planar. Note that $\Gamma(\mathbb{Z}_5)$ is a complete graph with 5 vertices. By Lemma 1.3(1), $\gamma(L(\Gamma(\mathbb{Z}_5))) = \gamma(L(K_5)) > 1$. In Figure 3, we mark the edges are $e_1, e_2, \cdots, e_{10}$. Then in $L(\Gamma(\mathbb{Z}_5))$, we let the vertex v_i be the corresponding point of the edge e_i in $\Gamma(\mathbb{Z}_5)$. Figure 4 shows that $L(\Gamma(\mathbb{Z}_5))$ can be embedded in the surface $\mathbb{S}_2$. So, $\gamma(L(\Gamma(\mathbb{Z}_5))) = 2$.

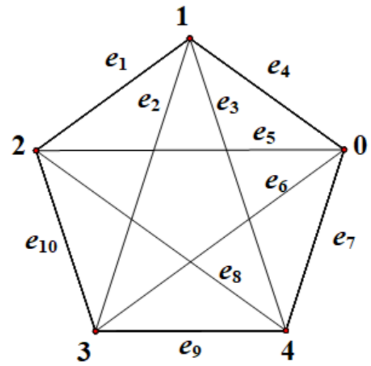

Figure 3. $\Gamma(\mathbb{Z}_5)$

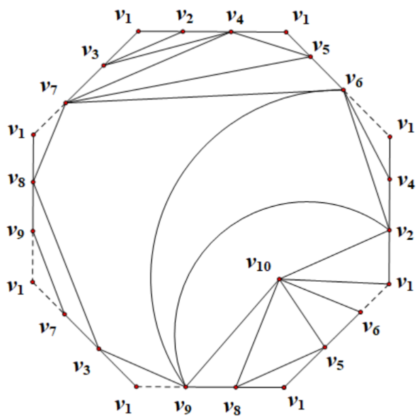

Figure 4. The embedding of $L(\Gamma(\mathbb{Z}_5))$ in $\mathbb{S}_2$

$\Gamma(\mathbb{Z}_7)$ is a complete graph with 7 vertices. According to Lemma 1.3(1), $\gamma(L(\Gamma(\mathbb{Z}_7))) = \gamma(L(K_7)) = \lceil \frac{(7+1)(7-3)(7-4)}{12} \rceil = 8$. $\Gamma(F_8)$ is a complete graph with 8 vertices. According to Lemma 1.3(1), $\gamma(L(\Gamma(F_8))) = \gamma(L(K_8)) > \lceil \frac{(8+1)(8-3)(8-4)}{12} \rceil = 15$. Since the comaximal graphs of $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{(x)^3}$, $\frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$ are isomorphic, we only need to consider the case of $R = \mathbb{Z}_8$. $\Gamma(\mathbb{Z}_8)$ contains 8 vertices and 22 edges, by direct computation, $L(\Gamma(\mathbb{Z}_8))$ contains 22 vertices and 108 edges. By Lemma 1.1, $\gamma(L(\Gamma(\mathbb{Z}_8))) \geq \lceil \frac{108}{6} - \frac{22}{2} + 1 \rceil = 8$. Now, $\Gamma(F_9)$ is a complete graph with 9 vertices, by Lemma 1.3(1), $\gamma(L(\Gamma(F_9))) = \gamma(L(K_9)) >$

$\lceil \frac{(9+1)(9-3)(9-4)}{12} \rceil = 25$. Since $\Gamma(\frac{\mathbb{Z}_3[x]}{(x)^2})$ is isomorphic to $\Gamma(\mathbb{Z}_9)$, we only need to consider the case of $R = \mathbb{Z}_9$. $\Gamma(\mathbb{Z}_9)$ contains 9 vertices and 33 edges, and $L(\Gamma(\mathbb{Z}_9))$ contains 33 vertices and 113 edges. By Lemma 1.1, we have $\gamma(L(\Gamma(\mathbb{Z}_9))) \geq \lceil \frac{113}{6} - \frac{33}{2} + 1 \rceil = 4$.

All cases are considered and determined, so our proof is complete. □

3. The genus of $L(\Gamma_2(R))$

As we can see, a unit is adjacent to all vertices in $\Gamma(R)$, next, we consider a subgraph of $\Gamma(R)$. Let $\Gamma_2(R)$ be a subgraph of $\Gamma(R)$ induced by $R\backslash\{U(R) \cup J(R)\}$, where $U(R)$ is the unit group of R and $J(R)$ is the Jacobson radical of R. In this section, we investigate the genus of $L(\Gamma_2(R))$. Our aim is to find all finite commutative rings whose genus of $L(\Gamma_2(R))$ is 0,1 and 2, respectively. Before giving the main conclusion of this section, we give a few lemmas.

Lemma 3.1. *Let $R \cong R_1 \times R_2 \times \cdots \times R_s$. If $s \geq 4$, then $\gamma(L(\Gamma_2(R))) \geq 3$.*

Proof. Assume that $s = 4$. Let $a_1 = (1,1,1,0)$, $a_2 = (1,1,0,1)$, $a_3 = (1,0,1,1)$, $a_4 = (0,1,1,1)$, $b_1 = (1,1,0,0)$, $b_2 = (1,0,1,0)$, $b_3 = (1,0,0,1)$, $b_4 = (0,1,1,0)$, $b_5 = (0,1,0,1)$, $b_6 = (0,0,1,1)$. Let G be a subgraph of $\Gamma_2(R)$ induced by $S = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, b_5, b_6\}$. Then $E(G) = \{a_ia_j | i \neq j\} \cup \{a_1b_i | i = 3,5,6\} \cup \{a_2b_i | i = 2,4,6\} \cup \{a_3b_i | i = 1,4,5\} \cup \{a_4b_i | i = 123\} \cup \{b_1b_6,\ b_2b_5,\ b_3b_4\}$, we have $|V(G)| = 10$, $|E(G)| = 21$. Thus $|V(L(G))| = 21$, $|E(L(G))| = 78$. By Lemma 1.1, we have $\gamma(L(G)) \geq \lceil \frac{78}{6} - \frac{21}{2} + 1 \rceil = 4$. So $\gamma(L(\Gamma_2(R))) \geq \gamma(L(G)) \geq 4$.

When $s \geq 5$, by above discussion, we know that $\Gamma_2(R)$ has a subgraph isomorphic to G, which is induced by $S = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, b_5, b_6\}$. So $\gamma(L(\Gamma_2(R))) \geq \gamma(L(G)) \geq 4$. □

Lemma 3.2. *Let $R \cong R_1 \times R_2 \times R_3$.*

(1) *If $|U(R_i)| \geq 3$(for some $1 \leq i \leq 3$), then $\gamma(L(\Gamma_2(R))) \geq 3$;*

(2) *If $|U(R_i)| \geq 2, |U(R_j)| \geq 2$(for some $1 \leq i \neq j \leq 3$), then $\gamma(L(\Gamma_2(R))) \geq 3$.*

Proof. (1) We may suppose that $|U(R_3)| \geq 3$ and $1, u, v \in U(R_3)$. Let $a_1 = (1,1,0)$, $a_2 = (1,0,1)$, $a_3 = (1,0,u)$, $a_4 = (1,0,v)$, $a_5 = (0,1,1)$, $a_6 = (0,1,u)$, $a_7 = (0,1,v)$, $b_1 = (1,0,0)$, $b_2 = (0,1,0)$. Let G be a subgraph of $\Gamma_2(R)$ induced by $\{a_1, a_2, \cdots, a_7, b_1, b_2\}$. Then $E(G) = \{a_ia_j | i \neq j\} \cup \{a_ib_1 | i = 5,6,7\} \cup \{a_ib_2 | i = 2,3,4\}$. Then $|V(G)| = 9$, $|E(G)| = 21$. Thus $|V(L(G))| = 21$, $|E(L(G))| = 75$. By Lemma 1.1, we have $\gamma(L(G)) \geq \lceil \frac{75}{6} - \frac{21}{2} + 1 \rceil = 3$. So $\gamma(L(\Gamma_2(R))) \geq \gamma(L(G)) \geq 3$.

(2) We may assume that $|U(R_2)| \geq 2$, $|U(R_3)| \geq 2$, and $1, u \in U(R_2)$, $1, v \in U(R_3)$. Let $a_1 = (1,1,0)$, $a_2 = (1,u,o)$, $a_3 = (1,0,1)$, $a_4 = (1,0,v)$, $b_1 = (0,1,1)$, $b_2 = (0,1,v)$, $b_3 = (0,u,1)$, $b_4 = (0,u,v)$. Let G be a subgraph of $\Gamma_2(R)$ induced by $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$. Then $E(G) = \{a_ia_j | i \neq j\} \cup \{a_ib_j\}$, there are $|V(G)| = 8$, $|E(G)| = 20$. Thus $|V(L(G))| = 20$, $|E(L(G))| = 84$. By Lemma 1.1, we have $\gamma(L(G)) \geq \lceil \frac{84}{6} - \frac{20}{2} + 1 \rceil = 5$. Therefore $\gamma(L(\Gamma_2(R))) \geq \gamma(L(G)) \geq 5$. □

Lemma 3.3. *Let $R \cong R_1 \times R_2$.*

(1) *If $|U(R_i)| \geq 9$(for some $1 \leq i \leq 2$), then $\gamma(L(\Gamma_2(R))) \geq 3$;*

(2) *If $|U(R_1)| \geq 2, |U(R_2)| \geq 7$, then $\gamma(L(\Gamma_2(R))) \geq 3$;*

(3) *If $|U(R_1)| \geq 3, |U(R_2)| \geq 6$, then $\gamma(L(\Gamma_2(R))) \geq 3$;*

(4) *If $|U(R_1)| \geq 4, |U(R_2)| \geq 5$, then $\gamma(L(\Gamma_2(R))) \geq 3$.*

Proof. (1) We may suppose that $|U(R_2)| \geq 9$, and $u_1, u_2, \cdots, u_9 \in U(R_2)$. Let $U = \{(1,0)\}$, $V = \{(0,u_i) | 1 \leq i \leq 9\}$. Then each element in U is adjacent to every element in V. Since $|U| = 1$ and $|V| = 9$, so $K_{1,9}$ is a subgraph of $\Gamma_2(R)$. From Lemma 1.3(2), we have $\gamma(L(\Gamma_2(R))) \geq \gamma(L(K_{1,9})) = 3$.

(2) Let $u_1, u_2 \in U(R_1)$ and $v_1, v_2, \cdots, v_7 \in U(R_2)$. Let $U = \{(u_1,0),\ (u_2,0)\}$, $V = \{(0,v_i) | 1 \leq i \leq 7\}$. Then every element in U is adjacent to every element in V. Since

$|U| = 2$ and $|V| = 7$, we know that $K_{2,7}$ is a subgraph of $\Gamma_2(R)$. By Lemma 1.3(3), we have $\gamma(L(\Gamma_2(R))) \geq \gamma(L(K_{2,7})) \geq 4$.

(3) Let $u_1, u_2, u_3 \in U(R_1)$, $v_1, v_2, \cdots, v_6 \in U(R_2)$. Let $U = \{(u_1, 0),\ (u_2, 0),\ (u_3, 0)\}$, $V = \{(0, v_i) | 1 \leq i \leq 6\}$. Then every element in U is adjacent to every element in V. Since $|U| = 3$ and $|V| = 6$, we have that $H = K_{3,6}$ is a subgraph of $\Gamma_2(R)$. Thus we have $|V(H)| = 9$, $|E(H)| = 18$. So, $|V(L(H))| = 18$, $|E(L(H))| = 63$, and by Lemma 1.1, $\gamma(L(H)) \geq \lceil \frac{63}{6} - \frac{18}{2} + 1 \rceil = 3$. Therefore $\gamma(L(\Gamma_2(R))) \geq \gamma(L(H)) \geq 3$.

(4) Let $u_1, u_2, u_3, u_4 \in U(R_1)$, $v_1, v_2, \cdots, v_5 \in U(R_2)$. Let $U = \{(u_i, 0) | 1 \leq i \leq 4\}$, $V = \{(0, v_i) | 1 \leq i \leq 5\}$. Then every element in U is connected to every element in V, and $|U| = 4$, $|V| = 5$, so $H = K_{4,5}$ is a subgraph of $\Gamma_2(R)$. Thus we have $|V(H)| = 9$, $|E(H)| = 20$. There are $|V(L(H))| = 20$, $|E(L(H))| = 70$, by the Lemma 1.1, $\gamma(L(H)) \geq \lceil \frac{70}{6} - \frac{20}{2} + 1 \rceil = 4$. So $\gamma(L(\Gamma_2(R))) \geq \gamma(L(H)) \geq 4$. □

The above lemmas give some sufficient conditions for $\gamma(L(\Gamma_2(R))) \geq 3$. Next, we give the main result in this section for classifying finite commutative rings R whose $\gamma(L(\Gamma_2(R)))$ is 0, 1, and 2, respectively.

Theorem 3.4. *Let R be a finite commutative ring and $\Gamma_2(R)$ be not an empty graph. Then:*

(1) $\gamma(L(\Gamma_2(R))) = 0$ *if and only if R is isomorphic to one of the following rings:* $\mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)},\ \mathbb{Z}_2 \times \mathbb{F}_4,\ \mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_3 \times \mathbb{F}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$;

(2) $\gamma(L(\Gamma_2(R))) = 1$ *if and only if R is isomorphic to one of the following rings:* $\mathbb{Z}_2 \times \mathbb{Z}_7,\ \mathbb{Z}_2 \times \mathbb{F}_8,\ \mathbb{Z}_3 \times \mathbb{Z}_4,\ \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)},\ \mathbb{Z}_3 \times \mathbb{Z}_5,\ \mathbb{F}_4 \times \mathbb{F}_4$;

(3) $\gamma(L(\Gamma_2(R))) = 2$ *if and only if R is isomorphic to one of the following rings:* $\mathbb{Z}_3 \times \mathbb{Z}_7,\ \mathbb{Z}_4 \times \mathbb{F}_4,\ \frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{F}_4,\ \mathbb{F}_4 \times \mathbb{Z}_5$.

Proof. Let $R \cong R_1 \times R_2 \times \cdots \times R_s$. By Lemma 3.1, we have $\gamma(L(\Gamma_2(R))) \geq 3$ when $s \geq 4$. As $\Gamma_2(R)$ is an empty graph if R is not a finite commutative local ring. So, we just consider the cases that $s = 2$ and $s = 3$.

Assume now that $s = 3$. According to the Lemma 3.2, R may be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x)^2}$. In $L(\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))$, let the vertex v_i corresponds to the edge e_i in $\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ (as shown in the figure 5). So $L(\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))$ can be drawn on a plane (as shown in the figure 6). Therefore $\gamma(L(\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))) = 0$.

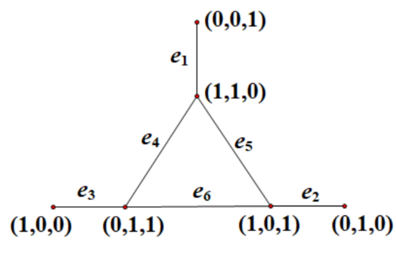

Figure 5. $\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. We have $|V(\Gamma_2(R))| = 9$. Let $a = (1, 1, 0)$, $b_1 = (1, 0, 1)$, $b_2 = (1, 0, 2)$, $c_1 = (0, 1, 1)$, $c_2 = (0, 1, 2)$, $d_1 = (0, 0, 1)$, $d_2 = (0, 0, 2)$, $d_3 = (0, 1, 0)$, $d_4 = (1, 0, 0)$. Then $E(\Gamma_2(R)) = \{ab_i, ac_j, b_ic_j\} \cup \{ad_1, ad_2, b_1d_3, b_2d_3, c_1d_4, c_2d_4\}$, i.e. $|E(\Gamma_2(R))| = 16$. Thus $|V(L(\Gamma_2(R)))| = 16$. It is easy to check that $|E(L(\Gamma_2(R)))| = 57$. By Lemma 1.1, we get $\gamma(L(\Gamma_2(R))) \geq \lceil \frac{57}{6} - \frac{16}{2} + 1 \rceil = 3$. When $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, let $a_1 = (1, 1, 0)$, $a_2 = (1, 1, 2)$, $b_1 = (1, 0, 1)$, $b_2 = (1, 0, 3)$, $c_1 = (0, 1, 1)$, $c_2 = (0, 1, 3)$, $d_1 =$

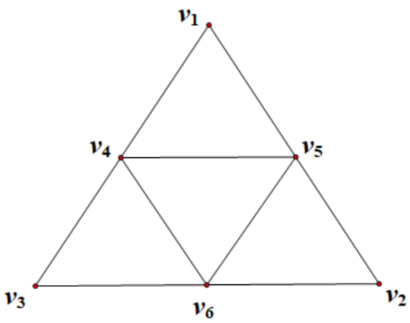

Figure 6. $L(\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))$

$(0,0,1)$, $d_2 = (0,0,3)$, $d_3 = (0,1,0)$, $d_4 = (1,0,0)$. Let G be a subgraph of $\Gamma_2(R)$ induced by $\{a_ib_j, c_kd_l\}$. Then $E(G) = \{a_ib_j, a_ic_k, b_jc_k\} \cup \{a_id_j | j = 1,2\} \cup \{b_jd_3, c_kd_4\}$. Thus $|V(L(G))| = 20$, $|E(L(G))| = 74$, by Lemma 1.1, we have $\gamma(L(G)) \geq \lceil \frac{74}{6} - \frac{20}{2} + 1 \rceil = 4$. So $\gamma(L(\Gamma_2(R))) \geq \gamma(L(G)) \geq 4$.

Assume now that $s = 2$. According to the Lemma 3.3, R may be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times S_1$, $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_7$, $\mathbb{Z}_2 \times S_2$, $\mathbb{Z}_2 \times \mathbb{F}_8$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times S_1$, $\mathbb{Z}_3 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_7$, $\mathbb{Z}_3 \times S_2$, $S_1 \times S_1$, $S_1 \times \mathbb{F}_4$, $S_1 \times \mathbb{Z}_5$, $S_1 \times \mathbb{Z}_7$, $S_1 \times S_2$, $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_5 \times \mathbb{Z}_5$, where S_1 is a commutative local ring of order 4 but not a field, and S_2 is a commutative local ring of order 8 but not a field.

If $R \cong R_1 \times R_2$, where R_1, R_2 are finite commutative local rings. Denote M_i the unique maximal ideal of R_i $(i = 1,2)$. Let $U = U(R_1) \times M_2$ and $V = M_1 \times U(R_2)$. Then every element in U is adjacent to every element in V in $\Gamma_2(R)$. If $|U| = m$ and $|V| = n$, then $\Gamma_2(R) = K_{m,n}$.

So, we have $\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2) = K_2$; $\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_3) = K_{1,2}$; $\Gamma_2(\mathbb{Z}_2 \times \mathbb{F}_4) = K_{1,3}$; $\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_5) = K_{1,4}$; $\Gamma_2(\mathbb{Z}_2 \times S_1) = \Gamma_2(\mathbb{Z}_3 \times \mathbb{Z}_3) = K_{2,2}$; $\Gamma_2(\mathbb{Z}_3 \times \mathbb{F}_4) = K_{2,3}$. By Lemma 1.3(2)(3), we know the line graphs of these graphs are all planar.

We know that $\Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_7) = K_{1,6}$; $\Gamma_2(\mathbb{Z}_2 \times \mathbb{F}_8) = K_{1,7}$; $\Gamma_2(\mathbb{Z}_3 \times S_1) = \Gamma_2(\mathbb{Z}_3 \times \mathbb{Z}_5) = K_{2,4}$; $\Gamma_2(\mathbb{F}_4 \times \mathbb{F}_4) = K_{3,3}$. By Lemma 1.3(2)(3)(4), we know that their line graphs all have genus one.

We know that $\Gamma_2(\mathbb{Z}_3 \times \mathbb{Z}_7) = \Gamma_2(S_1 \times \mathbb{F}_4) = K_{2,6}$; $\Gamma_2(\mathbb{F}_4 \times \mathbb{Z}_5) = K_{3,4}$. By Lemma 1.3(3)(4), we know their line graphs have genus 2.

Note that $\Gamma_2(S_1 \times \mathbb{Z}_5) = K_{2,8}$. Then by Lemma 1.3(3), we have $\gamma(L(\Gamma_2(S_1 \times \mathbb{Z}_5))) = 5$. $\Gamma_2(S_1 \times \mathbb{Z}_7) = K_{2,12}$. By Lemma 1.3(3), we have $\gamma(L(\Gamma_2(S_1 \times \mathbb{Z}_7))) \geq 15$.

If $R \cong \mathbb{Z}_2 \times S_2$ or $S_1 \times S_1$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$, then $\Gamma_2(R) = K_{4,4}$. By Lemma 1.3(4), we have $\gamma(L(\Gamma_2(R))) = \gamma(L(K_{4,4})) \geq 3$.

If $R \cong \mathbb{Z}_3 \times S_2$, then $\Gamma_2(R) = K_{4,8}$. By Lemma 1.3(4), we have $\gamma(L(\Gamma_2(R))) = \gamma(L(K_{4,8})) \geq \gamma(L(K_{4,4})) \geq 3$. If $R \cong S_1 \times S_2$, then $\Gamma_2(R) = K_{8,8}$. By Lemma 1.3(4), we have $\gamma(L(\Gamma_2(R))) = \gamma(L(K_{8,8})) \geq \gamma(L(K_{4,4})) \geq 3$. □

Author Contributions

All authors contributed to the study conception and design. The first version of the manuscript was written by H. Su and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Data Availability

The authors have not used any data for the preparation of this manuscript.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.
Ethical Approval This paper does not contain any studies with human participants or animals performed by any of the authors.

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