International Electronic Journal of Geometry
Volume 16 No. 2 Page 697–706 (2023)
DOI: https://doi.org/10.36890/iejg.1258996

On Four Dimensional Hermitian Manifolds

Beldjilali Gherici*

(Communicated by Uday Chand De)

Abstract

The present paper is devoted to 4-dimentional Hermitain manifold. We give a new necessary and sufficient condition of integrability and we introduce a new class of locally conformal Kähler manifolds that we consider a twin of the Vaisman ones. Then, some basic properties of this class is discussed, also the existence of such manifolds is shown with concrete examples.

Keywords: *Almost Hermitian manifolds, almost contact metric manifolds, locally conformal Kähler manifolds.*

AMS Subject Classification (2020): *Primary 53C55; Secondary 53C25.*

1. Introduction

The study of complex manifolds has attracted the attention of many authors interested in different fields in mathematics and physics since complex structures have proved to be an important tool in the description and geometrization of several phenomena.

Because of their nice topological properties Kähler manifolds have been studied much more extensively than other kinds of almost Hermitian manifolds. In the study of non-Kähler almost Hermitian manifolds it is natural to consider those whose almost complex structure satisfies similar but weaker conditions than those of Kahler manifolds.

Although known since 1954 from P. Libermann's paper [4], locally conformal Kähler (l.c.K.) structures have been intensively studied only since 1976 after the impetus given by I. Vaisman in [9]. A great number of research papers has appeared since then studying the main properties of l.c.K. manifolds, generalized Hopf (g.H.) manifolds, the relations with contact metric manifolds and some important classifications of submanifolds in g.H. manifolds. In 1998, the monograph by S. Dragomir and L. Ornea [1] brought together all known results in this field at that moment. After the book, the geometers continued to study l.c.K. manifolds and many other interesting results have appeared so far.

A Vaisman manifold is a particular l.c.K. manifold (M, J, g) with its fundamental form $\Omega(.,.) := g(., J.)$ satisfying the relation $\mathrm{d}\Omega = \omega \wedge \Omega$, for a nonzero one-form ω, which is parallel with respect to the Levi-Civita connection of the metric g. The one-form ω is called the Lee form. Note that many of the known l.c.K manifolds are in fact Vaisman. Recently, in [8], the authors studied a new class of locally conformal Kähler manifolds which will generalize the Vaisman manifold.

The present paper is devoted to Hermitian structure on a 4-dimensional manifold. In Section 2, we review basic definitions and results that are needed to state and prove our results. In Section 3, we derive certain necessary and sufficient conditions for almost Hermitian structure on M to be integrable and we point out some of their consequences. In Section 4, we establish an interesting class of locally conformal Kähler manifolds which is a twin of the Vaisman manifolds, called by us $\mathcal{B}$-Hermitian manifolds. We build concrete examples and we show that there is a one-to-one correspondence between $\mathcal{B}$-Hermitian and Kenmotsu structures.

2. Review definitions and needed results

Throughout the paper, the Lie algebra of all C^∞ vector fields on M will be denoted by $\mathfrak{X}(M)$.

Received : *02–03–2023,* ***Accepted :*** *07–09–2023*
*** Corresponding author**

2.1. Almost Hermitian manifolds

An almost complex manifold M is a differentiable manifold equipped with a $(1,1)$ tensor field J which satisfies $J^2 = -I$, where I is the identity. Such a manifold is even-dimensional. M^{2n} is an almost Hermitian manifold provided it is almost complex and has a Riemannian metric g for which

$$g(JX, JY) = g(X, Y)$$

for all $X, Y \in \mathfrak{X}(M^{2n})$ where C^∞ denotes the Lie algebra of all vector fields on M. To describe the geometry of an almost Hermitian manifold M^{2n}, it is useful to consider two special tensors. The first, called the Nijenhuis tensor, is a $(1,2)$ tensor field N_J defined by

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]. \tag{2.1}$$

An almost complex structure J is integrable if its Nijenhuis tensor N_J vanishes. A well known theorem that J is integrable if and only if (see [5])

$$(\nabla_{JX} J)JY = (\nabla_X J)Y, \tag{2.2}$$

where ∇ denote the Levi-Civita connection corresponding to g. In the case, the almost Hermitian manifold M^{2n} is a Hermitian manifold.
The second tensor is a 2-form Ω, called the Kähler form, and it is defined by

$$\Omega(X, Y) = g(X, JY)$$

for all $X, Y \in \mathfrak{X}(M^{2n})$.

In the classification of Gray-Hervella [2] of almost Hermitian manifolds an interesting class that arises in the scheme is the class corresponding to W_4. This is the class of almost Hermitian manifolds (M^{2n}, J, g) satisfying the identity

$$\begin{aligned} 2(n-1)(\nabla_X \Omega)(Y, Z) &= g(X, Z)\delta\Omega(Y) - g(X, Y)\delta\Omega(Z) \\ &\quad + g(X, JZ)\delta\Omega(JY) - g(X, JY)\delta\Omega(JZ), \end{aligned} \tag{2.3}$$

where $\delta\Omega$ denotes the codifferential of the form Ω. For an adapted local frame given by $\{e_i\}_{1\leq i\leq 4}$, $\delta\Omega$ is given by

$$\begin{aligned} \delta\Omega(X) &= \big(\nabla_{e_i}\Omega\big)(e_i, X) \\ &= -g\big((\nabla_{e_i} J)e_i, X\big). \end{aligned} \tag{2.4}$$

An important fact about the class W_4 is noteworthy, any manifold in W_4 automatically has an integrable almost complex structure.

Definition 2.1. [1, 7] A Hermitian manifold (M, J, g) is called locally conformal Kähler (shortly, l.c.K) manifold if there exists a closed one-form ω (called the Lee form) such that:

$$\mathrm{d}\Omega = \omega \wedge \Omega.$$

It is well known that a locally conformal Kähler manifold belong to the class W_4. In [1], the $2n$-dimensional l.c.K manifolds are characterized by:

$$2(\nabla_X J)Y = \theta(Y)X - \omega(Y)JX - g(X, Y)A - \Omega(X, Y)B, \tag{2.5}$$

where $\theta = \omega \circ J$ and $A = -JB$ are respectively the anti-Lee form and the anti-Lee vector field.

The most important subclass of l.c.K manifolds is defined by the parallelism of the Lee form with respect to the Levi-Civita connection of g

Definition 2.2. [9] An l.c.K manifold (M, J, ω, g) is called a Vaisman manifold if $\nabla\omega = 0$.

Also, the almost Hermitian manifold M^{2n} is an almost Kähler manifold if Ω is closed, i.e., $\mathrm{d}\Omega = 0$. If both $\mathrm{d}\Omega = 0$ and $N_J = 0$ are satisfied, then M^{2n} is called a Kähler manifold. Recall that $\mathrm{d}\Omega = 0$ and $N_J = 0$ are equivalent to

$$\nabla J = 0.$$

Torse forming vector fields were introduced by K. Yano [11] satisfies

$$\nabla_X \xi = aX + \eta(X)\xi, \tag{2.6}$$

for some smooth function a and 1-form η on M. The 1-form η is called the generating form and the function a is called the conformal scalar.

Further, a complex analogue of a torse forming vector field is called K-torse forming vector field and it was introduced by S.Yamaguchi and W. N. Yu [10],

$$\nabla_X \xi = aX + bJX + \eta_1(X)\xi + \eta_2(X)J\xi, \tag{2.7}$$

where a and b are functions, η_1 and η_2 are 1-forms on M. The functions a and b (resp. 1-forms η_1 and η_2) appearing in (2.7) will be called the associated functions (resp. forms) of ξ. Moreover if the associated functions a and b satisfy $a^2 + b^2 \neq 0$ in M, then we call such a vector field a proper K-torse-forming vector field. For the existence of torse-forming vector field on Riemannian manifold see for example [10].

Finally, we recall the Koszul's formula for the metric g which is used to calculate the components of the Levi-Civita connection ∇

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ &\quad -g\big(X,[Y,Z]\big) + g\big(Y,[Z,X]\big) + g\big(Z,[X,Y]\big). \end{aligned} \tag{2.8}$$

For more background on almost complex manifolds, we refer to [1, 12].

2.2. Almost contact metric manifolds

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a $(1,1)$ tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\eta(\xi) = 1, \quad \varphi^2(X) = -X + \eta(X)\xi \quad \textit{and} \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.9}$$

for any vector fields X,Y on M. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$.

Such a manifold is said to be a contact metric manifold if $\mathrm{d}\eta = \phi$, where $\phi(X,Y) = g(X, \varphi Y)$ is called the fundamental 2-form of M.
On the other hand, the almost contact metric structure of M is said to be normal if

$$N^{(1)}(X,Y) = N_\varphi(X,Y) + 2\mathrm{d}\eta\,(X,Y)\xi = 0, \tag{2.10}$$

for any $X, Y \in \mathfrak{X}(M)$, where d denotes the exterior derivative and N_φ denotes the Nijenhuis torsion of φ, given by

$$[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

A 3-dimensional almost contact metric structure (φ, ξ, η, g) on M is said to be Kenmotsu structure [3], if and only if

$$\mathrm{d}\eta = 0, \quad \mathrm{d}\phi = 2\phi \wedge \eta \quad \textit{and} \quad N^{(1)} = 0. \tag{2.11}$$

This manifold can be characterized through their Levi-Civita connection, by requiring

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X. \tag{2.12}$$

Moreover, in [6] the 3-dimensional Kenmotsu manifolds is also characterized by

$$\nabla_X \xi = -\varphi^2 X. \tag{2.13}$$

3. 4-dimensional almost Hermitian manifolds

Through the rest of this paper M always denotes the 4-dimensional differentiable manifolds. Here, we have

Proposition 3.1. *[2, 6] A 4-dimensional almost Hermitian manifold M is Hermitian if and only if the covariant derivative of Ω is of the form*

$$\begin{aligned} 2(\nabla_X\Omega)(Y,Z) = & g(X,Z)\delta\Omega(Y) - g(X,Y)\delta\Omega(Z) \\ & + g(X,JZ)\delta\Omega(JY) - g(X,JY)\delta\Omega(JZ). \end{aligned} \tag{3.1}$$

Let ω be the Lee form of M defined by $\omega = \delta\Omega \circ J$ and B the contravariant field of ω called the Lee field. Then, the identity (3.1) can be expressed equivalently in the following:

$$2(\nabla_X J)Y = g(X,Y)JB - g(X,JY)B - \omega(Y)JX + \omega(JY)X. \tag{3.2}$$

Remark 3.1. By comparing the two equations (2.5) and (3.2), one can check that any 4-dimensional Hermitian manifold with $\mathrm{d}\omega = 0$ is an l.c.K manifold.

From (3.2), we find $B = -J\sum_{i=1}^{4}(\nabla_{e_i}J)e_i$ and this Lee field is not necessary unitary. If $\omega \neq 0$ everywhere, we put $\xi_1 = \mathrm{e}^{-\sigma}B$ where $\mathrm{e}^{\sigma} = |B|$ and $\xi_2 = -J\xi_1$ we get immediately that $\eta_1 = \mathrm{e}^{-\sigma}\omega$ and $\eta_2 = \mathrm{e}^{\sigma}\omega \circ J$ such that

$$\eta_i(X) = g(\xi_i, X), \quad \textit{and} \quad \eta_i(\xi_j) = \delta_{ij} \quad \textit{for all} \quad i,j \in \{1,2\}.$$

Then, the formula 3.2 becomes

$$2\mathrm{e}^{-\sigma}(\nabla_X J)Y = g(JX,Y)\xi_1 - \eta_1(Y)JX - g(X,Y)\xi_2 + \eta_2(Y)X, \tag{3.3}$$

For a 4-dimensional manifold M with an almost Hermitian structure (J,g) we can also construct a useful local orthonormal basis. Let $\mathcal{U}$ be a coordinate neighborhood on M and e any unit vector field on $\mathcal{U}$ orthogonal to ξ_1 and ξ_2. Then Je is a unit vector field orthogonal to both ξ_1, ξ_2 and e. Then, we obtain a local orthonormal basis $\{\xi_1, \xi_2, e, Je\}$, called a J-basis. Let's start with the following main result:

Theorem 3.1. *For an almost Hermitian structure (J,g) on M, we have*

$$\begin{aligned} (\nabla_X J)Y = & g\big(J\nabla_X\xi_1 + \nabla_X\xi_2, Y\big)\xi_1 - \eta_1(Y)\big(J\nabla_X\xi_1 + \nabla_X\xi_2\big) \\ & + g\big(J\nabla_X\xi_2 - \nabla_X\xi_1, Y\big)\xi_2 - \eta_2(Y)\big(J\nabla_X\xi_2 - \nabla_X\xi_1\big). \end{aligned} \tag{3.4}$$

Proof. Since $\eta_1 \wedge \eta_2 \wedge \Omega$ is up to a constant factor the volume element on M, it is parallel with respect to ∇, i.e.,

$$\nabla_X(\eta_1 \wedge \eta_2 \wedge \Omega) = 0.$$

Knowing that

$$\begin{aligned} 4(\eta_1 \wedge \eta_2 \wedge \Omega)(X,Y,U,V) = & (\eta_1 \wedge \eta_2)(X,Y)\Omega(U,V) \\ & - (\eta_1 \wedge \eta_2)(X,U)\Omega(Y,V) \\ & + (\eta_1 \wedge \eta_2)(X,V)\Omega(Y,U) \\ & + (\eta_1 \wedge \eta_2)(Y,U)\Omega(X,V) \\ & - (\eta_1 \wedge \eta_2)(Y,V)\Omega(X,U) \\ & + (\eta_1 \wedge \eta_2)(U,V)\Omega(X,Y), \end{aligned}$$

and also,

$$3(\mathrm{d}\Omega)(X,Y,Z) = (\nabla_X\Omega)(Y,Z) + (\nabla_Y\Omega)(Z,X) + (\nabla_Z\Omega)(X,Y),$$

then, the equation

$$\big(\nabla_X(\eta_1 \wedge \eta_2 \wedge \Omega)\big)(X,Y,\xi_1,\xi_2) = 0,$$

gives

$$\begin{aligned} (\nabla_Z\Omega)(X,Y) = & \eta_1(X)\big(\Omega(Y,\nabla_Z\xi_1) + (\nabla_Z\eta_2)Y\big) \\ & - \eta_1(Y)\big(\Omega(X,\nabla_Z\xi_1) + (\nabla_Z\eta_2)X\big) \\ & + \eta_2(X)\big(\Omega(Y,\nabla_Z\xi_2) - (\nabla_Z\eta_1)Y\big) \\ & - \eta_2(Y)\big(\Omega(X,\nabla_Z\xi_2) - (\nabla_Z\eta_1)X\big), \end{aligned} \tag{3.5}$$

hence

$$(\nabla_Z J)Y = g\big(J\nabla_Z\xi_1 + \nabla_Z\xi_2, Y\big)\xi_1 - \eta_1(Y)\big(J\nabla_Z\xi_1 + \nabla_Z\xi_2\big) + g\big(J\nabla_Z\xi_2 - \nabla_Z\xi_1, Y\big)\xi_2 - \eta_2(Y)\big(J\nabla_Z\xi_2 - \nabla_Z\xi_1\big),$$

which leads to (3.4). □

Now, we shall introduce a new necessary and sufficient condition of integrability for 4-dimensional almost Hermitian structures.

Theorem 3.2. *A 4-dimensional almost Hermitian manifold M is Hermitian if and only if*

$$(\nabla_X J)\xi_1 = \frac{e^\sigma}{2}\big(-JX - \eta_1(X)\xi_2 + \eta_2(X)\xi_1\big), \tag{3.6}$$

or equivalently,

$$(\nabla_X J)\xi_2 = \frac{e^\sigma}{2}\big(X - \eta_1(X)\xi_1 + \eta_2(X)\xi_2\big). \tag{3.7}$$

Proof. Let (M, J, g) be an almost Hermitian manifold. Firstly, we have

$$\begin{aligned}(\nabla_X J)\xi_2 &= (\nabla_X J)J\xi_1\\ &= -\nabla_X\xi_1 - J\big((\nabla_X J)\xi_1 + J\nabla_X\xi_1\big)\\ &= -J(\nabla_X J)\xi_1,\end{aligned}$$

which demonstrates the equivalence between (3.6) and (3.7). Now, suppose that

$$(\nabla_X J)\xi_1 = \frac{e^\sigma}{2}\big(-JX - \eta_1(X)\xi_2 + \eta_2(X)\xi_1\big).$$

for all X vector field on M. Using $(\nabla_X J)\xi_1 = -\nabla_X\xi_2 - J\nabla_X\xi_1$ we get

$$J\nabla_X\xi_1 + \nabla_X\xi_2 = \frac{e^\sigma}{2}\big(JX + \eta_1(X)\xi_2 - \eta_2(X)\xi_1\big), \tag{3.8}$$

and by applying J we obtain

$$J\nabla_X\xi_2 - \nabla_X\xi_1 = \frac{e^\sigma}{2}\big(-X + \eta_1(X)\xi_1 + \eta_2(X)\xi_2\big). \tag{3.9}$$

Substituting (3.8) and (3.9) in (3.4) we obtain (3.3), then the structure is Hermitian.

Conversely, assuming that (M, J, g) is a Hermitian manifold, this is equivalent to

$$2e^{-\sigma}(\nabla_X J)Y = g(JX, Y)\xi_1 - \eta_1(Y)JX - g(X, Y)\xi_2 + \eta_2(Y)X.$$

Setting $Y = \xi_1$ gives

$$2e^{-\sigma}(\nabla_X J)\xi_1 = \eta_2(X)\xi_1 - JX - \eta_1(X)\xi_2,$$

and hence

$$(\nabla_X J)\xi_1 = \frac{e^\sigma}{2}\big(-JX - \eta_1(X)\xi_2 + \eta_2(X)\xi_1\big).$$

This completes the proof. □

It is convenient to rewrite equation (3.6) in term of ∇B as follows

Corollary 3.1. *A 4-dimensional almost Hermitian manifold M is Hermitian if and only if*

$$2(\nabla_X J)B = -|B|^2 JX + \omega(X)JB + \omega(JX)B. \tag{3.10}$$

Proof. The proof is direct just using the notation given above in formula (3.6). □

By examining formula 3.10 we can extract two important cases:
1) For $\nabla_X B = 0$ we get the Vaisman state.
3) For $\nabla_X JB = 0$, we get

$$2\nabla_X B = |B|^2 X - \omega(X)B + \omega(JX)JB.$$

This means that B is a K-torse forming vector field that was included by S.Yamaguchi [10]. Since this case is new and interesting, we dedicate the next section to study its properties with illustrative examples.

4. A special type of 4-dimensional Hermitian manifolds

Let's start with providing the following definition:

Definition 4.1. A 4-dimensional almost Hermitian manifold (M, J, g) is called $\mathcal{B}$-Hermitian manifold if there exists a global vector field B such that for all X vector field on M

$$2\nabla_X B = |B|^2 X - \omega(X)B + \omega(JX)JB. \tag{4.1}$$

$\mathcal{B}$-Hermitian manifold is a type of l.c.K manifolds and we can consider it as a counterpart to Vaisman manifold. To prove the existence of such manifolds we present the following examples:

Example 4.1. Let $\mathcal{A}$ be the 4-dimensional Lie algebra whose skew-symmetric multiplication is given by

$$[E_1, E_3] = E_3, \qquad [E_1, E_4] = E_4 \qquad \textit{and} \qquad [E_i, E_j] = 0 \quad \textit{in other cases},$$

where $\{E_1, E_2, E_3, E_4\}$ is certain fixed basis of $\mathcal{A}$.

Consider a connected Lie subgroup G of general linear group $GL(k, \mathbb{R})$ for certain natural k, such that the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathcal{A}$.
Let $s : \mathcal{A} \to \mathfrak{g}$ be the isomorphism. Let $\{e_1, e_2, e_3, e_4\}$ be the basis og $\mathfrak{g}$ formed by left invariant vector fields on G such that $s(E_i) = e_i,\ 1 \leq i \leq 4$. Then we have

$$[e_1, e_3] = e_3, \qquad [e_1, e_4] = e_4 \qquad \textit{and} \qquad [e_i, e_j] = 0 \quad \textit{in other cases}.$$

Let (J, g) be the left invariant almost Hermitian structure on G defined by

$$Je_1 = e_2, \quad Je_2 = -e_1, \quad Je_3 = e_4, \quad Je_4 = -e_3,$$

and

$$g(e_i, e_j) = \delta_{ij}, \qquad 1 \leq i, j \leq 4.$$

By Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\nabla_{e_3} e_1 = -e_3, \quad \nabla_{e_3} e_3 = e_1, \quad \nabla_{e_i} e_j = 0 \quad \textit{in other cases}.$$

we have $JB = \sum_{i=1}^{4} (\nabla_{e_i} J)e_i$ which gives $B = -2e_1$. Hence

$$\nabla_{e_1} B = \nabla_{e_2} B = 0, \quad \nabla_{e_3} B = 2e_3, \quad \nabla_{e_4} B = 2e_4 \qquad \textit{and} \quad \nabla_{e_i} JB = 0.$$

So, we can easily verify that

$$2\nabla_{e_i} B = |B|^2 e_i - \omega(e_i)B + \omega(Je_i)JB,$$

or, equivalently

$$\nabla_{e_i} e_1 = -e_i + \delta_{1i} e_1 + \delta_{2i} e_2.$$

Therefore, (J, g) is a $\mathcal{B}$-Hermitian structure on G

The second type of examples are closely related to the conformal transformation. It is well known that any conformal transformation

$$\tilde{g} = \mathrm{e}^{2\rho} g, \qquad \rho \in \mathcal{C}^\infty(M) \quad \textit{and} \quad \mathrm{d}\rho \neq 0$$

of the metric g in a Kähler manifold (M, J, g) gives rise to an l.c.K manifold $(M, J, \tilde{g})$. Easily, one can get $\tilde{\Omega} = \mathrm{e}^{2\rho}\Omega$ implies $\mathrm{d}\tilde{\Omega} = 2\mathrm{d}\rho \wedge \tilde{\Omega}$ which gives $\omega = 2\mathrm{d}\rho$ and $B = 2\mathrm{grad}\rho$.

Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections associated with the metrics g and $\tilde{g}$ respectively. As is well know, they are connected by

$$\tilde{\nabla}_X Y = \nabla_X Y + X(\rho)Y + Y(\rho)X - g(X, Y)\mathrm{grad}\rho. \tag{4.2}$$

Now, let's calculate $\tilde{\nabla} B$. Using (4.2), we get

$$\tilde{\nabla}_X B = \nabla_X B + \frac{1}{2}|B|^2. \tag{4.3}$$

On the other hand, using (3.2) taking into account $\tilde{\nabla}_X JB = 0$, one can get

$$\begin{aligned} J\tilde{\nabla}_X B &= -(\tilde{\nabla}_X J)B \\ &= -\frac{1}{2}\big(g(X,B)JB - g(X,JB)B - \omega(B)JX\big) \\ &= -\frac{1}{2}\big(\omega(X)JB + \omega(JX)B - |B|^2 JX\big), \end{aligned}$$

which gives

$$\tilde{\nabla}_X B = \frac{1}{2}\big(-\omega(X)B + \omega(JX)JB + |B|^2 X\big) \tag{4.4}$$

From (4.3) and (4.4) we conclude that in order for the $(M, J, \tilde{g})$ to be $\mathcal{B}$-Hermitian manifold, the following condition must be fulfilled

$$2\nabla_X B = -\omega(X)B + \omega(JX)JB, \tag{4.5}$$

or, equivalently

$$\nabla_X \mathrm{grad}\rho = -X(\rho)\mathrm{grad}\rho + JX(\rho)J\mathrm{grad}\rho. \tag{4.6}$$

Therefore, summing up the arguments above, we have the following proposition:

Proposition 4.1. *Let (J, g) be a Kählerian structure on M and ρ a non-zero function on M. Then, the structure $(J, \tilde{g})$ defined above is $\mathcal{B}$-Hermitian structure on M if and only if the function ρ satisfies the equation (4.5).*

According to [7], Vaisman geometry is intimately related to Sasakian one. In what follows, we prove that the $\mathcal{B}$-Hermitian manifolds are closely related to the Kenmotsu ones.

Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional Kenmotsu manifold and $\tilde{M} = \mathbb{R} \times M$ be the product manifold of M and a real line $\mathbb{R}$ with natural coordinate system ∂_t equipped with the following almost Hermitian structure $(J, \tilde{g})$ defined by

$$\tilde{g} = dt^2 + g, \qquad J\partial_t = \xi \qquad \textit{and} \quad JX = \varphi X - \eta(X)\partial_t.$$

The manifold $(\tilde{M}, J, \tilde{g})$ possesses a fundamental 2-form, Ω, the Kähler form, defined by

$$\begin{aligned} \Omega\big((a\partial t, X), (b\partial t, Y)\big) &= \tilde{g}\big((a\partial t, X), J(b\partial t, Y)\big) \\ &= \tilde{g}\big((a\partial t, X), (-\eta(Y)\partial t, \varphi Y + b\xi)\big), \end{aligned}$$

we can check that is very simply as follows:

$$\Omega = \phi - 2\mathrm{d}t \wedge \eta, \tag{4.7}$$

where ϕ denotes the fundamental 2-form of the Kenmotsu structure (φ, ξ, η). We have immediately that,

$$\mathrm{d}\Omega = 2\eta \wedge \Omega.$$

Taking $\omega = 2\eta$ we get $B = 2\xi$ and $|B| = 2$. We compute

$$\begin{cases} 2\tilde{\nabla}_{\partial_t} B = 0 \\ 2\tilde{\nabla}_X B = 4\nabla_X \xi, \end{cases} \tag{4.8}$$

on the other hand,

$$\begin{cases} |B|^2\partial_t - \omega(\partial_t)B + \omega(J\partial_t)JB = 0 \\ |B|^2 X - \omega(X)B + \omega(JX)JB = -4\varphi^2 X, \end{cases} \tag{4.9}$$

which gives $\nabla_X \xi = -\varphi^2 X$ and this is true as long as $(M, \varphi, \xi, \eta, g)$ is a Kenmotsu manifold. Therefore, we have the following proposition:

Proposition 4.2. *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional Kenmotsu manifold. Then, the product $\tilde{M} = \mathbb{R} \times M$ equipped with the almost Hermitian $(J, \tilde{g})$ defined above is a $\mathcal{B}$-Hermitian manifold.*

Now, we will show that from a 4-dimensional $\mathcal{B}$-Hermitian manifold (more general, 4-dimensional Hermitian manifold) we can build a 5-dimensional Kenmotsu one.

Let (N, J, h) be a 4-dimensional Hermitian manifold. According to the remark 3.1, (N, J, h) is a 4-dimensional l.c.K manifold. Then, we have

$$N_J = 0, \quad \mathrm{d}\Omega = \omega \wedge \Omega \quad and \quad \mathrm{d}\omega = 0. \tag{4.10}$$

On the product $M = \mathbb{R} \times N$ one can define an almost contact structure (φ, ξ, η) by setting

$$\varphi\partial_r = 0, \quad \varphi X = JX - 2\omega(JX)\partial_r, \quad \xi = \partial_r, \quad \eta = dr + 2\omega, \tag{4.11}$$

and a Riemannian metric g by

$$g = fh + \eta \otimes \eta, \tag{4.12}$$

for any vector filed $X \in \mathfrak{X}(M)$ and ∂_r denote the unit tangent field to $\mathbb{R}$ where f is a positive function on $\mathbb{R}$.

Proposition 4.3. *The structure (φ, ξ, η, g) constructed on the product M is an almost contact metric structure.*

Proof. It is easy to see that $\eta(\xi) = 1$ and $\varphi^2\partial_r = -\partial_r + \eta(\partial_r)\xi = 0$. For all $X \in \mathfrak{X}(M)$, we compute:

$$\begin{aligned} \varphi^2 X &= \varphi(JX - 2\omega(JX)\xi) \\ &= \varphi(JX) - 2\omega(JX)\varphi\xi \\ &= J^2X - 2\omega(J^2X)\xi \\ &= -X + 2\omega(X)\xi \\ &= -X + \eta(X)\xi. \end{aligned}$$

Let $X, Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned} g(\varphi X, \varphi Y) &= g(JX - 2\omega(JX)\xi, JY - 2\omega(JY)\xi) \\ &= g(JX, JY) - 2\omega(JY)g(JX, \xi) - 2\omega(JX)g(\xi, JY) \\ &\quad +4\omega(JX)\omega(JY)g(\xi, \xi), \end{aligned}$$

by the definition of the metric g with $\eta = dr + 2\omega$, we obtain

$$\begin{aligned} g(\varphi X, \varphi Y) &= h(JX, JY) + 4\omega(JX)\omega(JY) - 4\omega(JX)\omega(JY) \\ &\quad -4\omega(JX)\omega(JY) + 4\omega(JX)\omega(JY) \\ &= h(JX, JY), \end{aligned}$$

as $h(JX, JY) = h(X, Y)$ and $g(X, Y) = h(X, Y) + 4\omega(X)\omega(Y)$, we conclude that

$$\begin{aligned} g(\varphi X, \varphi Y) &= g(X, Y) - 4\omega(X)\omega(Y) \\ &= g(X, Y) - \eta(X)\eta(Y). \end{aligned}$$

As $\varphi\xi = 0$, $g(X, \xi) = 2\omega(X) = \eta(X)$, $\eta(\xi) = 1$ and $g(\xi, \xi) = 1$, we get

$$g(\varphi X, \varphi\xi) = g(X, \xi) - \eta(X)\eta(\xi) = 0,$$

$$g(\varphi\xi, \varphi\xi) = g(\xi, \xi) - \eta(\xi)\eta(\xi) = 0.$$

According to (2.9), (φ, ξ, η, g) is an almost contact metric structureon M. □

Since the vanishing of the tensor field $N^{(1)}$ of (φ, ξ, η, g) is a necessary and sufficient condition for normality, we seek to express the condition of integrability in terms of N_J, the Nijenhuis torsion of J .

Since $N^{(1)}$ is a tensor field of type $(1, 2)$, it suffices to compute $N^{(1)}\big((\partial_r, 0), (0, X)\big)$ and $N^{(1)}\big((0, X), (0, Y)\big)$ for vector fields X and Y on M. A direct calculation gives

$$N^{(1)}\big((\partial_r, 0), (0, X)\big) = 0.$$

For $N^{(1)}\big((0, X), (0, Y)\big)$, we have

$$\begin{aligned} N^{(1)}\big((0, X), (0, Y)\big) &\equiv N^{(1)}(X, Y) \\ &= \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + 2\mathrm{d}\eta(X, Y)\xi, \end{aligned}$$

using definition of φ in (4.11) with $\mathrm{d}\eta(X,Y) = 2\mathrm{d}\omega(X,Y) = 0$, one can get

$$N^{(1)}\big((0,X),(0,Y)\big) = N_J(X,Y) - 2\omega\big(N_J(X,Y)\big)\partial_r,$$

since $N_J = 0$ then $N^{(1)} = 0$ therefore the almost contact metric structure (φ, ξ, η) is normal.
On the other hand, the fundamental 2-form ϕ of (φ, ξ, η) is

$$\phi\Big(\big(a\frac{\partial}{\partial_r}, X\big), \big(b\frac{\partial}{\partial_r}, Y\big)\Big) = g\Big(\big(a\frac{\partial}{\partial_r}, X\big), \varphi\big(b\frac{\partial}{\partial_r}, Y\big)\Big),$$

we can check that is very simply as follows:

$$\phi = f\Omega, \tag{4.13}$$

then we have

$$\begin{cases} \phi = f\Omega \\ \eta = dr + 2\omega \end{cases} \quad \Rightarrow \quad \begin{cases} \mathrm{d}\phi = (\frac{f'}{f}dr + 2\omega) \wedge \phi \\ \mathrm{d}\eta = 0. \end{cases}$$

Taking $\omega = 2\theta$
We can claim the following Proposition:

Proposition 4.4. *Let (N, J, h) be a 4-dimensional Hermitian manifold. The almost contact metric structure (φ, ξ, η, g) defined above on $M = \mathbb{R} \times N$ is a Kenmotsu structure if and only if $\mathrm{d}\Omega = 2\theta \wedge \Omega$ and $f = ce^{2r}$ where $\omega = 2\theta$ and $c > 0$.*

Proof. The necessity was observed above. For the sufficiency, first note that

$$\begin{cases} \mathrm{d}\phi\big((\partial_r, 0), (0,X), (0,Y)\big) = f'\Omega(X,Y), \\ \mathrm{d}\phi\big((0,X), (0,Y), (0,Z)\big) = f\mathrm{d}\Omega(X,Y,Z), \\ \mathrm{d}\eta\big((0,X),(0,Y)\big) = 0, \end{cases} \tag{4.14}$$

Suppose that (φ, ξ, η, g) is a Kenmotsu structure on M i.e. we have $\mathrm{d}\phi = 2\eta \wedge \phi$ and $\mathrm{d}\eta = 0$. From equations (4.14) we obtain

$$f = ce^{2r} \quad and \quad \mathrm{d}\Omega = \omega \wedge \Omega,$$

which shows that (M, J, g) is a Locally conformally Kähler manifold with $f = ce^{2r}$ and $c > 0$. □

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Dragomir, S., Ornea, L.: Locally Conformal Kähler Geometry. Progress in Mathematics, vol. 55. Birkhäuser Boston, MA (1998). https://doi.org/10.1007/978-1-4612-2026-8

[2] Gray, A., Hervella, L. M.: The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura Appl. (4) **123**, 35-58 (1980).

[3] Kenmotsu, K.: A class of almost contact Riemannian manifolds. Tohoku Math. J., **24**, 93-103 (1972).

[4] Libermann, P.: Sur le problème d'équivalence de certaines structures infinitésimales régulières. Ann. Mat. Pura Appl. **36**, 27-120 (1954). (In French.)

[5] Nagao, M., Kotô, S.: Curvature in almost Kähler spaces. Memoirs of The Faculty of Education. Niigata University (1973).

[6] Olszak, Z. : Curvature properties of four-dimentional Hermitian manifolds. Ann. Soc. Math. Pol., Ser. I, Commentat. Math. **36**, 169-179 (1987).

[7] Ornea, L., Verbitsky, M.: Structure theorem for compact Vaisman manifolds. Math. Res. Lett. **10**, 799-805 (2003).
[8] Oubbich,e N., Beldjilali, G., Bouzir, H., Delloum, A.: New Class of Locally Conformal Kähler Manifolds. Mediterr. J. Math. (2023), doi.org/10.1007/s00009-023-02288-3
[9] Vaisman, I.: On locally conformal almost Kähler manifolds. Isr. J. Math. **24**, 338-351 (1976)
[10] Yamaguchi, S.: On Kaehlerian torse-forming vector fields. Kodai Math. J. **2**(1), 103-115 (1979).
[11] Yano, K.: On the torse-forming directions in Riemannian spaces. Proc. Imp. Acad. Tokyo. **20**, 340-345 (1944).
[12] Yano, K., Kon, M.: Structures on Manifolds. Series in Pure Math., **3**, World Sci., (1984).

Affiliations

BELDJILAI GHERICI
ADDRESS: Laboratory of Quantum Physics and Mathematical Modeling (LPQ3M), University of Mascara, Algeria.
E-MAIL: gherici.beldjilali@univ-mascara.dz
ORCID ID: 0000-0002-8933-1548