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CERTAIN RESULTS CONCERNING (p, q)-PARAMETERIZED BETA LOGARITHMIC FUNCTION AND THEIR PROPERTIES

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Abstract. The primary object of this article is to introduce (p, q)-beta logarithmic function with extended beta function by using the logarithmic mean. We evaluate different properties and representations of beta logarithmic function. Further, it is evaluated logarithmic distribution, hypergeometric and confluent hypergeometric functions via logarithmic mean are evaluated and their essential properties are studied. Numerous formulas of (p, q)-beta logarithmic functions such as integral formula, derivative formula, transformation formula and generating function are analyzed.

1. Introduction and Preliminaries

The ordinary hypergeometric functions have been the subject of comprehensive research by various eminent mathematician. These functions play a vital role in different branches of mathematics. Applications of special functions (higher order transcendental functions such as Bessel function, Whittaker function, Wright functions etc.) are found in a broad variety of engineering sub-fields. The Euler beta function plays an important role in special function which introduced by Legendre, Whittaker and Watson etc. Using techniques to unify and generalize specialized

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functions has been an active and interesting area of research. An extension of the Euler beta function was proposed in 1997 by Chaudhry et al. [3] as well as a number of other researchers.

Definition 1. *The beta function (also called the Euler's integral of the first kind) is defined as (see [11, 13]):*

$$B(\xi,\zeta)=\frac{\Gamma(\xi)\ \Gamma(\zeta)}{\Gamma(\xi+\zeta)}=\int_0^1 t^{\xi-1}(1-t)^{\zeta-1}dt,\ \ (Re(\xi)>0,\ \ Re(\zeta)>0) \tag{1}$$

where Γ *(.) is gamma function, the Euler integral of the second kind (commonly used as extension of factorial function to complex numbers defined for all complex numbers except for the non-positive integers).*

As we know that the gamma and beta functions play a crucial role in the development of theory of higher order transcendental functions and their various generalizations are given by the various number of researchers (see [1], [2], [3], [4], [5], [7], [8], [9], [12], [15]).

Gamma function is defined by the convergent improper integral as:

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt,\ \ (Re(x)>0).$$

The underlying extension of Euler's beta function established by Chaudhry et al. [3] is defined as

$$B_p(\xi,\zeta)=\int_0^1 t^{\xi-1}(1-t)^{\zeta-1}\,exp\left[-\frac{p}{t(1-t)}\right]dt,\ (Re(p)>0,Re(\xi)>0,Re(\zeta)>0). \tag{2}$$

For $p=0$, the extended beta function reduces to the classical beta function.

In 2004, Chaudhry et al. [4] used new extended beta function $B(\beta,\zeta;\rho)$ to introduced extended Gauss hypergeometric and confluent hypergeometric functions which are defined by their series representation as

$$F_\rho\,(\xi,\zeta;\eta;z)=\sum_{n=0}^{\infty}(\xi)_n\ \frac{B_\rho(\zeta+n,\eta-\zeta)}{B(\zeta,\eta-\zeta)}\,\frac{z^n}{n!} \tag{3}$$

$$(\rho\geq 0,\ \ |z|<1,\ \ Re(\eta)>\Re(\zeta)>0),$$

and

$$\Phi_\rho\,(\zeta;\eta;z)=\sum_{n=0}^{\infty}\frac{B_\rho(\zeta+n,\eta-\zeta)}{B(\zeta,\eta-\zeta)}\,\frac{z^n}{n!} \tag{4}$$

$$(\rho\geq 0,\ \ |z|<1,\ \ Re(\eta)>Re(\zeta)>0).$$

In 2014, Choi et al. [5] introduced another extension of beta function, denoted by $B_{p,q}(\xi,\zeta)$ and is defined by

$$B_{p,q}(\xi,\zeta)=\int_0^1 t^{\xi-1}(1-t)^{\zeta-1}exp\left[\frac{-p}{t}-\frac{q}{(1-t)}\right]dt, \tag{5}$$

$$(Re(p)>0,\ Re(q)>0),(Re(\xi)>0,\ Re(\zeta)>0).$$

The integral representation for extended Gauss hypergeometric function and extended confluent hypergeometric function are defined as follows :

$$F_{p,q}\left(\xi,\zeta;\eta;z\right)=\frac{1}{B(\zeta,\eta-\zeta)}\int_0^1 t^{\zeta-1}\left(1-t\right)^{\eta-\zeta-1}\left(1-zt\right)^{-\xi} exp\left[\frac{-p}{t}-\frac{q}{(1-t)}\right]dt, \tag{6}$$

$$(p,q\geq 0;\quad |\arg(1-z)|<\pi;\ Re(\eta)>Re(\zeta)>0),$$

and

$$\Phi_{p,q}\left(\zeta;\eta;z\right)=\frac{1}{B(\zeta,\eta-\zeta)}\int_0^1 t^{\zeta-1}\left(1-t\right)^{\eta-\zeta-1} exp\left(zt-\frac{p}{t}-\frac{q}{(1-t)}\right)dt, \tag{7}$$

$$\{p,q\geq 0,\quad Re(\eta)>Re(\zeta)>0\}.$$

Definition 2. *The logarithmic mean for $x,y>0$ (quotient of difference of two non-negative numbers by their logarithmic value) is defined as (see [14])*

$$L(x,y)=\int_0^1 x^{1-t}y^t\ dt=\begin{cases}\frac{x-y}{log(x)-log(y)} & x\neq y,\\ x & x=y.\end{cases} \tag{8}$$

It can be easily seen that the logarithmic mean satisfies the following properties (see [6], [10]):

• The logarithmic mean always lies between the geometric mean and arithmetic mean.

• For $x=y$ all three means that are geometric mean, arithmetic mean and logarithmic mean are same.

• The limiting condition of the logarithmic mean is given as:

$$\lim_{y\to x}L(x,y)=L(x,x)=x.$$

• The logarithmic mean satisfies the following property that is:

$$\frac{1}{L(x,y)}=\int_0^1\frac{dt}{tx+(1-ty)}.$$

• The infinite product of the logarithmic mean of any two positive real numbers are given as:

$$L(x,y)=\prod_{m=1}^{\infty}\left(\frac{x^{2-m}+y^{2-m}}{2}\right).$$

2. Construction of $(\mathbf{p},\mathbf{q})$- Beta Logarithmic Function

For any fixed $x, y > 0$ the function $x^{1-t}y^t$ is continuous in $[0,1]$ and so it is bounded on $[0,1]$. It means that there exist $c \geq 0$ and for any $x, y, \xi, \zeta > 0$, we have

$$\begin{aligned} 0 &\leq x^{1-t}\, y^t\, t^{\xi-1}\,(1-t)^{\zeta-1}\, exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right] \\ &\leq c\, t^{\xi-1}\,(1-t)^{\zeta-1}\, exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right],\ \forall\, t \in (0,1). \end{aligned}$$

Thus, $x^{1-t}\ y^t\ t^{\xi-1}\,(1-t)^{\zeta-1}\, exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right]$ is integrable on $(0,1)$. We introduce the underlying definition that defines the relation between beta function and logarithmic mean.

Definition 3. *For any $x, y, \xi, \zeta \in \mathbb{R}^+$, we define*

$$B^m_{p,q}\ L(x,y;\xi,\zeta) = \int_0^1 x^{1-t}\ y^t\ t^{\xi-1}\,(1-t)^{\zeta-1}\, exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right]dt, \tag{9}$$

$$(p, q \geq 0,\ Re(\xi) > 0,\ Re(\zeta) > 0),$$

which we call the (p,q) beta-logarithmic function.

Remark 1. *Substituting $x = y = 1$ in (9), we get extended beta function*

$$B^m_{p,q}\ L(1,1;\xi,\zeta) = B^m_{p,q}\ (\xi,\zeta),\ (Re(\xi) > 0,\ Re(\zeta) > 0)$$

where,

$$B^m_{p,q}(\xi,\zeta) = \int_0^1 t^{\xi-1}(1-t)^{\zeta-1} exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right] dt,\ (Re(p) > 0,\ Re(q) > 0). \tag{10}$$

Remark 2. *By setting $x = y = 1$, $p = q = 0$ and $m = 1$ in (9), we get the Euler Beta function (1) (see [11], [13])*

$$B^1_{0,0}\ L(1,1;\xi,\zeta) = B(\xi,\zeta),\ (Re(\xi) > 0,\ Re(\zeta) > 0.$$

Remark 3. *If we take $\xi = \zeta = 1$, $p = q = 0$ and $m = 1$ in (9), we get logarithmic mean (8) (see [14]).*

$$B^1_{0,0}\ L(x,y;1,1) = L(x,y),\ (x, y > 0).$$

3. Properties of $(\mathbf{p},\mathbf{q})$- Beta Logarithmic Function

In this section, we analyze different properties and representations of a new form of beta function that we call the (p,q) beta logarithmic function. This function is a combined study of a new extended beta function and the logarithmic mean.

Proposition 1. *For $x, y, \xi, \zeta, p, q > 0$, the following assertions hold true:*

$$B^{m}_{p,q}\ L(x, y; \xi, \zeta) = B^{m}_{p,q}\ L(y, x; \xi, \zeta), \tag{11}$$

$$B^{m}_{p,q}\ L(x, x; \xi, \zeta) = x\, B^{m}_{p,q}\ (\xi, \zeta), \tag{12}$$

and

$$B^{m}_{p,q}\ L(\delta x, \delta y; \xi, \zeta) = \delta B^{m}_{p,q}\ L(x, y; \xi, \zeta). \tag{13}$$

Proof. The result (11) may be reached by altering the variable t by $1-u$ in equation (9). The assertions (12) and (13) may be produced by easy computation in equation (9). □

Proposition 2. *For any $x, y, \xi, \zeta, p, q > 0$, the following assertions hold true:*

$$B^{m}_{p,q}\ L(x, y; \xi + 1, \zeta) + B^{m}_{p,q}\ L(x, y; \xi, \zeta + 1) = B^{m}_{p,q}\ L(x, y; \xi, \zeta). \tag{14}$$

Proof. By using the definition (9) to the left side of (14), we get the required assertion (14). □

Corollary 1. *If we set $x = y = 1$ in (14), we obtained the well known result introduced by M. Raïssouli et al. [14]*

$$B^{m}_{p,q}(\xi + 1, \zeta) + B^{m}_{p,q}(\xi, \zeta + 1) = B^{m}_{p,q}(\xi, \zeta). \tag{15}$$

Proposition 3. *For any $x, y, \xi, \zeta > 0$, $p, q \geq 0$, the following assertions hold true:*

$$\begin{aligned} min(x, y) B^{m}_{p,q}(\xi, \zeta) \leq B^{m}_{p,q} L(x, y; \xi, \zeta) &\leq x B^{m}_{p,q}(\xi, \zeta + 1) + y B^{m}_{p,q}(\xi + 1, \zeta) \\ &\leq max(x, y) B^{m}_{p,q}(\xi, \zeta). \end{aligned} \tag{16}$$

Proof. From the underlying inequality

$$min(x, y) \leq \sqrt{xy} \leq L(x, y) \leq \left(\frac{x + y}{2}\right) \leq max(x, y) \text{ and } B^{m}_{p,q}(\xi, \zeta) > 0,$$

we get the following relation

$$min(x, y) B^{m}_{p,q}(\xi, \zeta) \leq B^{m}_{p,q} L(x, y; \xi, \zeta). \tag{17}$$

By using the underlying well known Young's inequality

$$x^{1-t} y^{t} \leq x(1 - t) + yt, \quad \forall\ t \in [0, 1]$$

we get

$$\begin{aligned} B^{m}_{p,q}\ L(x,y;\xi,\zeta) &\le \int_0^1 \ x^{1-t}\ y^t\ t^{\xi-1}\,(1-t)^{\zeta-1}\,exp\Big[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \Big]dt \\ &\le x\left(\int_0^1 t^{\xi-1}\,(1-t)^{\zeta-1}\,exp\Big[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \Big]dt\right) \\ &+ y\left(\int_0^1 \ t^{\xi-1}\,(1-t)^{\zeta-1}\,exp\Big[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \Big]dt\right) \\ &\le x\ \big(B^{m}_{p,q}(\xi,\zeta+1)\big)+\big(y\ B^{m}_{p,q}(\xi+1,\zeta)\big) \\ &\le\ max(x,y)\ \big(B^{m}_{p,q}(\xi,\zeta+1)+B^{m}_{p,q}(\xi+1,\zeta)\big) \end{aligned}$$

by using the relation (15), we achieved the required result. □

Proposition 4. *For any $x, y, \xi, \zeta > 0$, $p, q \ge 0$ the following assertion holds true:*

$$B^{m}_{p,q}\ L(x,y;\xi,\zeta)=\sum_{n=0}^{\infty}B^{m}_{p,q}(x,y;\xi+n,\zeta+1). \tag{18}$$

Proof. We have

$$B^{m}_{p,q}\ L(x,y;\xi,\zeta)=\int_0^1 \ x^{1-t}\ y^t\ t^{\xi-1}\,(1-t)^{\zeta-1}\exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \right]dt,$$

By using the series representation $(1-t)^{-1}=\sum_{n=0}^{\infty}t^n$, for $t\in(0,1)$ with the arguments of uniform convergence of this power series, we have

$$\begin{aligned} B^{m}_{p,q}\ L(x,y;\xi,\zeta) &= \int_0^1 \ x^{1-t}\ y^t\ t^{\xi-1}\,(1-t)^{\zeta-1}\exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \right]dt \\ &= \int_0^1 \ x^{1-t}\ y^t\ t^{\xi-1}\,(1-t)^{\zeta}\ (1-t)^{-1}\exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \right]dt \\ &= \sum_{n=0}^{\infty}\int_0^1 \ x^{1-t}\ y^t\ t^{\xi-1}\,(1-t)^{\zeta}\ t^n\exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \right]dt \\ &= \sum_{n=0}^{\infty}\int_0^1 \ x^{1-t}\ y^t\ t^{\xi+n-1}\,(1-t)^{\zeta}\exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \right]dt, \end{aligned}$$

using the definition (9) in the above expression, we achieved the desired result. □

Theorem 1. *Let $x, y, \xi, \zeta > 0$, $p, q \ge 0$, the following representation holds true:*

$$B^{m}_{p,q}\ L(x,y;\xi,\zeta)=\sum_{r,n=0}^{\infty}\frac{B^{m}_{p,q}(\xi+n,\zeta+r)}{n!r!}(log(x))^r(log(y))^n. \tag{19}$$

Proof. Using the following power series expansion

$$x^{1-t} = \sum_{r=0}^{\infty} \frac{(log(x))^r}{r!}(1-t)^r,\ y^t = \sum_{n=0}^{\infty} \frac{(log(y))^n}{n!} t^n$$

using the above expansion in the result (9), we have

$$B_{p,q}^{m}\ L(x,y;\xi,\zeta) = \int_0^1\ x^{1-t}\ y^t\ t^{\xi-1}\,(1-t)^{\zeta-1} \exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\ \right]dt$$

$$= \int_0^1 \sum_{r,n=0}^{\infty} \frac{(log(x))^r (log(y))^n}{r!n!}\ t^{\xi+n-1}\,(1-t)^{\zeta+r-1} \exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\ \right]dt$$

$$= \int_0^1 \frac{t^{\xi+n-1}\,(1-t)^{\zeta+r-1}}{r!n!}\ (log(x) \sum_{r,n=0}^{\infty})^r (log(y))^n \exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\ \right]dt,$$

using the definition (10) in the above expression, we achieved the required result (19). □

4. The $(\mathbf{p},\mathbf{q})$-Beta Logarithmic Random Variable

In this section, we define beta-logarithmic distribution of (9) and obtain its mean, variance and moment generating function.

Definition 4. *For $x, y, \xi, \zeta > 0$, $p, q \geq 0$, the beta-logarithmic distribution is defined as:*

$$f(t) = \begin{cases} \frac{1}{B_{p,q}^{m}\ L(\xi,\zeta)}\ x^{1-t}\ y^t\ t^{\xi-1}(1-t)^{\zeta-1} \exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\ \right], & (0 < t < 1), \\ 0, & \textit{otherwise.} \end{cases} \tag{20}$$

The k^{th}- moment of a random variable X for any real number k is given as:

$$\mathbb{E}(X^k) = \frac{B_{p,q}^{m}\ L(x,y;\xi+k,\zeta)}{B_{p,q}^{m}\ L(x,y;\xi,\zeta)}, \tag{21}$$

$$(\ p,q \geq 0,\ \ x,y,\xi,\zeta > 0).$$

For $k = 1$, we obtain the mean as a particular case of (21) given by

$$\mu = \mathbb{E}(X) = \frac{B_{p,q}^{m}\ L(x,y;\xi+1,\zeta)}{B_{p,q}^{m}\ L(x,y;\xi,\zeta)}. \tag{22}$$

The variance of the distribution is defined as: $\sigma^2 = \mathbb{E}(X^2) - \{\mathbb{E}(X)\}^2$

$$\sigma^2 = \frac{B_{p,q}^{m}\ L(x,y;\xi,\zeta)\, B_{p,q}^{m}\ L(x,y;\xi+2,\zeta) - \left\{B_{p,q}^{m}\ L(x,y;\xi+1,\zeta)\right\}^2}{\left\{B_{p,q}^{m}\ L(x,y;\xi,\zeta)\right\}^2}. \tag{23}$$

The moment generating function of the distribution is defined as

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, \mathbb{E}(X^n) = \frac{1}{B_{p,q}^m L(x,y;\xi,\zeta)} \sum_{n=0}^{\infty} B_{p,q}^m L(x,y;\xi+n,\zeta) \frac{t^n}{n!}. \tag{24}$$

Here, we recall the following known lemma.

Lemma 1. *Let Y be a random variable with values that exist inside a finite range $[x, y]$. Then, we have for all $\mathcal{E} \in [x, y]$,*

$$\left| P(Y \leq \mathcal{E}) - \frac{y - E(Y)}{y - x} \right| \leq \frac{1}{2} + \frac{|\mathcal{E} - \frac{x+y}{2}|}{y - x}. \tag{25}$$

Proposition 5. *Let X represent a beta-logarithmic random variable with parameters $(x, y; \xi, \zeta)$. Then, for any $k, \mathcal{E} > 0$, the following assumptions are true:*

$$\left| P(X \leq \mathcal{E}) - \frac{B_{p,q}^m\ L(x,y;\xi,\zeta+1)}{B_{p,q}^m\ L(x,y;\xi,\zeta)} \right| \leq \frac{1}{2} + \left| \mathcal{E} - \frac{1}{2} \right| \tag{26}$$

and

$$P(X^k \geq \mathcal{E}) \leq \frac{B_{p,q}^m L(x,y;\xi+k,\zeta)}{\mathcal{E}\ B_{p,q}^m L(x,y;\xi,\zeta)} \tag{27}$$

Proof. With the help of (14) and (22), we have

$$E(X) = 1 - \frac{B_{p,q}^m\ L(x,y;\xi,\zeta+1)}{B_{p,q}^m L(x,y;\xi,\zeta)}, \tag{28}$$

using the above relation in inequality (25), we achieved the desired result (26).

The second inequality (27) can be deduced by using the Markov's inequality

$$P(X^k \geq \mathcal{E}) \leq \frac{E(X^k)}{\mathcal{E}}$$

and the definition of $E(X^k)$, we get the desired result (27). □

5. Hypergeometric and Confluent Hypergeometric Representation by (p, q)-Beta Logarithmic Function

Many researchers gave the extension of hypergeometric and confluent hypergeometric functions (see [4], [5], [12]). Here, we introduce a new hypergeometric and confluent hypergeometric functions in terms of (p,q)-beta logarithmic function.

The (p, q)-beta logarithmic hypergeometric function is defined as:

$$F_{p,q}^m\ L(\xi,\zeta;\eta;z) = \sum_{n=0}^{\infty} (\xi)_n\ \frac{B_{p,q}^m\ L(x,y;\zeta+n,\eta-\zeta)}{B(\zeta,\eta-\zeta)}\ \frac{z^n}{n!}, \tag{29}$$

$$(p, q \geq 0,\ |z| < 1,\ Re(\eta) > Re(\zeta) > 0,\ x, y > 0).$$

The (p,q)-beta logarithmic confluent hypergeometric logarithmic function is defined as:

$$\Phi_{p,q}^{m}\ L\left(\xi;\zeta;z\right)=\sum_{n=0}^{\infty}\frac{B_{p,q}^{m}\ L(x,y;\xi+n,\eta-\zeta)}{B(\zeta,\eta-\zeta)}\frac{z^n}{n!}, \tag{30}$$

$$(p,q\geq 0,\ x,y,>0,\ Re(\eta)>Re(\zeta)>0, Re(\xi)>0, |z|<1).$$

5.1. **Integral formula.**

Theorem 2. *The following integral formula for the (p,q)-beta logarithmic hypergeometric and (p,q)-beta logarithmic confluent hypergeometric function holds true:*

$$F_{p,q}^{m}\ L\left(\xi,\zeta;\eta;z\right)=\frac{1}{B(\zeta,\eta-\zeta)}\times\int_0^1 x^{1-t}y^t\ t^{\zeta-1}\left(1-t\right)^{\eta-\zeta-1}\left(1-zt\right)^{-\xi}\exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\right]dt, \tag{31}$$

$$(|\arg(1-z)|<\pi;\ p,q\geq 0;\ x,y,\in\mathbb{R}^{+};\ Re(\eta)>Re(\zeta)>0),$$

and

$$\phi_{p,q}^{m}\ L\left(\zeta;\eta;z\right)=\frac{1}{B(\zeta,\eta-\zeta)}\int_0^1 x^{1-t}y^t\ t^{\zeta-1}\left(1-t\right)^{\eta-\zeta-1}e^{zt}exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\right]dt \tag{32}$$

$$(p,q\geq 0;\ x,y,\in\mathbb{R}^{+};\ Re(\eta)>Re(\zeta)>0).$$

Proof. By applying the definition of beta logarithmic function (9) into (29) and by rearranging the order of integral and summation, we get

$$F_{p,q}^{m}\ L\left(\xi,\zeta;\eta;z\right)=\frac{1}{B(\zeta,\eta-\zeta)}\times\int_0^1 x^{1-t}y^t\ t^{\zeta-1}\left(1-t\right)^{\eta-\zeta-1}exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\right]\sum_{n=0}^{\infty}(\xi)_n\frac{(zt)^n}{n!}dt. \tag{33}$$

Applying the binomial theorem in (33), we obtained the desired result (31).

Similarly, we can obtain (32). □

5.2. **Derivative formula.**

Theorem 3. *The following derivative formula for (p,q)-beta logarithmic hypergeometric and (p,q)-beta logarithmic confluent hypergeometric functions hold true:*

$$\frac{d^n}{dz^n}\left\{F_{p,q}^{m}\ L\left(\xi,\zeta;\eta;z\right)\right\}=\frac{(\xi)_n(\zeta)_n}{(\eta)_n}F_{p,q}^{m}\ L\left(\xi+n,\zeta+n;\eta+n;\ z\right), \tag{34}$$

and

$$\frac{d^n}{dz^n}\left\{\phi_{p,q}^{m}\ L\left(\zeta;\eta;z\right)\right\}=\frac{(\zeta)_n}{(\eta)_n}\phi_{p,q}^{m}\ L\left(\zeta+n;\eta+n;\ z\right), \tag{35}$$

where

$$(p,q \geq 0,\ \ Re(\eta) > Re(\zeta) > 0);\ \ n \in \mathbb{N}_0.$$

Proof. We know well known relation of Euler-Beta function,

$$B(\zeta,\eta-\zeta) = \frac{\eta}{\zeta}\ B(\zeta+1,\eta-\zeta), \tag{36}$$

Differentiating (29) with respect to variable z, we get

$$\begin{aligned}\frac{d}{dz}\left\{F_{p,q}^{m}\ L(\xi,\zeta;\eta;z)\right\} &= \sum_{n=0}^{\infty}(\xi)_n\ \frac{B_{p,q}^{m}\ L(x,y;\zeta+n,\eta-\zeta)}{B(\zeta,\eta-\zeta)}\ \frac{z^{n-1}}{n-1!}\\ &= \sum_{n=0}^{\infty}(\xi)_{n+1}\ \frac{B_{p,q}^{m}\ L(x,y;\zeta+n+1,\eta-\zeta)}{B(\zeta,\eta-\zeta)}\ \frac{z^n}{n!},\end{aligned}$$

Using $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ and (36) in the above expression, we obtain

$$\frac{d}{dz}\left\{F_{p,q}^{m}\ L(\xi,\zeta;\eta;z)\right\} = \frac{\xi\zeta}{\eta}\sum_{n=0}^{\infty}(\xi+1)_n\ \frac{B_{p,q}^{m}\ L(x,y;\zeta+n+1,\eta-\zeta)}{B(\zeta+1,\eta-\zeta)}\ \frac{z^n}{n!},$$

where $(\alpha)_n$ is the Pochhammar symbol defined as

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases}1 & (n=0;\alpha\in\mathbb{C}\setminus\{0\})\\ \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1) & (n\in\mathbb{N};\alpha\in\mathbb{C}),\end{cases}$$

Now continuing the same process up-to $(n-1)$, we get the required result (34). Similarly, by applying the same process on (30), we get the required result (35). □

Remark 4. *If we take p = q = 1 and m =1 in the expression* (34) *and* (35)*, we obtain a similar result in [4].*

5.3. **Transformation formulas.**

Theorem 4. *The following formulae for the hypergeometric logarithmic and confluent hypergeometric logarithmic functions hold true:*

$$F_{p,q}^{m}\ L\left(\xi,\zeta;\eta;\ z\right) = (1-z)^{-\xi}\,F_{p,q}^{m}\ L\left(\xi,\eta-\zeta;\eta;\ -\frac{z}{1-z}\right), \tag{37}$$

$$F_{p,q}^{m}\ L\left(\xi,\zeta;\eta;\ 1-\frac{1}{z}\right) = z^{\xi}\,F_{p,q}^{m}L\left(\xi,\eta-\zeta;\eta;\ 1-z\right), \tag{38}$$

$$F_{p,q}^{m}\ L\left(\xi,\zeta;\eta;\ \frac{z}{1+z}\right) = (1+z)^{\xi}\,F_{p,q}^{m}\ L\left(\xi,\eta-\zeta;\eta;\ -z\right), \tag{39}$$

$$\Phi_{p,q}^{m}\ L\left(\zeta,\eta;\ z\right) = e^{z}\,\Phi_{p,q}^{m}\ L\left(\eta-\zeta;\eta;\ -z\right). \tag{40}$$

$$(p,q \geq 0, x, y, \in \mathbb{R}^{+};\ \ |z| < 1;\ Re(\eta) > Re(\zeta) > 0).$$

Proof. Substituting t by $1-t$ in $(1-zt)^{-\xi}$ and replacing the following equation

$$[1-z(1-t)]^{-\xi} = (1-z)^{-\xi}\left(1+\frac{z}{1-z}t\right)^{-\xi}$$

in (31) we obtain

$$F^{m}_{p,q}\ L(\xi,\zeta;\eta;\ z) = \frac{(1-z)^{-\xi}}{B(\zeta,\eta-\zeta)} \times \int_0^1 t^{\zeta-1}(1-t)^{\eta-\zeta-1}\left(1+\frac{z}{1-z}t\right)^{-\xi} exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\right]dt, \tag{41}$$

further, we have

$$F^{m}_{p,q}\ L(\xi,\zeta;\eta;\ z) = \frac{(1-z)^{-\xi}}{B(\zeta,\eta-\zeta)} \times \int_0^1 t^{\zeta-1}(1-t)^{\eta-\zeta-1}\left(1-\frac{-z}{1-z}t\right)^{-\xi} exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\right]dt. \tag{42}$$

In view of (31), we get the required result (37).
Substituting z by $1-\frac{1}{z}$ and $\frac{z}{1+z}$ in (37) yield (38) and (39) respectively. □

Similarly applying the same process in (37) by simple calculation, we can establish (40).

Theorem 5. *The following relation holds true:*

$$F^{m}_{p,q}\ L(\xi,\zeta;\eta;\ 1) = \frac{B^{m}_{p,q}\ (x,y;\xi,\eta-\xi-\zeta)}{B(\zeta,\eta-\zeta)} \tag{43}$$

$$(p,q\geq 0;\ x,y\in\mathbb{R}^+;\ Re(\eta-\xi-\zeta)>0).$$

Proof. Putting $z=1$ in (31) and using the definition (9), we obtain desired result (43). □

6. Generating function of $F^{m}_{p,q}L(\xi,\zeta;\eta;\ z)$

Theorem 6. *The generating function for $F^{m}_{p,q}\ L(\xi,\zeta;\eta;\ z)$ holds the underlying relation*

$$\sum_{k=0}^{\infty}(\xi)_k\ F^{m}_{p,q}\ L(\xi+k,\zeta;\eta;z)\,\frac{t^k}{k!} = (1-z)^{-\xi}\,F^{m}_{p,q}\left(\xi,\zeta;\eta;\ \frac{z}{1-t}\right) \tag{44}$$

$$(p,q\geq 0, |t|<1).$$

Proof. Let left hand side of (44) be denoted by L, then from (29), we have

$$L = \sum_{k=0}^{\infty} (\xi)_k \left(\sum_{n=0}^{\infty} \frac{(\xi + k)_n \, B_{p,q}^m \; L(x, y; \zeta + n, \eta - \zeta)}{B(\zeta, \eta - \zeta)} \frac{z^n}{n!} \right) \frac{t^k}{k!}.$$

Using the identity $(\alpha)_n(\alpha + n)_k = (\alpha)_k(\alpha + k)_n$, we get

$$L = \sum_{n=0}^{\infty} (\xi)_n \, \frac{B_{p,q}^m \; L(a, b; \zeta + n, \eta - \zeta)}{B(\zeta, \eta - \zeta)} \left(\sum_{k=0}^{\infty} (\xi + n)_k \frac{t^k}{k!} \right) \frac{z^n}{n!}.$$

Since, we know that $\sum_{n=0}^{\infty} (\xi + n)_n \, \frac{t^n}{n!} = (1 - t)^{-\xi - n}$, we obtain

$$L = \sum_{n=0}^{\infty} (\xi)_n \, \frac{B_{p,q}^m \; L(x, y; \zeta + n, \eta - \zeta)}{B(\zeta, \eta - \zeta)} (1 - t)^{-\xi - n} \, \frac{z^n}{n!}$$

$$L = (1 - t)^{-\xi} \sum_{n=0}^{\infty} (\xi)_n \, \frac{B_{p,q}^m \; L(x, y; \zeta + n, \eta - \zeta)}{B(\zeta, \eta - \zeta)} \left(\frac{z}{1 - t} \right)^n \frac{1}{n!}. \tag{45}$$

Finally by using (29) in (45), we get the desired result (44). □

7. Conclusions

In this article we define a (p, q)-beta logarithmic function which links with logarithmic mean and generalized beta function (see [3], [4]). Here, we analyze yet another extension of the Euler beta function and study a variety of properties, including integral representation, summation formula and derivative formula of the (p, q)-beta logarithmic function. Some analytical properties of this new extended function are developed and discuss its probabilistic concept as an application. Further, we get the beta distribution and the other statistical formula that go along with it. Finally, we expand the definition of hypergeometric and confluent hypergeometric function and explore the different features of the extended definition.

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