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Common Solutions to Stein Inequalities

Şerife YILMAZ[*1], Birgül AKSOY[2]

Abstract

In this paper for linear discrete time switched systems, the problem of existence of a common solution to Stein inequalities is considered. A sufficient condition for robust Schur stability of a matrix polyope by using Schur complement lemma and a necessary and sufficient condition for the existence of a common solution of Stein equation are given. As in the case of continuous time systems, the problem of existence of a common solution is reduced to a convex optimization one. An efficient solution algorithm which requires solving a linear minimax problem at each step is suggested. The algorithm is supported with a number of examples from the literature and observed that the method desired results fastly.

Keywords: Discrete-time system, Schur stability, Stein equation, common quadratic Lyapunov function, Schur complement

1. INTRODUCTION

Consider discrete-time system

$$x(k+1) = Ax(k), \ (k = 1, 2, 3, \dots) \qquad (1)$$

where $x(k) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$, where $\mathbb{R}^{n \times n}$ is the set of $n \times n$ dimensional real matrices. If all eigenvalues of A lie in the open unit disc of the complex plane then A is called Schur (stable) matrix. The Schur stability of A guarantees asymptotical stability of the system (1). An equivalent condition for the asymptotical stability of (1) is the existence of a positive definite matrix $P > 0$ such that

$$A^T PA - P < 0, \qquad (2)$$

where "< 0" means the negative definiteness. The matrix inequality (2) is called the Stein inequality and for $Q > 0$ the matrix equation $A^T PA - P = -Q$ is called the Stein equation for (1). In this case the function $V(x) = x^T Px$ is a discrete time Lyapunov function for system (1).

Consider discrete-time switched system

$$x(k+1) = A_i x(k) \qquad (3)$$

where A_i, $(i = 1, 2, \dots, N)$ are Schur matrices. If the following system of matrix inequalities

$$A_i^T PA_i - P < 0, \ (i = 1, 2, \dots, N) \qquad (4)$$

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is satisfied for a $P > 0$ then the positive definite matrix P is called a common solution and the function $V(x) = x^T Px$ is called common Lyapunov function of system (3). In this paper we consider the problem of existence and evaluation of a common Lyapunov function for (3).

Stein equations and inequalities have been investigated in many works (see [1-3]).

In [1], necessary and sufficient conditions for the existence of a common quadratic Lyapunov function (CQLF) are given for two second-order discrete-time systems. In [2], the existence of parameter-dependent quadratic Lyapunov functions for finite set of Hurwitz stable matrices is considered via LMI method. In [3], the relationship between Lyapunov and Stein equations is given by using the Cayley transformation. Common solutions to matrix inequalities corresponding to continuous-time switched systems have been investigated in [4-11]. In [4], it is shown that provided the CQLF for pairwise commutative matrices by solving a sequence of Stein equations. In [5], LMI method is given for the common solution of matrix inequalities. In [6], it is shown that a CQLF exists if the Lie-algebra generated by the matrices is solvable. In [7], necessary and sufficient conditions for the existence a CQLF for a pair of second-order linear continuous-time systems are derived, and a constructive procedure is described for finding a CQLF when it exists.

In this paper we investigate system of matrix inequalities (4) and give a necessary and sufficient condition for the existence of a common solution $P > 0$. Our approach is based on the results obtained in [8].

2. COMMON SOLUTIONS

Consider the following matrix polytope

$$\mathcal{A} = \{A(\alpha) = \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_N A_N : \alpha_1 + \alpha_2 + \cdots + \alpha_N = 1, 0 \le \alpha_i \le 1, i = 1, 2, \dots, N\} \quad (5)$$

with the A_i, $(i = 1, 2, \dots, N)$. Here the matrices A_i are called extreme (generator) matrices of the matrix polytope $\mathcal{A}$. Firstly we show that if $P > 0$ is a common solution to (4) then it is a common solution to all polytope $\mathcal{A}$, that is for all $A \in \mathcal{A}$ the matrix inequality

$$A^T PA - P < 0 \quad (6)$$

is satisfied.

In the Hurwitz case the proof of this statement is straightforward due to linearity of the Lyapunov inequality $A_i^T P + PA_i < 0$ with respect to A_i. In the Schur case we will use the following proposition named the Schur complement lemma.

Lemma 1 ([2]): Let a matrix $M \in \mathbb{R}^{(n+m)\times(n+m)}$ be partitioned as

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix},$$

where $M_1 \in \mathbb{R}^{n\times n}$, $M_2 \in \mathbb{R}^{n\times m}$, $M_3 \in \mathbb{R}^{m\times m}$, M_1 and M_3 are symmetric and M_3 is invertible. Then $M > 0 \Leftrightarrow M_3 > 0$ and $M_1 - M_2 M_3^{-1} M_2^T > 0$.

Theorem 1: Let the polytope $\mathcal{A}$ (5) be given and $P > 0$ satisfies (4). Then for all $A \in \mathcal{A}$ the inequality (6) is satisfied and consequently the family $\mathcal{A}$ is robustly Schur stable.

Proof: Take $M_1 = M_3 = P > 0$, $M_2 = A_i^T P$. Then

$$M_1 - M_2 M_3^{-1} M_2^T = P - A_i^T P P^{-1} (A_2^T P)^T = P - A_i^T P A_i > 0,$$

since P satisfies (4). By Lemma 1

$$\begin{bmatrix} P & A_i^T P \\ PA_i & P \end{bmatrix} > 0. \quad (7)$$

Multiplication both sides of (7) by α_i and summation from $i = 1$ to $i = N$ gives

$$\begin{bmatrix} P & A^T(\alpha)P \\ PA(\alpha) & P \end{bmatrix} > 0 \qquad (8)$$

(note that due to $\alpha_1 + \alpha_2 + \cdots + \alpha_N = 1$ there exists i such that $\alpha_i > 0$). Applying Lemma 1 in the reverse direction we obtain

$$A^T(\alpha)PA(\alpha) - P < 0$$

which proves that for all $A \in \mathcal{A}$ the inequality (6) is satisfied.

For a given matrix A the transformation

$$A \to \tilde{A} = (A + I)^{-1}(A - I)$$

is called the Cayley transformation, where I is the identity matrix.

If all eigenvalues of $A \in \mathbb{R}^{n\times n}$ lie in the open left half plane then A is called Hurwitz (stable) matrix.

Theorem 2 ([3]):
1) If A is Schur matrix then $\tilde{A}$ is Hurwitz matrix.

2) If $P > 0$ satisfies Lyapunov inequality $\tilde{A}^T P + P\tilde{A} < 0$ then the same P satisfies Stein inequality $A^T PA - P < 0$.

Due to Theorem 2 we have the following

Theorem 3: Assume that the matrices A_i, $(i = 1, 2, \ldots, N)$ are given and $\tilde{A}_i = (A_i + I)^{-1}(A_i - I)$. Then P satisfies (4) if and only if

$$\tilde{A}_i^T P + P\tilde{A}_i < 0 \;\; (i = 1, 2, \ldots, N). \qquad (9)$$

Now consider the existence and evaluation problem of $P > 0$ satisfying (4). By Theorem 3, we can construct a new family $\{\tilde{A}_1, \ldots, \tilde{A}_N\}$ which is Hurwitz and use algorithm from [8]. On the other hand, by definition, at this approach at each i the inverse of $(A_i + I)$ should be calculated and if an eigenvalue of A_i is near -1 then $\tilde{A}_i$ may have infinitely large entries which is an undesirable situation. An example of this situation is given below.

Example 1: Consider the matrix

$$A = \begin{bmatrix} -20.8043 & 21.8113 & 3.9371 \\ -17.8239 & 18.5309 & 2.8413 \\ 1.9749 & -2.0749 & -0.6846 \end{bmatrix}.$$

Its eigenvalues are

$$-0.999363, -0.984077, -0.974559$$

and

$$(A + I)^{-1}(A - I) = \begin{bmatrix} -9.34 \times 10^7 & 1.16 \times 10^8 & 1.15 \times 10^8 \\ -8.7 \times 10^7 & 1.08 \times 10^8 & 1.07 \times 10^8 \\ 1.23 \times 10^7 & -1.53 \times 10^7 & -1.52 \times 10^7 \end{bmatrix}.$$

Theorem 4: Let the matrix $A \in \mathbb{R}^{n\times n}$ be Schur matrix and there exists a symmetric matrix $P_* \in \mathbb{R}^{n\times n}$ such that $A^T P_* A - P_* < 0$. Then $P_* > 0$.

Proof: Denote $Q = P_* - A^T P_* A > 0$. Then $A^T P_* A - P_* = -Q$. This equality can be written as

$$[(A^T - I)P_*(A + I) + (A^T + I)P_*(A - I)] = -2Q$$

Left multiplication by $(A^T + I)^{-1}$, right multiplication by $(A + I)^{-1}$ gives (the inverses exist, since A is Schur matrix)

$$(A^T + I)^{-1}(A^T - I)P_* + P_*(A - I)(A + I)^{-1} = -2(A^T + I)^{-1}Q(A + I)^{-1}. \qquad (10)$$

Since the matrix $C = (A + I)^{-1}$ is nonsingular,

$$(A^T + I)^{-1}Q(A + I)^{-1} = [(A + I)^{-1}]^T Q(A + I)^{-1} = C^T QC > 0.$$

Since $(A - I)(A + I)^{-1} = (A + I)^{-1}(A - I)$, from (10) it follows that

$$\tilde{A}^T P_* + P_*\tilde{A} < 0. \qquad (11)$$

By Theorem 1 $\tilde{A}$ is Hurwitz matrix and by Lyapunov theorem $P_* > 0$.

3. THE EXISTENCE THEOREM

In this section we give number of propositions and the existence theorem without proofs. The proofs of these results are similar to the proofs in the Hurwitz case which have been carried out in [8], therefore are omitted.

Define $d = \frac{n(n+1)}{2}$, $X = [-1,1]^d$, variable symmetric matrix

$$S(x) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_{n+1} & \cdots & x_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{2n-1} & \cdots & x_d \end{bmatrix}$$

and convex positive homogeneous function

$$\begin{aligned} \varphi(x) = &\max_i \lambda_{\max}(A_i^T S(x) A_i - S(x)) \\ = &\max_i \max_{\|v\|=1} v^T [A_i^T S(x) A_i - S(x)] v \\ = &\max_{i,\|v\|=1} f(x,i,v), \end{aligned} \qquad (12)$$

where $f(x,i,v) = v^T [A_i^T S(x) A_i - S(x)] v$.

Proposition 1 ([8]): Take any point $\hat{x} \in X$. Let the pair $(\hat{\imath}, \hat{v})$ be a maximizing pair in (12) and $\hat{g}$ be the gradient vector of the linear map $x \mapsto f(x, \hat{\imath}, v)$. Then $\varphi(\hat{x}) = \langle \hat{x}, \hat{g} \rangle$ and $\varphi(x) \geq \langle x, \hat{g} \rangle$ for all $x \in X$.

Proposition 2: $\|\hat{g}\| \leq \sqrt{2}(M^2+1)$, where $M = \max_{1\leq i\leq N} \|A_i\|$, and $\|A_i\|$ is the Frobenius norm of A_i.

Proof: The norm $\|f\|$ of the linear functional $f(x,\hat{\imath},\hat{v}) = \langle x, \hat{g} \rangle$ equals $\|\hat{g}\|$ and therefore ([9], p. 188)

$$\begin{aligned} \|\hat{g}\| &= \|f\| \\ &= \max_{\|x\|\leq 1} |f(x,\hat{\imath},\hat{v})| \\ &= \max_{\|x\|\leq 1} \left|\hat{v}^T \left[A_{\hat{\imath}}^T S(x) A_{\hat{\imath}} - S(x)\right] \hat{v}\right| \\ &\leq \max_{\|x\|\leq 1} \left(\left|\hat{v}^T A_{\hat{\imath}}^T S(x) A_{\hat{\imath}} \hat{v}\right| + |\hat{v}^T S(x) \hat{v}| \right) \\ &\leq \max_{\|x\|\leq 1} \left(\left\|A_{\hat{\imath}}^T S(x) A_{\hat{\imath}}\right\| + \|S(x)\| \right) \\ &\leq (M^2+1) \max_{\|x\|\leq 1} \|S(x)\|. \end{aligned}$$

Since $\|S(x)\|^2 \leq 2(x_1^2 + x_2^2 + \cdots + x_d^2) \leq 2$ for $\|x\| \leq 1$, $\|\hat{g}\| \leq \sqrt{2}(M^2+1)$.

Let x^1 be any nonzero vector from X, and g^1 is calculated as in Proposition 1, $s_1 = \min_{x\in X} \langle x, g^1 \rangle$ and x^2 be a minimizer, that is $s_1 = \langle x^2, g^1 \rangle$. Having chosen $x_1, x_2, \dots, x_k$, let x^{k+1} be a minimizer in

$$s_k = \min_{x\in X} \max_{1\leq i\leq k} \langle x, g^i \rangle. \qquad (13)$$

Proposition 3: Assume that $x^1 \in X$, $x^1 \neq 0$. There exists a common solution $P > 0$ to (4) if and only if $\varphi(x^{k_*}) < 0$ for some $k_* \in \{1,2,3,\dots\}$. If $\lim_{k\to\infty} s_k = 0$ then common solution does not exist.

Solution of the linear minimax problem (13) can be obtained from the solution of the following linear programming (LP) problem.

Proposition 4: Let (x^*, t^*) be a solution of the following LP problem

$$\begin{aligned} & t \to \min \\ & -1 \leq x_i \leq 1, \qquad j = 1,2,\dots,d \\ & -\sigma \leq t \leq \sigma, \quad \sigma = \sqrt{2d}(M^2+1) \\ & \langle x, g^i \rangle \leq t \qquad i = 1,2,\dots,k. \end{aligned}$$

Then $s_k = t^*$ and x^* is minimizer in (12).

The proofs of Propositions 3 and 4 follows the same scheme as the proofs of Theorem 5 and Proposition 6 in [8].

The following examples are taken from [1] and there it has been analytically shown the existence of a common solution for Example

2 and the nonexistence of a common solution for Example 3.

Example 2 ([1]): Consider the following Schur stable matrices

$$A_1 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.35 & -0.25 \\ 0.50 & 0.85 \end{bmatrix}.$$

Let $x^1 = (1,0,0)^T \in X = [-1,1]^3$. Maximization of $\lambda_{max}\left(A_m^T S(x^1) A_m - S(x^1)\right)$ with respect to m gives $m = 2$ and

$$\lambda_{max}\left(A_2^T S(x^1) A_2 - S(x^1)\right) = 0.0705.$$

The unit eigenvector of $A_2^T S(x^1) A_2 - S(x^1)$ is

$$v^1 = (-0.091901, 0.995768)^T.$$

Since $\varphi(x^1) = 0.0705$, this process is continued for a new point x^2. The gradient vector is

$$g^1 = (0.0705, -0.2670, -0.3508)^T,$$

$$s_1 = \min_{x \in X} \langle x, g^1 \rangle = -0.688$$

and the minimizer is $x^2 = (-1,1,1)^T$. Repeating the procedure according (13) we obtain the following.

$k = 2$, $m = 2$,
$\varphi(x^2) = 1.520, s_2 = -0.419$,
$k = 3$, $m = 1$,
$\varphi(x^3) = -0.172 < 0$,

where $x^3 = (1, 0.521365, 1)^T$. Consequently by Proposition 3 the matrix

$$S(x^3) = \begin{bmatrix} 1 & 0.521365 \\ 0.521365 & 1 \end{bmatrix}$$

is a common solution to Stein equation for the matrices $\{A_1, A_2\}$.

Example 3 ([1]): Consider the Schur stable matrices

$$A_1 = \begin{bmatrix} 0.8 & 0 \\ -1 & -0.8 \end{bmatrix}, A_2 = \begin{bmatrix} 0.8 & 0 \\ 1 & -0.8 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0.95 & -0.08 \\ 0.1 & 0.95 \end{bmatrix}.$$

It is known that these matrices have no CQLF [1]. Starting from $x^1 = (1,0,0)^T \in X = [-1,1]^3$ and carrying out calculations as in Example 2 we obtain $s_k = 0$ for $k \geq 4$. Therefore there is no CQLF for $\{A_1, A_2, A_3\}$ by Proposition 3.

Example 4: Given Schur stable matrices

$$A_1 = \begin{bmatrix} 0.27 & 0.78 & -0.36 \\ 0 & -0.77 & 0.1 \\ -0.3 & -0.2 & -0.81 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.85 & 0.13 & -0.15 \\ 0.07 & -0.92 & 0 \\ 0.15 & 0 & -0.97 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0.2 & 0.97 & 0.21 \\ -0.4 & -1.2 & 0.1 \\ 0.81 & 0.5 & -0.6 \end{bmatrix}.$$

Let $x^1 = (1,0,0,0,0,0)^T \in X = [-1,1]^6$. Maximization of $\lambda_{max}\left(A_m^T S(x^1) A_m - S(x^1)\right)$ with respect to m gives $m = 3$ and $\varphi(x^1) = \lambda_{max}\left(A_3^T S(x^1) A_3 - S(x^1)\right) = 1.005$.

The unit eigenvector of $A_3^T S(x^1) A_3 - S(x^1)$ is

$$v^1 = (0.100500, 0.972409, 0.210521)^T.$$

Since $\varphi(x^1) > 0$, this process is continued for a new point x^2. The gradient vector is

$$g^1 = (1.005, -2.585, 0.846, 0.461, -1.456, 0.15)^T,$$

$s_1 = -6.505$ and the minimizer is $x^2 = (-1,1,-1,-1,1,-1)^T$. The calculations give the following values:

$k = 2,\quad m = 3,$
$\varphi(x^2) = 2.746 > 0, s_2 = -1.077,$
$k = 3,\quad m = 1,$
$\varphi(x^3) = 0.928 > 0, s_3 = -0.558,$
$\vdots \quad \vdots$
$k = 14,\quad m = 2,$
$\varphi(x^{14}) = 0.020 > 0, s_{14} = -0.010,$
$k = 15,\quad m = 3,$
$\varphi(x^{15}) = -0.001647 < 0,$

where
$x^{15} = (0.684, 0.533, -0.170, 1, -0.204, 0.254)^T$

and the matrix

$$S(x^{15}) = \begin{bmatrix} 0.684 & 0.533 & -0.170 \\ 0.533 & 1 & -0.204 \\ -0.170 & -0.204 & 0.254 \end{bmatrix}$$

is a common solution for the Schur stable matrices $\{A_1, A_2, A_3\}$.

4. CONCLUSION

In this paper, we considered the problem of existence and evaluation of a common solution to Stein equation for a finite number of real Schur stable matrices. We gave a sufficient condition for robust Schur stability of a matrix polyope by using Schur complement lemma and a necessary and sufficient condition for the existence of a common solution of Stein equation. In addition we gave a procedure that finds one of these solutions effectively in case a common solution of the Stein equation exists. We supported the method with a number of examples from the literature and observed that the method desired results fastly.

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Authors' Contribution
The authors contributed equally to the study.

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The authors of the paper declare that they comply with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that they do not make any falsification on the data collected. In addition, they declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

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