INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 16 NO. 1 PAGE 244–253 (2023)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1260464

Inequalities on Isotropic Submanifolds in Pseudo-Riemannian Space Forms

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

Spacelike and timelike isotropic submanifolds of pseudo-Riemannian spaces have interesting properties, with important applications in Mathematics and Physics. The article presents inequalities for isotropic spacelike and timelike submanifolds of pseudo-Riemannian space forms and isotropic Lorentzian submanifolds are also considered.

Keywords: *Isotropic submanifolds, spacelike submanifolds, timelike submanifolds, Lorentzian submanifolds, pseudo-Riemannian space forms.*
AMS Subject Classification (2020): *Primary: 53C40 ; Secondary: 53C25.*

1. Preliminaries

One of the most important part of Differential Geometry is represented by the pseudo-Riemannian geometry. Also, the theory of submanifolds in pseudo-Riemannian manifolds is studied intensively. In this article we focus on isotropic spacelike, timelike and Lorentzian submanifolds.

Important work on this topics was done by great geometers. This article is dedicated to the memory of professor Duggal and then we would like to mention here one of his important contribution in this direction ([7]).

In this Preliminaries section we give few basic definitions and formulae. In section 2 we recall inequalities for isotropic spacelike submanifolds of a pseudo-Riemannian space form, an analogue of the generalized Euler inequality and a Ricci inequality. A consequence gives a characterization for totally umbilical isotropic submanifolds. In section 3 we study the isotropic timelike submanifolds in pseudo-Riemannian space forms. Section 4 gives results on isotropic Lorentzian submanifolds. Examples are presented in the last section.

The notion of an *isotropic immersion* was introduced by B. O'Neill [8]. An *isotropic immersion* is defined as an isometric immersion such that all its normal curvature vectors have the same length. Obviously, this is a generalization of the totally umbilical immersion. For isotropic submanifolds in the pseudo-Riemannian settings we cite [1].

Let $(\tilde{M}^m_\nu, g)$ be a pseudo-Riemannian manifold of dimension m and signature ν and $\phi : M^n_s \to \tilde{M}^m_\nu$ an isometric immersion of a pseudo-Riemannian manifold M^n_s of dimension n and signature s.

If $s = 0$, then M^n_0 is called a *spacelike* submanifold.

If $s = 1$, then M^n_1 is called a *Lorentzian* submanifold.

If $s = n$, then M^n_n is called a *timelike* submanifold.

We use the common notations for H - the mean curvature vector, R - the Riemannian curvature tensor, K - the sectional curvature, Ric - the Ricci curvature, τ - the scalar curvature, etc.

The immersion ϕ is called *pseudo-umbilical* if its second fundamental form h satisfies $g(h(X,Y),H) = \rho g(X,Y)$, for some function ρ. It follows that $\rho = g(H,H)$.

Also, ϕ is a *totally umbilical* immersion if any point $p \in M^n_s$ is *umbilic*, i.e., there exists a vector $\xi_p \in T^\perp_p M^n_s$ such that for all $u, v \in T_p M^n_s$ one has $h(u,v) = g(u,v)\xi_p$. Any totally umbilical immersion is pseudo-umbilical.

Received : *05–03–2023,* ***Accepted :*** *29–03–2023*
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The isometric immersion ϕ is called *isotropic at* $p \in M_s^n$ (see [1]) if

$$g(h(u,u), h(u,u)) = \lambda(p) \in \mathbb{R}$$

does not depend on the choice of the unit vector $u \in T_pM_s^n$ and ϕ is said to be *isotropic* if it is isotropic at any point of M_s^n.

2. Isotropic spacelike submanifolds in pseudo-Riemannian space forms

In [2] the following inequality (known as a *generalized Euler inequality* (G.E.)) for spacelike submanifolds was proven by Chen:

Corollary 2.1. *Let N be a spacelike submanifold of an indefinite real space form $R_r^{n+r}(c)$ of constant curvature c. Then*

$$||H||^2 \leq \frac{2\tau}{n(n-1)} - c, \tag{G.E.}$$

$n = \dim N$, $n \geq 2$, *with equality holding at a point $p \in N$ if and only if p is a totally umbilical point.*

In [4] the authors considered isotropic spacelike submanifolds of a pseudo-Riemannian space form and the following proposition was proven.

Proposition 2.1. *Let M_0^n be an isotropic spacelike submanifold of a pseudo-Riemannian space form $\tilde{M}_s^m(c)$. Then*

$$3ng(H,H) = (n+2)\lambda + 2(n-1)(\rho - c), \tag{2.1}$$

where $\rho = \frac{2\tau}{n(n-1)}$ is the normalized scalar curvature.

The following consequence of this result coincides with Corollary 4.3. from [1]:

Corollary 2.2. *Let $\phi : M_s^n(k) \to \tilde{M}_s^m(c)$ be an isotropic immersion of a pseudo-Riemannian space form $M_s^n(k)$ into a pseudo-Riemannian space form $\tilde{M}_s^m(c)$. Then*

$$3ng(H,H) = (n+2)\lambda + 2(n-1)(k - c).$$

In particular, if $k = c$ then $\lambda = 0$ if and only if $g(H,H) = 0$.

Remark 2.1. Corollary 4.3 from [1] also proved that ϕ is pseudo-umbilical, i.e. $g(h(X,Y), H) = \rho g(X,Y)$, for a function ρ. In our case, $\rho = g(H,H)$.

On the other hand, by using the (G.E.) inequality, in [4] the next proposition was proven.

Proposition 2.2. *Let M_0^n be an isotropic spacelike submanifold of a pseudo-Riemannian space form $\tilde{M}_s^{n+s}(c)$. Then $g(H,H) \geq \lambda$.*

Then, from Chen's Corollary (inequality (G.E.)) and Proposition 2.2. we obtained in [4] the following characterization of a totally umbilical isotropic spacelike submanifold of a pseudo-Riemannian space form.

Theorem 2.1. *Let M_0^n be an isotropic spacelike submanifold of a pseudo-Riemannian space form $\tilde{M}_s^{n+s}(c)$. Then M_0^n is totally umbilical if and only if $g(H,H) = \lambda$.*

Recall that for $X \in T_pM_0^n$ a unit vector and $\{X = e_1, e_2, ..., e_n\}$ an orthonormal basis of $T_pM_0^n$, the Ricci curvature of X is defined by

$$Ric(X) = \sum_{i=2}^{n} K(X \wedge e_i),$$

where $K(X \wedge e_i)$ is the sectional curvature of the plane spanned by X and e_i.

In [4] the authors proved the following Ricci inequality and the characterization of the equality case.

Theorem 2.2. *Let M_0^n be an isotropic spacelike submanifold of a pseudo-Riemannian space form $\tilde{M}_s^{n+s}(c)$ of constant curvature c. Then one has the following Ricci inequality:*

$$Ric(X) \geq \tau + \frac{1}{4}n(n+2)\lambda - \frac{n^2}{2}g(H,H) - \frac{1}{2}(n-1)(n-2)c. \tag{2.2}$$

The equality case holds for any vector $X \in T_pM_0^n$ if and only if p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.

Also, the following corollary was obtained in [4] :

Corollary 2.3. *Let M_0^n be an isotropic spacelike submanifold of $\mathbb{E}_s^{n+s}$. Then*

$$\min Ric(X) \geq \tau + \frac{1}{4}n(n+2)\lambda - \frac{n^2}{2}g(H,H). \tag{2.3}$$

3. Isotropic timelike submanifolds in pseudo-Riemannian space forms

In this section we will continue the previous study and consider the isotropic timelike submanifolds in pseudo-Riemannian space forms.

One denotes by $(\tilde{R}(Z,X)Y)^T$ the tangential component of $\tilde{R}(Z,X)Y$. We recall the formula

$$3A_{h(X,Y)}Z = \lambda[g(X,Y)Z + g(Y,Z)X + g(X,Z)Y] \tag{3.1}$$
$$+R(Z,X)Y - (\tilde{R}(Z,X)Y)^T + R(Z,Y)X - (\tilde{R}(Z,Y)X)^T,$$

for $X, Y, Z \in \mathcal{X}(M_s^n)$, which is an equivalent condition for the submanifold M_s^n to be isotropic (see [1], Theorem 3.3.).

A similar result with Corollary 2.1. from the previous section can be imediately proven for timelike submanifolds, by using the same technique.

Corollary 3.1. *Let N be a timelike submanifold of an indefinite real space form $R_r^{n+r}(c)$ of constant curvature c. Then*

$$||H||^2 \geq \frac{2\tau}{n(n-1)} - c, \tag{G.E.}$$

$n = \dim N$, $n \geq 2$, with equality holding at a point $p \in N$ if and only if p is a totally umbilical point.

This represents a corresponding *generalized Euler inequality* for timelike submanifolds.

Proposition 3.1. *Let M_n^n, $n \geq 2$, be an isotropic timelike submanifold of a pseudo-Riemannian space form $\tilde{M}_s^m(c)$. Then*

$$3ng(H,H) = (n+2)\lambda + 2(n-1)(\rho - c), \tag{3.2}$$

where $\rho = \frac{2\tau}{n(n-1)}$ is the normalized scalar curvature.

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_pM_n^n$, $p \in M_n^n$.

From (3.1) we have

$$3g(A_{h(X,Y)}Z, W) = \lambda[g(X,Y)g(Z,W) + g(Y,Z)g(X,W) + g(X,Z)g(Y,W)] \tag{3.3}$$
$$+g(R(Z,X)Y,W) - g(\tilde{R}(Z,X)Y,W) + g(R(Z,Y)X,W) - g(\tilde{R}(Z,Y)X,W),$$

for $X, Y, Z, W \in \mathcal{X}(M_n^n)$. For $X = Y = e_i$, $Z = W = e_j$ we obtain from (3.3):

$$3g(A_{h(e_i,e_i)}e_j, e_j) = \lambda[g(e_i,e_i)g(e_j,e_j) + g(e_i,e_j)g(e_i,e_j) + g(e_i,e_j)g(e_i,e_j)] \tag{3.4}$$
$$+g(R(e_j,e_i)e_i,e_j) - g(\tilde{R}(e_j,e_i)e_i,e_j) + g(R(e_j,e_i)e_i,e_j) - g(\tilde{R}(e_j,e_i)e_i,e_j).$$

By summation after $i, j \in \{1, ..., n\}$ we get

$$3n^2 g(H,H) = \lambda(n^2 + 2n) + 4\tau - 2n(n-1)c, \tag{3.5}$$

where the mean curvature vector H is given by $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$, the scalar curvature τ is calculated by $\tau = \sum_{1\le i<j\le n} g(R(e_j, e_i)e_i, e_j)$ and the curvature of $\tilde{M}_s^m(c)$ is $c = g(\tilde{R}(e_j, e_i)e_i, e_j), i \neq j$.

From (3.5), after dividing by n, one obtains

$$3ng(H,H) = (n+2)\lambda + \frac{4\tau}{n} - 2(n-1)c,$$

which is equivalent with (3.2), the equality to prove.

□

Proposition 3.2. *Let M_n^n, $n \ge 2$, be an isotropic timelike submanifold of a pseudo-Riemannian space form $\tilde{M}_n^{n+r}(c)$. Then $g(H,H) \le \lambda$.*

Proof. From Corollary 3.1. we get

$$g(H,H) \ge \rho - c,$$

with equality holding if and only if M_n^n is a totally umbilical submanifold.

From Proposition 3.1. we have

$$\rho - c = \frac{3ng(H,H) - (n+2)\lambda}{2(n-1)}.$$

Then

$$g(H,H) \ge \frac{3ng(H,H) - (n+2)\lambda}{2(n-1)},$$

which implies

$$[2(n-1) - 3n]g(H,H) \ge -(n+2)\lambda,$$

equivalent with $g(H,H) \le \lambda$. □

It follows that, combining Corollary 3.1. and Proposition 3.2. the following theorem is proved:

Theorem 3.1. *Let M_n^n, $n \ge 2$, be an isotropic timelike submanifold of a pseudo-Riemannian space form $\tilde{M}_n^{n+r}(c)$. Then M_n^n is totally umbilical if and only if $g(H,H) = \lambda$.*

The previous theorem represents a characterization of totallly umbilical isotropic timelike submanifolds of pseudo-Riemannian space forms.

Next result contains a Ricci inequality for isotropic timelike submanifolds of pseudo-Riemannian space forms and characterization of the equality case.

Theorem 3.2. *Let M_n^n, $n \ge 2$, be an isotropic timelike submanifold of a pseudo-Riemannian space form $\tilde{M}_n^{n+r}(c)$ of constant curvature c. Then one has the following Ricci inequality:*

$$Ric(X) \le \tau + \frac{1}{4}n(n+2)\lambda - \frac{n^2}{2}g(H,H) - \frac{1}{2}(n-1)(n-2)c. \tag{3.6}$$

The equality case holds for any vector $X \in T_pM_n^n$ if and only if p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.

Proof. Let M_n^n be an isotropic timelike submanifold of a pseudo-Riemannian space form $\tilde{M}_n^{n+r}(c)$. Denote by $\{X = e_1, e_2, \ldots, e_n\}$ an orthonormal basis of $T_pM_n^n$, $p \in M_n^n$.

By Gauss equation, we have

$$Ric(X) = \tau - \frac{1}{2}(n-1)(n-2)c + \sum_{q=n+1}^{n+r} \sum_{2\le i<j\le n} [h_{ii}^q h_{jj}^q - (h_{ij}^q)^2], \tag{3.7}$$

where h_{ij}^q are the coefficients of the second fundamental form.
On the other hand, the formula (6) from [1] gives the relation

$$n^2 g(H,H) = n(n+2)\lambda - 2g(h,h), \tag{3.8}$$

for every isotropic immersion, where $g(h,h)$ is expressed by

$$g(h,h) = \sum_{i,j=1}^{n} \epsilon_i \epsilon_j g(h(e_i,e_j), h(e_i,e_j)), \epsilon_i = g(e_i,e_i) = -1, \forall i \in \{1,...,n\}.$$

By using (3.7), we can write

$$n^2 g(H,H) = n(n+2)\lambda - 2\sum_{q=n+1}^{n+r} [(h_{11}^q)^2 + (h_{22}^q + \ldots + h_{nn}^q)^2 \tag{3.9}$$

$$+2\sum_{1\le i<j\le n} (h_{ij}^q)^2] + 4\sum_{q=n+1}^{n+r} \sum_{2\le i<j\le n} h_{ii}^q h_{jj}^q$$

$$= n(n+2)\lambda + 4\tau - 4Ric(X) - 2(n-1)(n-2)c$$

$$-\sum_{q=n+1}^{n+r} [(h_{11}^q + \ldots + h_{nn}^q)^2 + (h_{11}^q - h_{22}^q - \ldots - h_{nn}^q)^2 - 4\sum_{j=2}^{n} (h_{1j}^q)^2]$$

$$\le n(n+2)\lambda + 4\tau - 4Ric(X) - 2(n-1)(n-2)c - n^2 g(H,H).$$

Then (3.9) is equivalent with

$$Ric(X) \le \tau + \frac{1}{4}n(n+2)\lambda - \frac{n^2}{2}g(H,H) - \frac{1}{2}(n-1)(n-2)c.$$

The equality case holds for X if and only if the inequality in (3.9) becomes an equality, i.e., $h_{1j}^q = 0$ and $h_{11}^q = h_{22}^q + \ldots + h_{nn}^q$, for any $q \in \{n+1,...,n+r\}$.
The equality case holds for every vector in $T_pM_n^n$ if and only if

$$h_{ij}^q = 0, \forall 1 \le i \ne j \le n, \forall q \in \{n+1,...,n+r\},$$

$$2h_{ii}^q = h_{11}^q + h_{22}^q + \ldots + h_{nn}^q; \forall q \in \{n+1,...,n+r\}.$$

Summing the above second equations, we get

$$(n-2)(h_{11}^q + h_{22}^q + \ldots + h_{nn}^q) = 0, \forall q \in \{n+1,...,n+r\}.$$

We distinguish 2 cases:
i) $n \ne 2$; then p is a totally geodesic point;
ii) $n = 2$; then p is a totally umbilical point.

□

Corollary 3.2. *Let M_n^n, $n \ge 2$, be an isotropic timelike submanifold of $\mathbb{E}_n^{n+r}$. Then*

$$\max Ric(X) \le \tau + \frac{1}{4}n(n+2)\lambda - \frac{n^2}{2}g(H,H). \tag{3.10}$$

4. Isotropic Lorentzian Submanifold of Pseudo-Riemannian Space Form

In this section we prove two geometric inequalities (Ricci inequalities) and characterizations of their equality cases for isotropic Lorentzian submanifolds in a pseudo-Riemannian space form.

For an isotropic Lorentzian submanifold M_1^n of a pseudo-Riemannian space form, we distinguish 2 cases:

Case 1. $M_1^n \subset \tilde{M}_1^{n+r}(c)$, $r \geq 1$, $n \geq 2$.

Case 2. $M_1^n \subset \tilde{M}_{s+1}^{n+s}(c)$, $s \geq 1$, $n \geq 2$.

Case 1. Let $p \in M_1^n$. Denote by $\{X = e_1, e_2, ..., e_n\}$ an orthonormal basis of $T_pM_1^n$, where $X = e_1$ is a timelike vector and $e_2, ..., e_n$ are orthonormal spacelike vectors.

Then

$$g(X, X) = g(e_1, e_1) = -1,$$

$$g(e_i, e_i) = 1, i \in \{2, ..., n\},$$

$$g(X, e_i) = g(e_1, e_i) = 0, i \in \{2, ..., n\}.$$

From relation (3.1), for $X = Y, Z = W = e_i, i \in \{2, .., n\}$ we obtain:

$$3g(A_{h(X,X)}e_i, e_i) = \lambda[g(X, X)g(e_i, e_i) + g(X, e_i)g(X, e_i) + g(X, e_i)g(X, e_i)]$$

$$+2g(R(e_i, X)X, e_i) - 2g(\tilde{R}(e_i, X)X, e_i).$$

We have $g(X, X) = g(e_1, e_1) = -1$ and $g(X, e_i) = 0, \forall i \in 2, ..., n.$

It follows that, by summation after $i \in 2, ..., n$, we get:

$$3g(h(X, X), \sum_{i=2}^{n} h(e_i, e_i)) = -(n-1)\lambda - 2\sum_{i=2}^{n} K(X \wedge e_i) + 2(n-1)c,$$

which is equivalent with:

$$3g(h(X, X), \sum_{i=2}^{n} h(e_i, e_i)) = -(n-1)\lambda - 2Ric(X) + 2(n-1)c,$$

from where we obtain

$$2Ric(X) = -3g(h(X, X), \sum_{i=2}^{n} h(e_i, e_i)) - (n-1)\lambda + 2(n-1)c. \tag{4.1}$$

On the other hand,

$$g(h(X, X), \sum_{i=2}^{n} h(e_i, e_i)) = \sum_{q=1}^{r} (h_{11}^q \sum_{i=2}^{n} h_{ii}^q), \tag{4.2}$$

where $h_{ij}^q = g(h(e_i, e_j), e_{n+q}), i, j \in \{1, ..., n\}, q \in \{1, ..., r\}.$

We will use an algebraic result; more precisely, in [5] the following Lemma (Lemma 2.3.) holds:

Lemma 4.1. *Let $f(x_1, x_2, ..., x_n)$ be a function on $\mathbb{R}^n$ defined by*

$$f(x_1, x_2, ..., x_n) = x_1 \sum_{j=2}^{n} x_j - x_1^2.$$

If $x_1 + x_2 + ... + x_n = 4a$, we have

$$f(x_1, x_2, ..., x_n) \leq \frac{1}{8}(x_1 + x_2 + ... + x_n)^2.$$

The equality sign holds if and only if $x_1 = a, x_2 + ... + x_n = 3a$.

Then we have from (4.2):

$$g(h(X,X),\sum_{i=2}^{n}h(e_i,e_i)) = h^1_{11}(h^1_{22}+h^1_{33}+...+h^1_{nn})$$

$$+h^2_{11}(h^2_{22}+h^2_{33}+...+h^2_{nn})+...+h^r_{11}(h^r_{22}+h^r_{33}+...+h^r_{nn})$$

$$\leq \frac{1}{8}(h^1_{11}+h^1_{22}+...+h^1_{nn})^2+(h^1_{11})^2$$

$$+\frac{1}{8}(h^2_{11}+h^2_{22}+...+h^2_{nn})^2+(h^2_{11})^2$$

$$..$$

$$+\frac{1}{8}(h^r_{11}+h^r_{22}+...+h^r_{nn})^2+(h^r_{11})^2$$

$$=\frac{1}{8}n^2||H||^2+||h(X,X)||^2=\frac{1}{8}n^2||H||^2+\lambda.$$

Applying this inequality to (4.1) one obtains:

$$Ric(X)\geq -\frac{3}{16}n^2||H||^2-\frac{n+2}{2}\lambda+(n-1)c.$$

Then we proved the following:

Theorem 4.1. *Let M^n_1, $n\geq 2$, be an isotropic Lorentzian submanifold of a pseudo-Riemannian space form $\tilde{M}^{n+r}_1(c)$. Then one has the following inequality:*

$$Ric(X)\geq -\frac{3}{16}n^2||H||^2-\frac{n+2}{2}\lambda+(n-1)c,$$

where X is a unit timelike vector tangent to M^n_1.

The equality case holds for the vector $X\in T_pM^n_1$ if and only if $nH=4h(X,X)$ at p.

Case 2. We study isotropic Lorentzian submanifolds M^n_1 of a pseudo-Riemannian space form $\tilde{M}^{n+s}_{s+1}(c)$

Let $p\in M^n_1$. One considers $\{e_1,e_2,...,e_n\}$ a basis of $T_pM^n_1$ and $\{e_{n+1},...,e_{n+s}\}$, $s\geq 1$ a basis of $T^{\perp}_pM^n_1$.

Obviously $e_{n+1},...,e_{n+s}$ are unit normal timelike vectors. By using the similar technique as in the previous subsection, one obtains:

Theorem 4.2. *Let M^n_1, $n\geq 2$, be an isotropic Lorentzian submanifold of a pseudo-Riemannian space form $\tilde{M}^{n+s}_{s+1}(c)$. Then one has the following inequality:*

$$Ric(X)\geq \frac{3}{16}n^2g(H,H)-\frac{n-4}{2}\lambda+(n-1)c,$$

where X is a unit timelike vector tangent to M^n_1.

The equality case holds for the vector $X\in T_pM^n_1$ if and only if $nH=4h(X,X)$ at p.

5. Examples

The following theorem was obtained in [6] :

Theorem 5.1. *Let M be an isotropic Lorentzian Lagrangian submanifold of an indefinite complex space form $\tilde{M}^n_1(4c)$, with $c\in\{-1,1\}$. Let p be a non totally geodesic point of M. Then in a neighborhood of p, M is locally isometric with a warped product manifold $I\times_{\frac{1}{\alpha}}N$. Moreover, if t is the standard variable on I, with $<\frac{\partial}{\partial t},\frac{\partial}{\partial t}>=-\epsilon$, then α depends only on t and satisfies the following ordinary differential equation:*

$$(\log\alpha)^{''}-((\log\alpha)^{'})^2=-c\epsilon-8\alpha^6$$

and N is a real space form with constant sectional curvature $\frac{c-4\epsilon\alpha^6-\epsilon((\log\alpha)^{'})^2}{\alpha^2}$.

The H-umbilical submanifolds (in the sense of Chen) generalize isotropic submanifolds. For $c = 0$, the following corollaries were proven in [3] (for the definitions of H-umbilical submanifolds and complex extensors please see [3]):

Corollary 5.1. *Let M be a Lagrangian submanifold of $\mathbb{C}^n_k$ with $n \geq 3$ and $k \geq 1$. Then, up to rigid motions, M is an open portion of a Lagrangian pseudo-Riemannian sphere or of a Lagrangian pseudo-hyperbolic space if and only if M is a Lagrangian H-umbilical submanifold with nonzero constant curvature.*

Corollary 5.2. *Let $L : M \to \mathbb{C}^n_1$ be a Lagrangian H-umbilical immersion in the Lorentzian complex Euclidean n-space with $n \geq 3$.*
(i) If M is of constant curvature, then, up to rigid motions of $\mathbb{C}^n_1$, one of the following three cases occurs:
(i-a) M is a flat Lorentzian n-manifold.
(i-b) M is an open portion of a Lagrangian hyperbolic space in $\mathbb{C}^n_1$.
(i-c) M is an open portion of a Lagrangian de Sitter spacetime in $\mathbb{C}^n_1$.
(ii) If M contains no open subset of constant curvature, then, up to rigid motions, L is locally one of the following two Lagrangian submanifolds:
(ii-a) L is a complex extensor of the unit hyperbolic space H^{n-1} via a unit speed curve in $\mathbb{C}^*$.
(ii-b) L is a complex extensor of the unit de Sitter spacetime S^{n-1}_1 via a unit speed curve in $\mathbb{C}^*$.

In the previous results, some characterizations of isotropic submanifolds in a pseudo-Riemannian space forms of signature 2 are given.

In our present article, we have obtained in section 4 two geometric inequalities (Ricci inequalities) and characterizations of their equality cases for isotropic Lorentzian submanifolds in pseudo-Riemannian space forms of signatures 1, respectively $s + 1$.

We will give an example of an isotropic Lorentzian submanifold of Lorentzian complex Euclidean 4-space which verifies the same equality case of the inequality proven in Theorem 4.1.

Let $M^4_1 = \mathbb{R} \times S^3$ be a Lorentzian submanifold of the pseudo-Riemannian space form $\mathbb{C}^4_1$. On M^4_1 we consider the metric

$$g = \cosh 2t(-dt^2 + g_0),$$

where g_0 is the standard Riemannian metric on S^3 (induced by the standard Euclidean inner product on $\mathbb{R}^4$).

Following [1], we define an isometric immersion (for $n = 4$) by

$$\phi : \mathbb{R} \times S^3 \to \mathbb{C}^4_1, \quad \phi(t, x_1, x_2, x_3, x_4) = \gamma(t)(x_1, x_2, x_3, x_4),$$

where

$$\gamma : \mathbb{R} \to \mathbb{C}$$

is the smooth plane curve

$$\gamma(t) = \sqrt{\cosh 2t}\exp\{i \arctan(\tanh t)\},$$

or, equivalently,

$$\gamma(t) = \sqrt{\cosh 2t}\left(\frac{1}{\sqrt{1 + (\tanh t)^2}} + i\frac{\tanh t}{\sqrt{1 + (\tanh t)^2}}\right).$$

Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis of $T_pM^4_1$, with

$$e_1 = \frac{1}{|\gamma|}(\partial_t, 0),$$
$$e_i = \frac{1}{|\gamma|}(0, \overline{e}_i), i \in \{2, 3, 4\},$$

where $\{\overline{e}_2, \overline{e}_3, \overline{e}_4\}$ is an orthonormal basis of the corresponding tangent space of S^3.

With respect to this orthonormal basis, the second fundamental form h of ϕ has the expression:

$$h(e_1, e_1) = h(e_2, e_2) = h(e_3, e_3) = h(e_4, e_4) = -aJe_1,$$

$$h(e_1, e_i) = -aJe_i, \ i \in \{2, 3, 4\},$$

$$h(e_j, e_k) = 0, \ 2 \leq j \neq k \leq 4,$$

where $a = -(\cosh 2t)^{-\frac{3}{2}}$ and J is the standard complex structure on $\mathbb{C}_1^4$.

This implies that ϕ is an isotropic immersion with the isotropy function $\lambda = -a^2$.

For $i \in \{2, 3, 4\}$, we compute the following sectional curvatures:

$$\begin{aligned} K(e_1 \wedge e_i) &= - < h(e_1, e_1), h(e_i, e_i) > + < h(e_1, e_i), h(e_1, e_i) > \\ &= - < -aJe_1, -aJe_1 > + < -aJe_i, -aJe_i > \\ &= a^2 + a^2 = 2a^2. \end{aligned}$$

Then $Ric(e_1) = 6a^2$.

If we introduce the above values in the relation given by Theorem 4.1. we remark that:

$$6a^2 = -\frac{3}{16}4^2(-a^2) - \frac{4+2}{2}(-a^2),$$

which means that the equality case is verified.

At the same time, from $H = -aJe_1$ one obtains

$$4H = -4aJe_1 = 4(-aJe_1) = 4h(e_1, e_1),$$

i.e. we verified once more the characterization of the equality case of the inequality given by Theorem 4.1.

Starting from this example, we expect that a similar Ricci inequality and characterization of its equality case can be done for isotropic Lorentzian submanifolds in pseudo-Riemannian space forms of signature 2.

To find examples of isotropic Lorentzian submanifolds in pseudo-Riemannian space forms of signature 1 satisfying the equality case in Theorem 4.1. remains an open problem.

Acknowledgements

The author would like to thank his Ph.D. supervisor, Prof. Dr. Adela Mihai, for proposing the subject. The author sincerely thanks to the Editor Prof. Dr. Cihan Ozgur and to the reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The author declares that he has no competing interests.

Author's contributions

Not applicable.

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