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A NOTE ON FRIENDLY AND SOLITARY GROUPS

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Abstract. In this paper, we extend the notions of friendly and solitary numbers to group theory and define friendly and solitary groups of type-1 and type-2. We provide many examples of friendly and solitary groups and study certain properties of the type-2 friends of cyclic p-groups, where p is a prime number.

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1. Introduction and main results

1.1. Introduction and definitions. In number theory, two or more natural numbers are said to be friendly if the abundancy index (the ratio between the sum of divisors of a number and the number itself) of each of these numbers is same. If the abundancy index of a number is not equal to that of any other natural number, then that number is solitary. We refer to [1] for more details on friendly and solitary numbers. The main aim of this paper is to extend these number-theoretic notions to group theory. To be more precise, in this paper, we define the notions of friendly and solitary groups and study their properties. Let us first discuss the motivation and logic behind defining these notions.

One of the oldest problems in number theory is related to perfect numbers, i.e., the numbers having abundancy index 2 (see [7,13] for a nice survey). The concept of perfect numbers is extended to that of the concepts of friendly numbers and Leinster groups that have drawn the attention of many researchers since they offer several interesting open problems (see [9,13]). For example, to find out an odd perfect number is a long-standing open problem but it has been shown that there exists a Leinster group of odd order.

It is known that the friendly numbers form a club [1], which helps in studying the properties of a subset of natural numbers rather than any particular natural number. We note that the concept of friendly numbers can be straight-forwardly defined for cyclic groups (see Definition 1.1), however, the club formation in the case of groups will be larger (in terms of size of the club) than that in the case of the numbers (this is because a cyclic group may have a non-cyclic friend, which is not the case with numbers). Therefore, one can study the properties of a larger club that includes cyclic as well as non-cylic groups together. To be more precise, we know only a few particular classes of non-abelian groups such as dihedral groups, quaternion groups, simple groups etc. However, our study would help in the formation of a club(s) (or a new class of groups) that includes cyclic as well as non-cyclic groups with at least one similarity, i.e., the common abundancy index. An interesting work in this direction is done in [14], where a class of groups having abundancy index less than or equal to 2 is considered and several characteristics of the whole class are studied.

To this end, we discuss a particular situation where the new larger clubs may play an important role. One of the well-known problems in abstract algebra is to compute the Wedderburn decomposition (WD) of a semisimple group algebra (see [11]). Using [12, Theorem 2.5], we know that the number of normal subgroups of a group plays an important role in uniquely deducing the WD. Therefore, by knowing the abundancy index of a club, results similar to that of [12] may be studied on a whole club rather than only on particular groups (as done in [14]). This drives us to define the notions of friendly and solitary groups and study their properties.

Next, we define the notions of friendly and solitary groups. Let G be a cyclic group of order n. Then it is well known that if $d \mid n$, there always exists a unique subgroup of G of order d [5]. This means that the notion of friendly numbers can be extended trivially for a cyclic group. However, for any natural number n, there may be a cyclic or a non-cyclic group of order n. Consequently, we define friendly groups of type-1 and type-2 as follows:

Definition 1.1. Friendly groups of type-1: Two or more finite cyclic (non-cyclic) groups are said to be *friendly of type-1* if the ratio between the sum of orders of their normal subgroups and the order of the group is same. Mathematically, let G_1 and G_2 be two finite cyclic (or non-cyclic) groups of order n_1 and n_2. Then G_1 and G_2 are said to be friendly if

$$\frac{\sum\limits_{H_1 \triangleleft G_1} |H_1|}{n_1} = \frac{\sum\limits_{H_2 \triangleleft G_2} |H_2|}{n_2}.$$

Definition 1.2. Friendly group of type-2: A cyclic (non-cyclic) group is said to be *friendly group of type*-2 with a non-cyclic (cyclic) group if the ratio between the sum of orders of their normal subgroups and the order of the group is same.

We discuss some friendly groups of type-1 and type-2 in the following example.

Example 1.3. (1) Cyclic groups C_{30} and C_{140} form a friendly pair of type-1 having abundancy index $\frac{12}{5}$. Further, cyclic groups C_{2480} and C_{6200} are also friendly with both C_{30} and C_{140}.
(2) Cyclic groups of odd orders can also be friendly with each other. For instance, C_{135} and C_{819} are type-1 friendly with abundancy index $\frac{16}{9}$.
(3) Cyclic groups of odd order can be friendly with cyclic groups of even order. For instance, C_{42} and C_{544635} are type-1 friendly with abundancy index $\frac{16}{7}$.
(4) $C_3 \times S_3$ and $(C_3 \times C_3) \rtimes C_2$ are two non-abelian type-1 friendly with abundancy index $\frac{20}{9}$.
(5) Groups having abundancy index 2 are also known as Leinster groups [9,10]. Thus, all abelian Leinster groups are friendly of type-1 with each other. For example, C_6 and C_{28} are all type-1 friendly Leinster groups with abundancy index 2.
(6) $C_5 \times S_3$ and C_6 are type-2 friendly Leinster with abundancy index 2.

Next, we define the notion of Solitary groups of type-1 (or type-2) and present their examples.

Definition 1.4. Solitary group of type-1 or type-2: A cyclic (non-cyclic) group which is not a friend of any cyclic (non-cyclic) group is called *solitary group of type*-1 (2).

Example 1.5. (1) Trivial group is a solitary group of both type-1 and type-2.
(2) All cyclic p-groups are solitary of type-1 (see [1]). More generally, all cyclic groups of order n with $(n, \sigma(n)) = 1$ are solitary of type-1. Here $\sigma(n)$ denotes the sum of all the divisors of n.
(3) C_{18} is a solitary group of type-1 with $(18, \sigma(18)) \neq 1$, see [1].

Since the converse of Lagrange's theorem is true for cyclic groups, we have already discussed that studying friendly cyclic groups of type-1 is equivalent to studying the friendly numbers. However, the study of friendly non-cyclic groups of type-1 or friendly groups of type-2 is entirely a new area considered in this paper.

Next, let us first discuss the importance of a computational algorithm to compute the abundancy index of a group. For cyclic groups, we know that the computation of abundancy index is same as that of natural numbers. However, for a non-cyclic group, in general, there is no straight-forward formula that tells about the number of normal subgroups and their order. Consequently, the computation of abundancy index for a non-cyclic group is a challenging task. We present a simple GAP [6] algorithm for the same in Appendix section. Its importance can be understood through the following example.

Example 1.6. The group $C_2 \times C_2 \times C_2 \times C_2$ has 67 normal subgroups and the group $C_2 \times C_2 \times C_2 \times C_2 \times C_2$ has 374 normal subgroups which are very difficult to compute manually.

Using Algorithm 1, we can see that abundancy index of the group $C_5 \rtimes C_4$ (its GAP identity is SmallGroup$(20, 3)$) is $\frac{9}{5}$ which means that $C_5 \rtimes C_4$ is a type-2 friend of C_{10}. This shows that a cyclic group can have a friend of type-2 even if its type-1 friend is not known [3].

1.2. Properties of abundancy index. Next, we discuss some properties of the abundancy index $A(G)$ of any finite group G. Let us define the set $A := \{A(G) \mid G$ is any group of finite order$\}$. Clearly, A is the set of abundancy indexes of all finite groups.

Proposition 1.7. *The following hold.*

(1) *For any non-trivial group, $A(G) > 1$ and $A(G) = 1$ iff $G = \{1\}$.*
(2) *For any cyclic group G of order n, $A(G) = \frac{\sigma(n)}{n}$, where $\sigma(n)$ is sum of positive divisors of n.*
(3) *For any simple group G, $A(G) = \frac{1+|G|}{|G|}$.*
(4) *For any normal subgroup N of G, $A(G/N) \leq A(G)$.*
(5) *For any finite groups G_1 and G_2 of co-prime orders, we have $A(G) = A(G_1) \times A(G_2)$, where $G = G_1 \times G_2$.*
(6) *The set A has infimum 1 but unbounded above.*

Proof. It is trivial to observe that the points (1)-(2) hold. For a simple group, only normal subgroups are identity and the whole group itself. Thus, (3) holds. Next, if N is normal in G, then the normal subgroups of G/N are of the form H/N, where H is a normal subgroup of G containing N. Consequently, $A(G/N) \leq A(G)$. Next, we prove point (5). Let a_1 and a_2 be the abundancy indexes of G_1 and G_2, respectively. This means

$$a_1 = \frac{\sum\limits_{H_1 \lhd G_1} |H_1|}{|G_1|}, \quad a_2 = \frac{\sum\limits_{H_2 \lhd G_2} |H_2|}{|G_2|}.$$

For the group $G = G_1 \times G_2$, the complete set of its normal subgroups is of the form $\{(H_1 \times H_2) : H_1 \lhd G_1, H_2 \lhd G_2\}$ since $\gcd(|G_1|, |G_2|) = 1$. This means that

$$A(G) = \frac{\sum\limits_{H_1 \lhd G_1, H_2 \lhd G_2} |H_1||H_2|}{|G_1||G_2|} = a_1 a_2 \quad \text{as} \quad \gcd(|G_1|, |G_2|) = 1.$$

This proves point (5). Finally, we prove point (6). We know that the alternating group A_n is simple for every $n > 5$. Using point (3), we have $A(A_n) = \frac{1+\frac{n!}{2}}{\frac{n!}{2}}$.

For $n \to \infty$, clearly $A(A_n) \to 1$ which together with point (1) imply that 1 is the infimum of A. Next, let us suppose that A be bounded above and $K \in \mathbb{Q}$ be its least upper bound. This means that there exists at least one group G_1 such that $A(G_1) > K-1$. Let $p_1, p_2, \cdots$ be the enumeration of primes in the increasing order starting from 2 and let p_t be the largest prime that appear in the prime factorization of $|G_1|$. Let G_2 be a cyclic group of order $p_{t+1}p_{t+2}\cdots p_r$, where r will be defined later on. Clearly, we have $\gcd(|G_1|, |G_2|) = 1$. Then, for $G = G_1 \times G_2$, point (5) derives that

$$A(G) = A(G_1) \times A(G_2) > (K-1)\left(1 + \frac{1}{p_{t+1}} + \frac{1}{p_{t+2}} + \cdots + \frac{1}{p_r}\right).$$

It follows from the last inequality that

$$A(G) > K \quad \text{provided} \quad (K-1)\left(\frac{1}{p_{t+1}} + \frac{1}{p_{t+2}} + \cdots + \frac{1}{p_r}\right) > 1.$$

Since the series $\sum\limits_{p \text{ prime}} \frac{1}{p}$ is divergent (so does its sub-series $\sum_{i=t+1}^{\infty} \frac{1}{p_i}$), we conclude that there exists a positive integer l such that $\sum_{i=t+1}^{l} \frac{1}{p_i} > \frac{1}{K-1}$. Therefore, by taking $r = l$, we note that $A(G) > K$, which is a contradiction. Therefore, the set A is unbounded above. This completes the proof. $\square$

1.3. Type-2 friends of cyclic p-groups. Since it is known that all cyclic p-groups have no type-1 friends, we are interested in their type-2 friends. In other words, cyclic p-groups are solitary of type-1. But we can find a sequence of abundancy indexes of cyclic groups which converges to the abundancy index of a cyclic p-group. We call it as *friend at infinity.*

Lemma 1.8. *The cyclic group C_{3^2} has a type-1 friend at infinity.*

Proof. Let $n_e = 3 \cdot 13^e$ for $e \geq 1$. Then, $A(n_e) = A(3) \times A(13^e) = \frac{4}{3} . \frac{(13^{e+1}-1)}{13^e.12}$. Using this we get

$$\lim_{e\to\infty} A(n_e) = \lim_{e\to\infty} \frac{(13^{e+1}-1)}{12.13^e} \cdot \frac{4}{3} = \frac{13}{9} = A(C_9).$$

This completes the proof. $\square$

We are now interested in looking for the type-2 friends of a cyclic p-group.

Lemma 1.9. *A cyclic p-group can not have an abelian type-2 friend.*

Proof. Let G be a friend of a cyclic group of order p^α. Then,

$$A(G) = \frac{1 + p + p^2 + \ldots + p^{\alpha-1} + p^\alpha}{p^\alpha} \leq \frac{2 \cdot p^\alpha}{p^\alpha} = 2.$$

Therefore, by Leinster's abelian quotient theorem [9], if G is abelian then it must be cyclic. Hence, G can not be a type-2 friend of a cyclic p-group. $\square$

By virtue of Lemma 1.9, in order to identify the type-2 friends of a cyclic p-group, we need to look into the class of non-abelian groups. For a cyclic group of order p, first we characterize the order of its type-2 friend in the following lemma.

Lemma 1.10. *Let G be a cyclic p-group of order p. Then any type-2 friend G_1 of G must be of order pn, where n is the sum of orders of all normal subgroups of G_1 except the whole group.*

Proof. Let G_1 be a type-2 friend of G. Then

$$A(G_1) = \frac{\sum\limits_{H_1 \triangleleft G_1} |H_1|}{|G_1|} = \frac{1+p}{p} = A(G).$$

This means that

$$|G_1| = p \times \frac{\sum\limits_{H_1 \triangleleft G_1} |H_1|}{p+1}.$$

Since $\gcd(p, p+1) = 1$, we conclude that $|G_1| = pn$, where n is a positive integer. Next, we explicitly find the value of n. Let x be the sum of orders of all non-trivial and proper normal subgroups of G_1. Then, we have

$$A(G_1) = \frac{1+x+np}{np} = \frac{1+p}{p} = A(G).$$

This means that $1 + x = n$. □

Next, we study certain properties of type-2 friends of a cyclic group of order p in the following propositions and theorems.

Proposition 1.11. *If G is a type-2 friend of a cyclic group of order p, then the following holds:*

(1) *G can not be simple,*
(2) *p does not divide $|G/G'|$, where G' denotes the commutator subgroup of G,*
(3) *For any prime divisor q of $|G/G'|$, $q > p$.*

Proof. If G is simple, then using Lemma 1.10 and Proposition 1.7, we have $A(G) = \frac{1+pn}{pn} \neq \frac{1+p}{p}$ for any positive integer $n \geq 2$. For (2), we suppose $p \mid |G/G'|$. Then (4) of Proposition 1.7 yields $A(G/G') \leq A(G)= \frac{1+p}{p} < 2$. By virtue of [9], it follows that G/G' is cyclic. Consequently, $A(G/G') > \frac{1+p}{p}$ which is not true. Finally, we prove (3). For any prime divisor q of $|G/G'|$, there must be a normal subgroup of G/G' of order q. Hence, we have $\frac{1+q}{q} < A(G/G') \leq A(G)=\frac{1+p}{p}$. This gives $q > p$. □

Proposition 1.12. *Let G be a type-2 friend of C_p having order pn such that p is an odd prime and $n \in \mathbb{Z}^+$, where $p \nmid n$. Then Sylow p-subgroup is normal in G.*

Proof. We have the following two cases depending on whether n is even or odd.
Case 1: n is odd.
We show the result via induction on the order of group G. It is straight-forward to see that the result is true for groups of order $p, 3p, 5p$. Let it be true for all such groups of order less than pn. By Fiet-Thompson Theorem [4], G is solvable. Then G' is a proper normal subgroup of G and $p \mid |G'|$ (cf. Proposition 1.11). In particular, the Sylow p-subgroup P is normal in G' and hence it is normal in G. This is because any two Sylow p-subgroups are conjugate in G and hence conjugate in G' as G' is normal in G.
Case 2: n is even.
Let $n = 2^{\alpha}m$, where $\alpha \geq 1$ and m odd. We again show the result via induction. One can note that the result is true for groups of order $2p, 4p, 6p$ (if $|G| = 4p$, where p is an odd prime, then G is solvable [2]. As $p \mid |G'|$ (cf. Proposition 1.11), $|G'| = p$ or $2p$. In either case, Sylow p-subgroup is normal in G). Suppose it is true for all such groups of order less than pn. As G is non-simple, let N be a maximal normal subgroup of G. Then G/N is either a cyclic group of prime order or a non-abelian simple group. So, either $|G/N| = q$, where $q \neq p$ for some prime divisor q of n or G/N is a non-abelian simple group. In either case, we have $|N| = p \cdot 2^{\beta} \cdot m_1$, where $2^{\beta}m_1 < n$. Consequently, the induction hypothesis implies that Sylow p-subgroup is normal in N and hence normal in G. This completes the proof. $\square$

Theorem 1.13. *Let G be a non-abelian group of order pn, where $p \nmid n$. Then G can not be a type-2 friend of C_p.*

Proof. Suppose that G is a type-2 friend of C_p. If $p = 2$, then G has a normal subgroup of order n (subgroup of all odd order elements). Hence, Lemma 1.10 gives $n = 1 + x \geq 1 + n > n$, where x is the sum of order of all non-trivial and proper normal subgroups of G. But this is a contradiction. Hence p can not be 2.

If $p \neq 2$, then by Proposition 1.12, Sylow p-subgroup P is normal in G. Therefore, by Schur-Zassenhaus Theorem [8], $G = P \rtimes_{\phi} G_1$, where G_1 is a subgroup of G of order n and $\phi : G_1 \to \mathrm{Aut}(P)$ is a group homomorphism that represents the action (by conjugation) of G_1 on P. Here $\mathrm{Aut}(P)$ represents the automorphism group of P. If ϕ is a trivial homomorphism, then G_1 is a normal subgroup of G and so $n = 1 + x \geq 1 + n > n$, which is a contradiction. If ϕ is non-trivial, then $(p-1) \mid n$ as $\mathrm{Aut}(P) \cong C_{p-1}$. Hence, n can not be odd. We note that the kernel of ϕ, i.e., $\mathrm{Ker}(\phi) \neq \{\mathrm{e}\}$, where e is the identity of G_1 (if $\mathrm{Ker}(\phi) = \{\mathrm{e}\}$, then $n \leq p - 1$ and hence Sylow p-subgroup is normal and $n = 1 + x \geq 1 + p > 1 + n > n$, which is not possible). Thus, ϕ induces an injective homomorphism $\tilde{\phi} : G_1/\mathrm{Ker}(\phi) \hookrightarrow C_{p-1}$, which implies that $|G_1/\mathrm{Ker}(\phi)|$ divides $p-1$. So, $|\mathrm{Ker}(\phi)| = \frac{nr}{p-1}$ for some positive integer r. Therefore, we have

$$n = 1 + x \geq 1 + p\left(\frac{nr}{p-1}\right) > 1 + nr > n,$$

which is a contradiction. Hence, G can not be a type-2 friend of C_p. $\square$

Theorem 1.14. *If G is a non-abelian group of order pn, where $p \mid n$, then G can not be a type-2 friend of C_p if the Sylow p-subgroup is either cyclic or its automorphism group is itself a p-group.*

Proof. Let G be a type-2 friend of C_p and $n = p^\alpha m$ for some $\alpha \geq 1$ and m is a positive integer coprime to p. First, we show that Sylow p-subgroup is normal in G by induction on the order of group. The result is true for groups of order $p^{\alpha+1}, 2p^{\alpha+1}, 3p^{\alpha+1}$ for any positive integer α (if $|G| = 3 \cdot 2^{\alpha+1}$, then G is solvable [2]. As $2 \nmid |G/G'|$ (cf. Proposition 1.11), $|G'| = 2^{\alpha+1}$ and hence Sylow p-subgroup is normal in G). As G is not simple, let N be a maximal normal subgroup of G. Then either $|G/N| = q$ for some prime $q \mid n$ or G/N is a non-abelian simple group. By induction hypothesis, $|N| = p^{\alpha+1}m_1$, for some $m_1 \mid m$ and Sylow p-subgroup P is normal in N and hence normal in G. Therefore, by Schur-Zassenhaus Theorem [8], $G = P \rtimes_\varphi G_1$, where G_1 is a subgroup of G of order m and φ denotes the action of G_1 on P by conjugation. Now, if φ is trivial, then G_1 is normal in G and $G/G_1 \cong P$. So,

$$p^\alpha m = n = 1 + x \geq 1 + m(1 + p + p^2 + \cdots + p^\alpha) \geq 1 + mp^\alpha > mp^\alpha = n,$$

which is absurd.

Next, we note that if $\mathrm{Aut}(P)$ is a p-group, then φ is trivial as p is coprime to m. If P is a cyclic p-group, then $\mathrm{Aut}(P) \cong C_{p^\alpha(p-1)}$ for p odd. Therefore, $|G_1/\mathrm{Ker}(\varphi)| \mid (p-1)p^\alpha$, which further implies that $|G_1/\mathrm{Ker}(\varphi)| \mid (p-1)$. So, $|\mathrm{Ker}(\varphi)| = \frac{ms}{p-1}$ for some positive integer s. Hence,

$$p^\alpha m = n = 1 + x \geq 1 + p^{\alpha+1}\left(\frac{ms}{p-1}\right) = 1 + p^\alpha m\left(\frac{ps}{p-1}\right) > n,$$

which is absurd. Thus, result holds. $\square$

Because of our main results, we are in a position to make the following conjecture:

Conjecture 1.15. *All cyclic p-groups for prime p are solitary of type-2.*

It is found that abelian (non-cyclic) p-groups are not solitary of type-2, in general. For example:

(1) $C_5 \times C_5$ and $C_{2^2 . 5^2 . 31} = C_{3100}$ are type-2 friends with abundancy index $\frac{56}{25}$.

(2) $C_2 \times C_2$ and $C_{2^9 \cdot 3 \cdot 31} = C_{47616}$ are type-2 friends with abundancy index $\frac{11}{4}$.

Appendix: In this section, we give a GAP algorithm to compute the abundancy index of any finite group. Let the order of group be n and let its GAP identity be SmallGroup(n, r) for some positive integer r. Comments are enclosed in between %.

Algorithm 1. **Algorithm to compute abundancy index of any group**.

```
G :=SmallGroup(n, r);
N :=NormalSubgroups(G);              % this lists all normal subgroups of G%
t :=Size(N);                         % total number of normal subgroups of G%
s := 0;                              % setting the initial counter to 0%
% initiating for loop to find the sum of orders of normal subgroups%
for i in [1..t] do s := s+Size(N[i]);   % N[i] is the ith normal subgroup of G.%
od;                                  % end for loop%
s := s/Size(G);                      % s is the required abundancy index. % □
```

2. Discussion

We have introduced the notions of friendly groups of type-1 and type-2 and solitary groups and discussed several examples of these groups. Further, we focused on the type-2 friends of cyclic p-groups and characterize the various possibilities for type-2 friends of the cyclic group C_p, where p is a prime number. This paper can be extended in a number of directions. One of the possible extensions is to look for type-2 friends of cyclic p-groups apart from C_p. Moreover, one can also work on the characterization of type-1 friends of non-cyclic groups.

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