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Research Article

The Bochner-Schoenberg-Eberlein module property for amalgamated duplication of Banach algebras

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Abstract

The Bochner-Schoenberg-Eberlein module property on commutative Banach algebras is a property related to extensions of multipliers on Banach algebras to module morphisms from Banach algebras into Banach modules. In this paper, we answer the problem (1) raised in [J. Algebra Appl., 21(8) (2022), 2250155, DOI: 10.1142/S0219498822501559]. We show that the Banach $\mathcal{A}\bowtie\mathfrak{A}$-module $X\times Y$ (X is a Banach $\mathcal{A},\mathfrak{A}$-module and Y is a Banach $\mathfrak{A}$-module) has a BSE-module property if and only if X is a BSE Banach $\mathcal{A},\mathfrak{A}$-module and Y is a BSE Banach $\mathfrak{A}$-module.

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. An $\mathcal{A}$-module morphism of $\mathcal{A}$ into X is called a multiplier of X and we denote it by $\mathcal{M}(\mathcal{A},X)$. If $T\in\mathcal{M}(\mathcal{A},X)$, then there exists a unique vector field $\widehat{T}$ on $\Delta(\mathcal{A})$ such that $\widehat{T(a)}=a\widehat{T}$, for all $a\in\mathcal{A}$. The notion of multipliers from Banach algebras into Banach modules is thoroughly investigated by Daws in [6]. A mapping $T:\mathcal{A}\longrightarrow\mathcal{A}$ is a *left (resp., right) multiplier* of $\mathcal{A}$ if $T(ab)=aT(b)$ ($T(ab)=T(a)b$), for all $a,b\in\mathcal{A}$. We denote the set of all left (resp., right) multipliers on $\mathcal{A}$ by $\mathcal{M}_l(\mathcal{A})$ (resp., $\mathcal{M}_r$). Moreover, T is called a *multiplier* of $\mathcal{A}$ if it is both left and right multiplier and the set of all multipliers of $\mathcal{A}$ is denoted by $\mathcal{M}(\mathcal{A})$, see [22], for more details related to multipliers on various versions of Banach algebras. A Banach algebra $\mathcal{A}$ is said to be *without order* if $x\mathcal{A}=\{0\}$ or $\mathcal{A}x=\{0\}$, then $x=0$. A bounded continuous function σ on $\Delta(\mathcal{A})$ is called a *BSE-function* if there exists a constant $C>0$ such that for every finite number of $\varphi_1,\dots,\varphi_n\in\Delta(\mathcal{A})$ and complex numbers $c_1,\dots,c_n$, the inequality

$$\left|\sum_{i=1}^{n}c_i\sigma(\varphi_i)\right|\leq C\left\|\sum_{i=1}^{n}c_i\varphi_i\right\|_{\mathcal{A}^*}$$

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holds, where $\mathcal{A}^*$ is the first dual of $\mathcal{A}$. The *BSE-norm* of σ i. e., $\|\cdot\|_{BSE}$ is defined to be the infimum of all such C. The set of all BSE-functions is denoted by $C_{BSE}(\Delta(\mathcal{A}))$. A Banach algebra $\mathcal{A}$ is called *BSE-algebra* if the BSE-functions on $\Delta(\mathcal{A})$ are precisely the Gel'fand transforms of the elements of $\mathcal{M}(\mathcal{A})$, i.e., $\widehat{\mathcal{M}(\mathcal{A})} = C_{BSE}(\Delta(\mathcal{A}))$. This notion is introduced by Takahasi and Hatori in [25] and it is characterized by Kaniuth and Ülger in [21]. There are many literatures that they have contained interesting results of BSE-algebras, see [1–4, 11, 12, 14–20, 26], for more details.

Takahasi in [24] generalized the BSE-property to Banach modules. Let A be a commutative Banach algebra with a bounded approximate identity and X be a symmetric Banach A-bimodule, i.e., $a \cdot x = x \cdot a$, for all $a \in A$ and $x \in X$. Let $\varphi \in \Delta(A)$. Denote $\ker \varphi$ by $M_\varphi = \{a \in A : \varphi(a) = 0\}$. There exists $e_\varphi \in A$ such that $\varphi(e_\varphi) = 1$. Now, define

$$X^\varphi = \overline{\mathrm{sp}}\{M_\varphi X + (1 - e_\varphi)X\},$$

where $\overline{\mathrm{sp}}$ is the closed linear span. Note that X^φ is independent of choice of e_φ. Then X^φ becomes a Banach A-submodule of X. Now define $X_\varphi = X/X^\varphi$ and $\hat{x}(\varphi) = x + X^\varphi$, for all $x \in X$. Hence, X_φ becomes a Banach A-bimodule. Let $\prod X_\varphi$ be the class of all functions σ defined on $\Delta(A)$ such that $\sigma(\varphi) \in X_\varphi$. An element of $\prod X_\varphi$ is called a *vector field* on $\Delta(A)$. The space $\prod X_\varphi$ is an A-module by the following action

$$(a \cdot \sigma)(\varphi) = \varphi(a)\sigma(\varphi), \quad (a \in A, \varphi \in \Delta(A), \sigma \in \prod X_\varphi).$$

Set

$$\prod{}^b X_\varphi = \left\{\sigma \in \prod X_\varphi : \|\sigma\|_\infty = \sup_{\varphi \in \Delta(A)} \|\sigma(\varphi)\| < \infty\right\}.$$

For each $\varphi \in \Delta(A)$, define $\pi_\varphi(x) = \hat{x}(\varphi)$, for all $x \in X$. A vector field $\sigma \in \prod X_\varphi$ is called *BSE* if there exists $\beta \in \mathbb{R}^+$ such that for any finite number of $\varphi_1, \ldots, \varphi_n \in \Delta(A)$ and the same number $f_1 \in (X_{\varphi_1})^*, \ldots, f_n \in (X_{\varphi_n})^*$, we have

$$\left|\sum_{i=1}^n \langle \sigma(\varphi_i), f_i\rangle\right| \leq \beta \left\|\sum_{i=1}^n f_i \circ \pi_{\varphi_i}\right\|_{X^*},$$

where $(X_{\varphi_i})^*$ denotes the dual space of the Banach space X_{φ_i}. Moreover, set

$$\prod{}_{\mathrm{BSE}} X_\varphi = \left\{\sigma \in \prod X_\varphi : \sigma \text{ is BSE}\right\}.$$

A vector field $\sigma \in \prod X_\varphi$ is called continuous if it is continuous at every $\varphi \in \Delta(A)$. The class of all continuous vector fields in $\prod X_\varphi$ is denoted by $\prod{}^c X_\varphi$ and set $\prod^c_{\mathrm{BSE}} X_\varphi = \prod_{\mathrm{BSE}} X_\varphi \bigcap \prod^c X_\varphi$. Let $\widehat{X} = \{\hat{x} : x \in X\}$ and $\widehat{\mathcal{M}(\mathcal{A}, X)} = \{\widehat{T} : T \in \mathcal{M}(\mathcal{A}, X)\}$. A Banach $\mathcal{A}$-module X is called *BSE* if $\widehat{M(A,X)} = \prod^c_{\mathrm{BSE}} X_\varphi$, for all $\varphi \in \Delta(A)$. In [24], some examples of Banach algebras that have BSE module property such group algebras on locally compact groups are given and in [2] authors characterized module property of module extensions of Banach algebras.

Let $\mathcal{A}$ and $\mathfrak{A}$ be two Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with the left and right compatible actions of $\mathfrak{A}$ on $\mathcal{A}$, i.e., for all $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$,

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \quad \text{and} \quad a(\alpha \cdot b) = (a \cdot \alpha)b.$$

Also, $\mathcal{A}$ is called a commutative Banach $\mathfrak{A}$-bimodule if $a \cdot \alpha = \alpha \cdot a$, for all $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. The amalgamated duplication of $\mathcal{A}$ along $\mathfrak{A}$, denoted by $\mathcal{A} \bowtie \mathfrak{A}$ is defined as the Cartesian product $\mathcal{A} \times \mathfrak{A}$ with the algebra product

$$(a, \alpha)(b, \beta) = (ab + \alpha \cdot b + a \cdot \beta, \alpha\beta),$$

and with the norm $\|(a, \alpha)\| = \|a\|_{\mathcal{A}} + \|\alpha\|_{\mathfrak{A}}$, for all $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathfrak{A}$. The Banach algebra $\mathcal{A} \bowtie \mathfrak{A}$ is introduced by Javanshiri and Nemati in [13] in light of D'Anna and Fontana work related to amalgamated duplication of a ring along an ideal [5]. Some results related to these algebras are given in [7,8,10]. In this paper we need the following results on $\mathcal{A} \bowtie \mathfrak{A}$:

Lemma 1.1. *[13, Lemma 3.1] If $\varphi \in \Delta(\mathcal{A})$, then there exists a unique linear functional $\widetilde{\varphi}$ in $\Delta(\mathfrak{A}) \cup \{0\}$ such that*

$$\varphi(a \cdot \beta) = \varphi(\beta \cdot a) = \varphi(a)\widetilde{\varphi}(\beta) \qquad (a \in \mathcal{A}, \beta \in \mathfrak{A}).$$

In particular, if either $\langle \mathcal{A} \cdot \mathfrak{A} \rangle = \mathcal{A}$ or $\langle \mathfrak{A} \cdot \mathcal{A} \rangle = \mathcal{A}$, then $\widetilde{\varphi} \neq 0$.

Proposition 1.2. *[13, Proposition 3.3] Let*

$$E := \{(\varphi, \widetilde{\varphi}) : \varphi \in \Delta(\mathcal{A})\} \quad \text{and} \quad F := \{(0, \psi) : \psi \in \Delta(\mathfrak{A})\}.$$

Set $E = \emptyset$ (respectively, $F = \emptyset$) if $\Delta(\mathcal{A}) = \emptyset$ (respectively, $\Delta(\mathfrak{A}) = \emptyset$). Then E and F are disjoint and $\Delta(\mathcal{A} \bowtie \mathfrak{A}) = E \cup F$.

According to [13, Remark 3.1], $\mathcal{A} \bowtie \mathfrak{A}$ is a commutative Banach algebra if and only if $\mathcal{A}$, $\mathfrak{A}$ are commutative Banach algebras and $\mathcal{A}$ is a symmetric Banach $\mathfrak{A}$-bimodule. In [9], authors investigated BSE property of $\mathcal{A} \bowtie \mathfrak{A}$ in a special case that $\mathcal{A}$ possess a nonzero idempotent that does not lie in the kernel of any character of $\mathcal{A}$. Authors in [9] asked the following problems:

(1) Let X and Y be Banach $\mathcal{A}$ and $\mathfrak{A}$-modules, respectively. Under which conditions $X \times Y$ is BSE Banach $\mathcal{A} \bowtie \mathfrak{A}$-module?
(2) Under which conditions $\mathcal{A} \bowtie \mathfrak{A}$ is BSE Banach $\mathcal{A} \bowtie \mathfrak{A}$-module?

We answer the problem (1) in the next section and problem (2) remains open because it is complicated and at this time we could not answer it.

2. BSE-module property of Banach $\mathcal{A} \bowtie \mathfrak{A}$-modules

In this section, we investigate the BSE-module property of $\mathcal{A} \bowtie \mathfrak{A}$-modules. Throughout this section, $\mathcal{A}$ and $\mathfrak{A}$ are commutative Banach algebras with bounded approximate identities such that $\mathcal{A}$ is a symmetric $\mathfrak{A}$-bimodule and $\langle \mathcal{A} \cdot \mathfrak{A} \rangle = \mathcal{A}$, by E and F, we mean the sets are obtained in Proposition 1.2 and X and Y are Banach $\mathcal{A}$-$\mathfrak{A}$ and $\mathfrak{A}$-modules, respectively. We consider $X \times Y$ as a Banach $\mathcal{A} \bowtie \mathfrak{A}$-module by the following module action:

$$(a, \alpha) \cdot (x, y) = (a \cdot x + \alpha \cdot x, \alpha \cdot y),$$

for all $(a, \alpha) \in \mathcal{A} \bowtie \mathfrak{A}$ and $(x, y) \in X \bowtie Y$. Moreover, we consider X as a Banach $\mathcal{A} \bowtie \mathfrak{A}$-module by the module action $(a, \alpha) \cdot x = a \cdot x + \alpha \cdot x$, for all $(a, \alpha) \in \mathcal{A} \bowtie \mathfrak{A}$ and $x \in X$. The existence of a bounded approximate identity for $\mathcal{A} \bowtie \mathfrak{A}$ is investigated in [8, Proposition 2.2 (ii)] and it was shown that $(a_\varpi, \beta_\varpi)_\varpi$ is a bounded approximate identity of $\mathcal{A} \bowtie \mathfrak{A}$ if and only if $\|a_\varpi\|_{\mathcal{A}} \longrightarrow 0$, $(\beta_\varpi)_\varpi$ is a bounded approximate identity of $\mathcal{A}$ in $\mathfrak{A}$ i. e., $a \cdot \beta_\varpi \to a$, for every $a \in \mathcal{A}$, in norm of $\mathcal{A}$. So, in the rest of this section we have assume that $\mathcal{A} \bowtie \mathfrak{A}$ has a bounded approximate identity. The proof of the following result is clear and so we have omitted it.

Lemma 2.1. *For any $\varphi \in E$ and $\psi \in F$, $M_{(0,\psi)} = M_\psi$ and*

$$M_{(\varphi,\widetilde{\varphi})} = (M_\varphi \times \{0\}) \cup \left(\{0\} \times M_{\widetilde{\varphi}}\right) \cup \left(M_\varphi \times M_{\widetilde{\varphi}}\right) \cup \{(a, \alpha) : \varphi(a) = -\widetilde{\varphi}(\alpha)\}.$$

By $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A}, X)$, we mean the space of all continuous linear maps such as $T : \mathcal{A} \longrightarrow X$ such that $T(\alpha \cdot a) = \alpha \cdot T(a)$, for all $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$.

Lemma 2.2. *$T \in \mathcal{M}(\mathcal{A} \bowtie \mathfrak{A}, X \times Y)$ if and only if there exist $T_{\mathcal{A},X} \in \mathcal{M}(\mathcal{A}, X) \cap \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A}, X)$, $T_{\mathfrak{A},X} \in \mathcal{M}(\mathfrak{A}, X)$ and $T_{\mathfrak{A},Y} \in \mathcal{M}(\mathfrak{A}, Y)$ such that*

$$T((a, \alpha)) = \left(T_{\mathcal{A},X}(a) + T_{\mathfrak{A},X}(\alpha), T_{\mathfrak{A},Y}(\alpha)\right), \tag{2.1}$$

for all $(a, \alpha) \in \mathcal{A} \bowtie \mathfrak{A}$.

Proof. Consider the mappings $\imath_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A} \rtimes \mathfrak{A}$ by $\imath_{\mathcal{A}}(a) = (a, 0)$, $\imath_{\mathfrak{A}} : \mathfrak{A} \longrightarrow \mathcal{A} \rtimes \mathfrak{A}$ by $\imath_{\mathfrak{A}}(\alpha) = (0, \alpha)$, $\rho_X : X \times Y \longrightarrow X$ by $\rho_X(x, y) = x$ and $\rho_Y : X \times Y \longrightarrow Y$ by $\rho_Y(x, y) = y$, for all $a \in \mathcal{A}$, $\alpha \in \mathfrak{A}$, $x \in X$ and $y \in Y$. Clearly, the above defined maps are linear. Now, we define $T_{\mathcal{A},X} = \rho_X \circ T \circ \imath_{\mathcal{A}}$, $T_{\mathcal{A},Y} = \rho_Y \circ T \circ \imath_{\mathcal{A}}$, $T_{\mathfrak{A},X} = \rho_X \circ T \circ \imath_X$ and $T_{\mathfrak{A},Y} = \rho_Y \circ T \circ \imath_Y$. It is easy to check that these mappings are linear. Then

$$T((a,\alpha)) = \left(T_{\mathcal{A},X}(a) + T_{\mathfrak{A},X}(\alpha), T_{\mathcal{A},Y}(a) + T_{\mathfrak{A},Y}(\alpha)\right), \tag{2.2}$$

for all $(a, \alpha) \in \mathcal{A} \rtimes \mathfrak{A}$. If $T \in \mathcal{M}(\mathcal{A} \rtimes \mathfrak{A}, X \times Y)$, then

$$T((a,\alpha)(b,\beta)) = (T_{\mathcal{A},X}(ab + a\cdot\beta + \alpha\cdot b) + T_{\mathfrak{A},X}(\alpha\beta), T_{\mathcal{A},Y}(ab + a\cdot\beta + \alpha\cdot b) + T_{\mathfrak{A},Y}(\alpha\beta)) \tag{2.3}$$

and

$$\begin{aligned}(a,\alpha)\cdot T((b,\beta)) =& (a\cdot T_{\mathcal{A},X}(b) + a\cdot T_{\mathfrak{A},X}(\beta) + \alpha\cdot T_{\mathcal{A},X}(b) + \alpha\cdot T_{\mathfrak{A},X}(\beta), \alpha\cdot T_{\mathcal{A},Y}(b)\\ &+ \alpha\cdot T_{\mathfrak{A},Y}(\beta)),\end{aligned} \tag{2.4}$$

for all $(a, \alpha), (b, \beta) \in \mathcal{A} \rtimes \mathfrak{A}$. Letting $a = b = 0$ in (2.3) and (2.4) implies that

$$T_{\mathfrak{A},X}(\alpha\beta) = \alpha\cdot T_{\mathfrak{A},X}(\beta) \quad \text{and} \quad T_{\mathfrak{A},Y}(\alpha\beta) = \alpha\cdot T_{\mathfrak{A},Y}(\beta).$$

Thus, $T_{\mathfrak{A},X} \in \mathcal{M}(\mathfrak{A}, X)$, because $\mathfrak{A}$ has a bounded approximate identity and $T_{\mathfrak{A},Y} \in \mathcal{M}(\mathfrak{A}, Y)$. Similarly, letting $\alpha = \beta = 0$ in (2.3) and (2.4) implies that

$$T_{\mathcal{A},X}(ab) = a\cdot T_{\mathcal{A},X}(b).$$

Therefore, $T_{\mathcal{A},X} \in \mathcal{M}(\mathcal{A}, X)$. Letting $b = 0$ and $\alpha = 0$ imply that

$$T_{\mathcal{A},Y}(a\cdot\beta) = 0.$$

Moreover, letting $a = 0$ and $\beta = 0$, imply that

$$T_{\mathcal{A},X}(\alpha\cdot b) = \alpha\cdot T_{\mathcal{A},X}(b) \quad \text{and} \quad T_{\mathcal{A},Y}(\alpha\cdot b) = \alpha\cdot T_{\mathcal{A},Y}(b).$$

Then the above equalities together with $\mathfrak{A}$ has a bounded approximate identity and continuity of $T_{\mathcal{A},Y}$ imply that $T_{\mathcal{A},Y} = 0$ and $T_{\mathcal{A},X} \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A}, X)$. The proof of the converse is clear. □

Lemma 2.3. *Let X be a Banach $\mathcal{A}$-module and Y be a Banach $\mathfrak{A}$-module. Then*

(i) $(X \times Y)^{(\varphi,\widetilde{\varphi})} = X^{(\varphi,\widetilde{\varphi})} \times Y^{\widetilde{\varphi}}$, *for every* $(\varphi, \widetilde{\varphi}) \in E$.
(ii) $(X \times Y)^{(0,\psi)} = Y^{\psi}$, *for every* $(0, \psi) \in F$.
(iii) $(X \times Y)_{(\varphi,\widetilde{\varphi})} \cong X_{(\varphi,\widetilde{\varphi})} \times Y_{\widetilde{\varphi}}$, *for every* $(\varphi, \widetilde{\varphi}) \in E$.
(iv) $(X \times Y)_{(0,\psi)} \cong Y_{\psi}$, *for every* $(0, \psi) \in F$.
(v) $\Pi^{c}_{\mathrm{BSE}}(X \times Y)_{(\varphi,\widetilde{\varphi})} = \Pi^{c}_{\mathrm{BSE}}X_{(\varphi,\widetilde{\varphi})} \times \Pi^{c}_{\mathrm{BSE}}Y_{\widetilde{\varphi}}$, *for every* $(\varphi, \widetilde{\varphi}) \in E$.
(vi) $\Pi^{c}_{\mathrm{BSE}}(X \times Y)_{(0,\psi)} = \Pi^{c}_{\mathrm{BSE}}Y_{\psi}$, *for every , for every* $(0, \psi) \in F$.

Proof. Let $e_\varphi \in A$ and $f_{\widetilde{\varphi}} \in \mathfrak{A}$ such that $\varphi(e_\varphi) = 1$ and $\widetilde{\varphi}(f_{\widetilde{\varphi}}) = 1$.

(i) Let $(x, y) \in (X \times Y)^{(\varphi,\widetilde{\varphi})}$. Then, for any $\varepsilon > 0$, there exist $(a_1, \alpha_1), \ldots, (a_n, \alpha_n) \in M_{(\varphi,\widetilde{\varphi})}$ and $(x_1, y_1), \ldots, (x_n, y_n), (r_1, s_1), \ldots, (r_m, s_m) \in X \times Y$, such that

$$\left\| (x,y) - \sum_{i=1}^{n} (a_i, \alpha_i)\cdot(x_i, y_i) - \left(1 - (e_\varphi, f_{\widetilde{\varphi}})\right) \sum_{j=1}^{m} (r_j, s_j) \right\| < \varepsilon. \tag{2.5}$$

Then (2.5) implies that

$$\left\|\left(x-\sum_{i=1}^{n}(a_i\cdot x_i+\alpha_i\cdot x_i)-(1-(e_\varphi,f_{\widetilde{\varphi}}))\sum_{j=1}^{m}r_j,\, y-\sum_{i=1}^{n}\alpha_i\cdot y_i-(1-f_{\widetilde{\varphi}})\sum_{j=1}^{m}s_j\right)\right\|$$

$$=\left\|x-\sum_{i=1}^{n}(a_i,\alpha_i)\cdot x_i-(1-(e_\varphi,f_{\widetilde{\varphi}}))\sum_{j=1}^{m}r_j\right\|+\left\|y-\sum_{i=1}^{n}\alpha_i\cdot y_i-(1-f_{\widetilde{\varphi}})\sum_{j=1}^{m}s_j\right\|$$

$$<\varepsilon.$$

Thus,

$$\left\|x-\sum_{i=1}^{n}(a_i,\alpha_i)\cdot x_i-(1-(e_\varphi,f_{\widetilde{\varphi}}))\sum_{j=1}^{m}r_j\right\|<\varepsilon$$

and

$$\left\|y-\sum_{i=1}^{n}\alpha_i\cdot y_i-(1-f_{\widetilde{\varphi}})\sum_{j=1}^{m}s_j\right\|<\varepsilon.$$

Then by the above inequalities we have $x\in X^{(\varphi,\widetilde{\varphi})}$ and $y\in Y^{\widetilde{\varphi}}$. Hence, $(X\times Y)^{\varphi}\subseteq X^{(\varphi,\widetilde{\varphi})}\times Y^{\widetilde{\varphi}}$. Now, let $(x,y)\in X^{(\varphi,\widetilde{\varphi})}\times Y^{\widetilde{\varphi}}$. Then for every $\varepsilon>0$, there exist $(a_1,\alpha_1),\ldots,(a_n,\alpha_n)\in M_{(\varphi,\widetilde{\varphi})}$, $\beta_1,\ldots,\beta_m\in\mathfrak{A}$, $x_1,\ldots,x_n,r_1,\ldots,r_t\in X$ and $y_1,\ldots,y_m,s_1,\ldots,s_k\in Y$ such that

$$\left\|x-\sum_{i=1}^{n}(a_i,\alpha_i)\cdot x_i-(1-(e_\varphi,f_{\widetilde{\varphi}}))\sum_{j=1}^{t}r_j\right\|<\frac{\varepsilon}{2}\text{ and }\left\|y-\sum_{i=1}^{m}\beta_i\cdot y_i-(1-f_{\widetilde{\varphi}})\sum_{j=1}^{k}s_j\right\|<\frac{\varepsilon}{2}.\tag{2.6}$$

If $m\geq n$, then we assume that $a_{n+1}=\cdots=a_m=0$ and similarly we do it for t and k. Set $n_1=\max\{n,m\}$ and $t_1=\max\{t,k\}$. For any $(\varphi,\widetilde{\varphi})\in E$, $M_{(\varphi,\widetilde{\varphi})}(X\times\{0\})+M_{(\varphi,\widetilde{\varphi})}(\{0\}\times Y)\subseteq M_{(\varphi,\widetilde{\varphi})}(X\times Y)$. Then by this fact and by (2.6), we have

$$\left\|(x,y)-\sum_{i=1}^{n_1}((a_i,\alpha_i)\cdot(x_i,0)+(a_i,\beta_i)\cdot(0,y_i))-\left(1-(e_\varphi,f_{\widetilde{\varphi}})\right)\sum_{j=1}^{t_1}(r_j,s_j)\right\|$$

$$=\left\|(x,y)-\left(\sum_{i=1}^{n_1}(a_i\cdot x_i+\alpha_i\cdot x_i)-\left(1-(e_\varphi,f_{\widetilde{\varphi}})\right)\sum_{j=1}^{t_1}r_j,\sum_{i=1}^{n_1}\beta_i\cdot y_i-(1-f_{\widetilde{\varphi}})\sum_{j=1}^{t_1}s_j\right)\right\|$$

$$=\left\|(x,y)-\left(\sum_{i=1}^{n}(a_i,\alpha_i)\cdot x_i-\left(1-(e_\varphi,f_{\widetilde{\varphi}})\right)\sum_{j=1}^{t}r_j,\sum_{i=1}^{m}\beta_i\cdot y_i-(1-f_{\widetilde{\varphi}})\sum_{j=1}^{k}s_j\right)\right\|$$

$$=\left\|\left(x-\sum_{i=1}^{n}(a_i,\alpha_i)\cdot x_i-\left(1-(e_\varphi,f_{\widetilde{\varphi}})\right)\sum_{j=1}^{t}r_j,\, y-\sum_{i=1}^{m}\beta_i\cdot y_i-(1-f_{\widetilde{\varphi}})\sum_{j=1}^{k}s_j\right)\right\|$$

$$=\left\|x-\sum_{i=1}^{n}(a_i,\alpha_i)\cdot x_i-\left(1-(e_\varphi,f_{\widetilde{\varphi}})\right)\sum_{j=1}^{t}r_j\right\|+\left\|y-\sum_{i=1}^{m}\beta_i\cdot y_i-(1-f_{\widetilde{\varphi}})\sum_{j=1}^{k}s_j\right\|$$

$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Thus, $(x,y)\in(X\times Y)^{(\varphi,\widetilde{\varphi})}$. Hence, (i) holds.

(ii) By Lemma 2.1 and similar argument in (i), we conclude that (ii) holds.

(iii) Define $\Lambda: X\times Y\longrightarrow X_{(\varphi,\widetilde{\varphi})}\times Y_{\widetilde{\varphi}}$ by $\Lambda(x,y)=\left(x+X^{(\varphi,\widetilde{\varphi})},y+Y^{\widetilde{\varphi}}\right)$, for all $(x,y)\in X\times Y$. Clearly, Λ is a continuous homomorphism between Banach spaces and, by applying (i),

$$\begin{aligned}\ker\Lambda=&\Big\{(x,y)\in X\times Y:\Lambda(x,y)=0_{X^{(\varphi,\widetilde{\varphi})}\times Y^{\widetilde{\varphi}}}=X^{(\varphi,\widetilde{\varphi})}\times Y^{\widetilde{\varphi}}\Big\}\\ =&X^{(\varphi,\widetilde{\varphi})}\times Y^{\widetilde{\varphi}}=(X\times Y)^{(\varphi,\widetilde{\varphi})}.\end{aligned}$$

Then

$$(X\times Y)_{(\varphi,\widetilde{\varphi})}\cong\frac{X\times Y}{(X\times Y)^{(\varphi,\widetilde{\varphi})}}\cong X_{(\varphi,\widetilde{\varphi})}\times Y_{\widetilde{\varphi}}.$$

Hence, (iii) holds. Similarly, one can show that (iv) holds.

(v) Define $\pi^X_{(\varphi,\widetilde{\varphi})}(x)=\hat{x}(\varphi,\widetilde{\varphi})$ and $\pi^Y_{\widetilde{\varphi}}(y)=\hat{y}(\widetilde{\varphi})$, for all $x\in X$ and $y\in Y$. Moreover, define $\pi_{(\varphi,\widetilde{\varphi})}(x,y)=(\pi^X_{(\varphi,\widetilde{\varphi})}(x),\pi^Y_{\widetilde{\varphi}}(y))$, for all $(x,y)\in X\times Y$. Suppose that $\sigma_X\in\Pi^c_{\mathrm{BSE}}X_{(\varphi,\widetilde{\varphi})}$, $\sigma_Y\in\Pi^c_{\mathrm{BSE}}Y_{\widetilde{\varphi}}$ and $(\varphi_1,\widetilde{\varphi}_1),\dots,(\varphi_n,\widetilde{\varphi}_n)\in E$. Then, there exist $\beta_1,\beta_2\in\mathbb{R}^+$ such that for all $f_1\in\left(X_{(\varphi_1,\widetilde{\varphi}_1)}\right)^*,\dots,f_n\in\left(X_{(\varphi_n,\widetilde{\varphi}_n)}\right)^*$ and $g_1\in\left(Y_{\widetilde{\varphi}_1}\right)^*,\dots,g_n\in\left(Y_{\widetilde{\varphi}_n}\right)^*$,

$$\left|\sum_{i=1}^n\langle\sigma_X(\varphi_i,\widetilde{\varphi}_i),f_i\rangle\right|\le\beta_1\left\|\sum_{i=1}^n f_i\circ\pi^X_{(\varphi_1,\widetilde{\varphi}_1)}\right\|_{X^*}\tag{2.7}$$

and

$$\left|\sum_{i=1}^n\langle\sigma_Y(\widetilde{\varphi}_i),g_i\rangle\right|\le\beta_2\left\|\sum_{i=1}^n g_i\circ\pi^Y_{\widetilde{\varphi}_i}\right\|_{Y^*}.\tag{2.8}$$

Set $\beta=2\max\{\beta_1,\beta_2\}$. We consider $\left(X_\varphi\times Y_{\widetilde{\varphi}}\right)^*=X^*_{(\varphi_i,\widetilde{\varphi}_i)}\times Y^*_{\widetilde{\varphi}}$ with the maximum norm, i.e., $\|(f,g)\|=\max\{\|f\|_{X^*_{(\varphi_i,\widetilde{\varphi}_i)}},\|g\|_{Y^*_{\widetilde{\varphi}}}\}$, for all $(f,g)\in X^*_{(\varphi_i,\widetilde{\varphi}_i)}\times Y^*_{\widetilde{\varphi}}$. Let $\mathcal{F}\in(X\times Y)^*_{(\varphi_1,\widetilde{\varphi}_1)},\dots,\mathcal{F}_n\in(X\times Y)^*_{(\varphi_n,\widetilde{\varphi}_n)}$. Then, there exist $f_1\in X^*_{(\varphi_i,\widetilde{\varphi}_i)},\dots,f_n\in X^*_{(\varphi_n,\widetilde{\varphi}_n)}$ and $g_1\in Y^*_{\varphi_1},\dots,g_n\in Y^*_{\varphi_n}$ such that $\mathcal{F}_i=(f_i,g_i)$, for $i=1,\dots,n$. Now, (2.7) and (2.8) imply that

$$\begin{aligned}\left|\sum_{i=1}^n\langle(\sigma_X,\sigma_Y)(\varphi_i,\widetilde{\varphi}_i)(\mathcal{F}_i)\rangle\right|=&\left|\sum_{i=1}^n\langle(\sigma_X(\varphi_i,\widetilde{\varphi}_i),\sigma_Y(\widetilde{\varphi}_i))(f_i,g_i)\rangle\right|\\ =&\left|\sum_{i=1}^n\langle\sigma_X(\varphi_i,\widetilde{\varphi}_i),f_i\rangle+\sum_{i=1}^n\langle\sigma_Y(\widetilde{\varphi}_i),g_i\rangle\right|\\ \le&\left|\sum_{i=1}^n\langle\sigma_X(\varphi_i,\widetilde{\varphi}_i),f_i\rangle\right|+\left|\sum_{i=1}^n\langle\sigma_Y(\widetilde{\varphi}_i),g_i\rangle\right|\\ \le&\beta_1\left\|\sum_{i=1}^n f_i\circ\pi^X_{(\varphi_i,\widetilde{\varphi}_i)}\right\|_{X^*}+\beta_2\left\|\sum_{i=1}^n g_i\circ\pi^Y_{\widetilde{\varphi}_i}\right\|_{Y^*}\\ \le&\beta\max\left\{\left\|\sum_{i=1}^n f_i\circ\pi^X_{(\varphi_i,\widetilde{\varphi}_i)}\right\|_{X^*},\left\|\sum_{i=1}^n g_i\circ\pi^Y_{\widetilde{\varphi}_i}\right\|_{Y^*}\right\}\\ =&\beta\left\|\left(\sum_{i=1}^n f_i\circ\pi^X_{(\varphi_i,\widetilde{\varphi}_i)},\sum_{i=1}^n g_i\circ\pi^Y_{\widetilde{\varphi}_i}\right)\right\|_{(X\times Y)^*}\\ =&\beta\left\|\sum_{i=1}^n(f_i,g_i)\circ\left(\pi^X_{(\varphi_i,\widetilde{\varphi}_i)},\pi^Y_{\widetilde{\varphi}_i}\right)\right\|_{(X\times Y)^*}\\ =&\beta\left\|\sum_{i=1}^n\mathcal{F}_i\circ\pi_{(\varphi_i,\widetilde{\varphi}_i)}\right\|_{(X\times Y)^*}.\end{aligned}$$

Since, σ_X is continuous on E and σ_Y is continuous on $\{\widetilde{\varphi} \in \Delta(\mathfrak{A}) : \varphi \in \Delta(A)\}$, (σ_X, σ_Y) is continuous on E. Thus $(\sigma_X, \sigma_Y) \in \Pi^c_{\mathrm{BSE}}(X \times Y)_{(\varphi,\widetilde{\varphi})}$. This implies that $\Pi^c_{\mathrm{BSE}} X_{(\varphi_i,\widetilde{\varphi}_i)} \times \Pi^c_{\mathrm{BSE}} Y_{\widetilde{\varphi}} \subseteq \Pi^c_{\mathrm{BSE}}(X \times Y)_{(\varphi,\widetilde{\varphi})}$.

Now, let $\sigma \in \Pi^c_{\mathrm{BSE}}(X \times Y)_{(\varphi,\widetilde{\varphi})}$, for any $(\varphi, \widetilde{\varphi}) \in E$. Then there exists $\beta \in \mathbb{R}^+$ such that for any $(\varphi_1, \widetilde{\varphi}_1), \ldots, (\varphi_n, \widetilde{\varphi}_n) \in E$ and $\mathcal{F}_1 \in (X \times Y)^*_{(\varphi_1,\widetilde{\varphi_1})}, \ldots, \mathcal{F}_n \in (X \times Y)^*_{(\varphi_n,\widetilde{\varphi_n})}$,

$$\left|\sum_{i=1}^{n} \langle \sigma(\varphi_i, \widetilde{\varphi_i}), \mathcal{F}_i \rangle \right| \leq \beta \left\| \sum_{i=1}^{n} \mathcal{F}_i \circ \pi_{(\varphi_i,\widetilde{\varphi_i})} \right\|_{(X \times Y)^*}. \tag{2.9}$$

Moreover, since $\sigma(\varphi, \widetilde{\varphi}) \in (X \times Y)_{(\varphi,\widetilde{\varphi})}$, by employing (iii), there exist $\sigma_X \in X_{(\varphi,\widetilde{\varphi})}$ and $\sigma_Y \in Y_{\widetilde{\varphi}}$ such that $\sigma(\varphi, \widetilde{\varphi}) = (\sigma_X(\varphi, \widetilde{\varphi}), \sigma_Y(\widetilde{\varphi}))$. We now show that $\sigma_X \in \Pi^c_{\mathrm{BSE}} X_{(\varphi,\widetilde{\varphi})}$ and $\sigma_Y \in \Pi^c_{\mathrm{BSE}} Y_{\widetilde{\varphi}}$.

Let $f_1 \in X^*_{(\varphi_1,\widetilde{\varphi}_1)}, \ldots, f_n \in X^*_{(\varphi_n,\widetilde{\varphi}_n)}$ and $g_1 \in Y^*_{\widetilde{\varphi}_1}, \ldots, g_n \in Y^*_{\widetilde{\varphi}_n}$. We define, $\mathcal{F}_i = (f_i, 0)$ and $\mathcal{G}_i = (0, g_i)$, for $i = 1, \ldots, n$. It is easy to verify that $\mathcal{F}_i, \mathcal{G}_i \in (\mathcal{A} \bowtie \mathfrak{A})_{(\varphi_i,\widetilde{\varphi}_i)}$. By employing (2.9), we have

$$\begin{aligned}
\left|\sum_{i=1}^{n} \langle \sigma_X(\varphi_i, \widetilde{\varphi}_i), f_i \rangle \right| &= \left|\sum_{i=1}^{n} \langle \sigma(\varphi_i, \widetilde{\varphi}_i), \mathcal{F}_i \rangle \right| \\
&\leq \beta \left\| \sum_{i=1}^{n} \mathcal{F}_i \circ \pi_{(\varphi_i,\widetilde{\varphi}_i)} \right\|_{(X \times Y)^*} \\
&= \beta \left\| \sum_{i=1}^{n} (f_i, 0) \circ (\pi^X_{(\varphi_i,\widetilde{\varphi}_i)}, \pi^Y_{\widetilde{\varphi}} \right\|_{(X \times Y)^*} \\
&= \beta \left\| \sum_{i=1}^{n} f_i \circ \pi^X_{(\varphi_i,\widetilde{\varphi}_i)} \right\|_{X^*}.
\end{aligned}$$

Moreover, σ_X is continuous on E, because σ is continuous on $\Delta(\mathcal{A} \bowtie \mathfrak{A})$. This implies that $\sigma_X \in \Pi^c_{\mathrm{BSE}} X_\varphi$. By a similar argumentation one can show that

$$\left|\sum_{i=1}^{n} \langle \sigma_Y(\widetilde{\varphi}_i), g_i \rangle \right| \leq \beta \left\| \sum_{i=1}^{n} g_i \circ \pi^Y_{\widetilde{\varphi}_i} \right\|_{Y^*}.$$

This means that $\sigma_Y \in \Pi^c_{\mathrm{BSE}} Y_{\widetilde{\varphi}}$. Thus, $\sigma \in (\sigma_X, \sigma_Y) \in \Pi^c_{\mathrm{BSE}} X_\varphi \times \Pi^c_{\mathrm{BSE}} Y_{\widetilde{\varphi}}$. Therefore, $\Pi^c_{\mathrm{BSE}}(X \times Y)_{(\varphi,\widetilde{\varphi})} \subseteq \Pi^c_{\mathrm{BSE}} X_\varphi \times \Pi^c_{\mathrm{BSE}} Y_{\widetilde{\varphi}}$. Hence, (v) holds.

By the above argument, the proof of (vi) is clear. □

Theorem 2.4. *$X \times Y$ is a BSE Banach $\mathcal{A} \bowtie \mathfrak{A}$-module if and only if X is a BSE Banach $\mathcal{A}, \mathfrak{A}$-module and Y is a BSE Banach $\mathfrak{A}$-module.*

Proof. Let $X \times Y$ is a BSE Banach $\mathcal{A} \bowtie \mathfrak{A}$-module. Let $\sigma_X \in \Pi^c_{\mathrm{BSE}} X_{(\varphi,\widetilde{\varphi})}$ and define $\sigma : \Delta(\mathcal{A} \bowtie \mathfrak{A}) \longrightarrow \bigcup_{E \cup F} (\mathcal{A} \bowtie \mathfrak{A})_\phi$ as follows:

$$\sigma(\phi) = \begin{cases} (\sigma_X(\varphi), 0), & \phi = (\varphi, \widetilde{\varphi}) \in E \\ 0, & \phi = (0, \psi) \in F. \end{cases}$$

Let $\phi_1, \ldots, \phi_n \in E \cup F$ and $\mathcal{F}_1 \in (X \times Y)_{\phi_1}, \ldots, \mathcal{F}_1 \in (X \times Y)_{\phi_n}$. Then there exist $f_i \in X^*_{\varphi_i}$ and $g_i \in Y_{\widetilde{\varphi}_i}$, whenever $\phi_i = (\varphi_i, \widetilde{\varphi}_i) \in E$ such that $\mathcal{F}_i = (f_i, g_i)$ and there exists

$h_j \in Y_\psi^*$ such that $\mathcal{F}_j = (0, h_j)$, whenever $\phi_j = (0, \psi_j)$. Then

$$\begin{aligned}\left|\sum_{i=1}^{n} \mathcal{F}_i\left(\sigma\left(\phi_i\right)\right)\right| =& \left|\sum_{i=1,\phi_i\in E}^{n} \mathcal{F}_i\left(\sigma\left(\phi_i\right)\right)\right| \\ =& \left|\sum_{i=1,(\varphi_i,\widetilde{\varphi}_i)\in E}^{n} \mathcal{F}_i\left(\sigma\left(\varphi_i,\widetilde{\varphi}_i\right)\right)\right| \\ =& \left|\sum_{i=1,\varphi_i\in\Delta(\mathcal{A})}^{n} f_i\left(\sigma_X\left(\varphi_i,\widetilde{\varphi}_i\right)\right)\right| \\ \leq& \beta\left\|\sum_{i=1,\varphi_i\in\Delta(\mathcal{A})}^{n} f_i\circ\pi^X_{(\varphi_i,\widetilde{\varphi}_i)}\right\|_{X^*} \\ =& \beta\left\|\sum_{i=1,(\varphi_i,\widetilde{\varphi}_i)\in E}^{n} \mathcal{F}_i\circ\pi_{(\varphi_i,\widetilde{\varphi}_i)}\right\|_{(X\times Y)^*}.\end{aligned}$$

Thus, $\sigma \in \Pi_{\mathrm{BSE}}(X\times Y)_\phi$, for all $\phi \in E\cup F$. From the continuity of σ_X on E, we obtain that σ is continuous on E. Moreover, for any $(0,\psi) \in F$, $\sigma(0,\psi) = 0$, so σ is continuous on F. Thus, σ is continuous on $E\cup F$ and consequently, it is in $\Pi^c_{\mathrm{BSE}}(X\times Y)_\phi$, for all $\phi \in E\cup F$.

According to $X\times Y$ is a BSE Banach $\mathcal{A}\rtimes\mathfrak{A}$-module, so there exists $T \in \mathcal{M}(\mathcal{A}\rtimes\mathfrak{A}, X\times Y)$ such that $\sigma = \widehat{T}$. Hence, $\widehat{T(a,\alpha)} = (a,\alpha)\widehat{T}$, for all $(a,\alpha) \in \mathcal{A}\rtimes\mathfrak{A}$. By Lemma 2.2, $T_{\mathcal{A},X} \in \mathcal{M}(\mathcal{A},X)\cap \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A},X)$, there exist $T_{\mathfrak{A},X} \in \mathcal{M}(\mathfrak{A},X)$ and $T_{\mathfrak{A},Y} \in \mathcal{M}(\mathfrak{A},Y)$ such that

$$T(a,\alpha) = \left(T_{\mathcal{A},X}(a) + T_{\mathfrak{A},X}(\alpha), T_{\mathfrak{A},Y}(\alpha)\right), \tag{2.10}$$

for all $(a,\alpha) \in \mathcal{A}\rtimes\mathfrak{A}$. Then,

$$\begin{aligned}(a,0)\cdot\sigma =& (a,\alpha)\widehat{T} = \widehat{T(a,\alpha)} \\ =& \left(T_{\mathcal{A},X} + \widehat{T_{\mathfrak{A},X}, T_{\mathfrak{A},Y}}\right)(a,\alpha) \\ =& \left(\widehat{T_{\mathcal{A},X}(a)}, 0\right) \\ =& \left(a\widehat{T}_{\mathcal{A},X}, 0\right)\end{aligned} \tag{2.11}$$

for all $a \in \mathcal{A}$. Moreover, for all $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$,

$$\begin{aligned}((a,0)\cdot\sigma)(\varphi,\widetilde{\varphi}) =& (\varphi,\widetilde{\varphi})(a,0)\sigma(\varphi,\widetilde{\varphi}) = \varphi(a)(\sigma_X(\varphi,\widetilde{\varphi}),0) \\ =& (\varphi(a)\sigma_X(\varphi,\widetilde{\varphi}),0) = (a\cdot\sigma_X)(\varphi,\widetilde{\varphi}).\end{aligned} \tag{2.12}$$

Then (2.11) and (2.12) imply that $a\cdot\sigma_X = a\widehat{T}_{\mathcal{A},X}$, for all $a \in \mathcal{A}$. Thus, $\sigma_X = \widehat{T}_{\mathcal{A},X} \in \widehat{\mathcal{M}(\mathcal{A},X)}$. Hence, $\Pi^c_{\mathrm{BSE}}X_{(\varphi,\widehat{\varphi})} \subseteq \widehat{\mathcal{M}(\mathcal{A},X)}$.

Let $\sigma_X \in \Pi^c_{\mathrm{BSE}}X_{(\varphi,\widehat{\varphi})}$ and $\sigma_Y \in \Pi^c_{\mathrm{BSE}}Y_{\widehat{\varphi}}$ and define $\sigma : \Delta(\mathcal{A}\rtimes\mathfrak{A}) \longrightarrow \bigcup_{E\cup F}(\mathcal{A}\rtimes\mathfrak{A})_\phi$ as follows:

$$\sigma(\phi) = \begin{cases} (\sigma_X(\varphi,\widetilde{\varphi}), \sigma_Y(\varphi)) & \phi = (\varphi,\widetilde{\varphi}) \in E \\ 0, & \phi = (0,\psi) \in F. \end{cases}$$

Then by a similar argumentation, one can verify that $\sigma \in \Pi^c_{\mathrm{BSE}}(X\times Y)_{(\varphi,\widehat{\varphi})}$. Thus, there exists $T \in \mathcal{M}(A\rtimes\mathfrak{A}, X\times Y)$ satisfies (2.10) and $\sigma = \widehat{T}$. Then, $(0,\alpha)\cdot\sigma = (0,\alpha)\widehat{T}$, for all $\alpha \in \mathfrak{A}$. By similar argumentations (2.11) and (2.12), we conclude that $\widehat{T}_{\mathfrak{A},X} =$

$\sigma_X \in \widehat{\mathcal{M}_{\mathfrak{A}}(\mathfrak{A},X)}$ and $\widehat{T}_{\mathfrak{A},Y} = \sigma_Y \in \widehat{\mathcal{M}(\mathfrak{A},Y)}$. This means that $\Pi^c_{\mathrm{BSE}} X_{(\varphi,\widehat{\varphi})} \subseteq \widehat{\mathcal{M}(\mathfrak{A},X)}$ and $\Pi^c_{\mathrm{BSE}} Y_{\widehat{\varphi}} \subseteq \widehat{\mathcal{M}(\mathfrak{A},Y)}$. Similar argumentations hold for $(0,\psi) \in F$.

Let $T \in \mathcal{M}(X \times Y)$. Then by Lemma 2.3, T is as in (2.10). Since $X \times Y$ is BSE as Banach $A \bowtie \mathfrak{A}$-module, there exists $\sigma \in \prod^c_{\mathrm{BSE}}(X \times Y)_{(\varphi,\widetilde{\varphi})}$ such that $\sigma = \widehat{T}$. By employing Lemma 2.3(v), $\sigma = (\sigma_X, \sigma_Y)$, where $\sigma_X \in \prod^c_{\mathrm{BSE}} X_{(\varphi,\widetilde{\varphi})}$, $\sigma_Y \in \prod^c_{\mathrm{BSE}} Y_{\widetilde{\varphi}}$ and $(\varphi,\widetilde{\varphi}) \in E$. Let $e_\varphi \in \mathcal{A}$ and $f_{\widetilde{\varphi}} \in \mathfrak{A}$ such that $\varphi(e_\varphi) = 1/2$ and $\widetilde{\varphi}(f_{\widetilde{\varphi}}) = 1/2$. Then $(\varphi,\widetilde{\varphi})\left(e_\varphi, f_{\widetilde{\varphi}}\right) = 1$ and so

$$
\begin{aligned}
(\sigma_X,\sigma_Y)(\varphi,\widetilde{\varphi}) = \sigma(\varphi,\widetilde{\varphi}) = \widehat{T}(\varphi,\widetilde{\varphi}) &= \widehat{T\left(e_\varphi, f_{\widetilde{\varphi}}\right)}(\varphi,\widetilde{\varphi}) \\
&= \widehat{\left(T_{\mathcal{A},X}(e_\varphi) + T_{\mathfrak{A},X}(f_{\widetilde{\varphi}}), T_{\mathfrak{A},Y}\left(f_{\widetilde{\varphi}}\right)\right)}(\varphi,\widetilde{\varphi}) \\
&= \left(\widehat{T_{\mathcal{A},X}(e_\varphi)} + \widehat{T_{\mathfrak{A},X}(f_{\widetilde{\varphi}})}, \widehat{T_{\mathfrak{A},Y}\left(f_{\widetilde{\varphi}}\right)}\right)(\varphi,\widetilde{\varphi}) \\
&= \widehat{T_{\mathcal{A},X}(e_\varphi)}(\varphi) + \widehat{T_{\mathfrak{A},X}(f_{\widetilde{\varphi}})}(\widetilde{\varphi}) + \widehat{T_{\mathfrak{A},Y}\left(f_{\widetilde{\varphi}}\right)}(\widetilde{\varphi}) \\
&= \frac{1}{2}\left(\widehat{T}_{\mathcal{A},X}(\varphi) + \widehat{T}_{\mathfrak{A},X}(\widetilde{\varphi}) + \widehat{T}_{\mathfrak{A},Y}(\widetilde{\varphi})\right) \\
&= \frac{1}{2}\left(\widehat{T}_{\mathcal{A},X} + \widehat{T}_{\mathfrak{A},X}, \widehat{T}_{\mathfrak{A},Y}\right)(\varphi,\widetilde{\varphi}).
\end{aligned}
$$

Hence, $\frac{1}{2}\left(\widehat{T}_{\mathcal{A},X} + \widehat{T}_{\mathfrak{A},X}\right) = \sigma_X \in \prod^c_{\mathrm{BSE}} X_{(\varphi,\widetilde{\varphi})}$ and $\widehat{T}_{\mathfrak{A},Y} = \sigma_Y \in \prod^c_{\mathrm{BSE}} Y_{\widetilde{\varphi}}$. These follow that $\widehat{\mathcal{M}(\mathcal{A},X)} + \widehat{\mathcal{M}(\mathfrak{A},X)} \subseteq \prod^c_{\mathrm{BSE}} X_{(\varphi,\widetilde{\varphi})}$ and $\widehat{M(\mathfrak{A},Y)} \subseteq \prod^c_{\mathrm{BSE}} Y_{\widetilde{\varphi}}$. Thus, X is a BSE Banach $\mathcal{A},\mathfrak{A}$-module and Y is a BSE Banach $\mathfrak{A}$-module.

Conversely, suppose that X is a BSE Banach $\mathcal{A},\mathfrak{A}$-module and Y is a BSE Banach $\mathfrak{A}$-module. Let $\sigma \in \prod^c_{\mathrm{BSE}}(X \times Y)_{(\varphi,\widetilde{\varphi})}$, where $\varphi \in \Delta(A)$. By Lemma 2.3(v), we have $\sigma = (\sigma_X, \sigma_Y)$, where $\sigma_X \in \prod^c_{\mathrm{BSE}} X_{(\varphi,\widetilde{\varphi})}$, $\sigma_Y \in \prod^c_{\mathrm{BSE}} Y_{\widetilde{\varphi}}$. Then there exist $T_{\mathcal{A},X} \in \mathcal{M}(\mathcal{A},X)$, $T_{\mathfrak{A},X} \in \mathcal{M}(\mathfrak{A},X)$ and $T_{\mathfrak{A},Y} \in \mathcal{M}(\mathfrak{A},Y)$ such that $\sigma_X = \widehat{T}_{\mathcal{A},X} + \widehat{T}_{\mathfrak{A},X}$ and $\sigma_Y = \widehat{T}_{\mathfrak{A},Y}$.

Now define $T : \mathcal{A} \bowtie \mathfrak{A} \longrightarrow X \times Y$ by $T(a,\alpha) = \left(T_{\mathcal{A},X}(a) + T_{\mathfrak{A},X}(\alpha), T_{\mathfrak{A},Y}(\alpha)\right)$, for all $(a,\alpha) \in \mathcal{A} \times \mathfrak{A}$. Then by Lemma 2.2, $T \in M(\mathcal{A} \bowtie \mathfrak{A}, X \times Y)$. This implies that $\sigma = \widehat{T} \in \widehat{\mathcal{M}(\mathcal{A} \bowtie \mathfrak{A}, X \times Y)}$. Hence, $\prod^c_{\mathrm{BSE}}(A \oplus_1 X)_{\widetilde{\varphi}} \subseteq \widehat{\mathcal{M}(\mathcal{A} \bowtie \mathfrak{A}, X \times Y)}$.

Now, let $\widehat{T} \in \widehat{\mathcal{M}(\mathcal{A} \bowtie \mathfrak{A}, X \times Y)}$. Thus, $T = (T_{\mathcal{A},X} + T_{\mathfrak{A},X}, T_{\mathfrak{A},Y})$, where $T_{\mathcal{A},X} \in \mathcal{M}(\mathcal{A},X)$, $T_{\mathfrak{A},X} \in \mathcal{M}(\mathfrak{A},X)$ and $T_{\mathfrak{A},Y} \in \mathcal{M}(\mathfrak{A},Y)$. Since X is a BSE Banach $\mathcal{A},\mathfrak{A}$-module and X is a BSE Banach $\mathfrak{A}$-module, there exist $\sigma_X \in \prod^c_{\mathrm{BSE}} X_{(\varphi,\widetilde{\varphi})}$ and $\sigma_Y \in \prod^c_{\mathrm{BSE}} Y_{\widetilde{\varphi}}$ such that $\widehat{T}_{\mathcal{A},X} + \widehat{T}_{\mathfrak{A},X} = \sigma_A$ and $\widehat{T}_{U,Y} = \sigma_Y$. Then by a similar argument, we have $\widehat{T} = (\sigma_X, \sigma_Y) \in \prod^c_{\mathrm{BSE}}(X \times Y)_{(\varphi,\widetilde{\varphi})}$. Thus $\widehat{\mathcal{M}(\mathcal{A} \bowtie \mathfrak{A}, X \times Y)} \subseteq \prod^c_{\mathrm{BSE}}(X \times Y)_{(\varphi,\widetilde{\varphi})}$. Hence, $X \times Y$ is a BSE Banach $\mathcal{A} \bowtie \mathfrak{A}$-module. □

Corollary 2.5. *Let $\mathfrak{A}$ be a without order Banach algebra. Then $\mathcal{A} \times \mathfrak{A}$ is a BSE Banach $\mathcal{A} \bowtie \mathfrak{A}$-module if and only if $\mathcal{A}$ is a BSE $\mathcal{A},\mathfrak{A}$-module and $\mathfrak{A}$ is a BSE $\mathfrak{A}$-module.*

Proof. Clearly, if $\mathfrak{A}$ is a without order Banach algebra. Thus, by Theorem 2.4, the proof holds. □

Corollary 2.6. *Let G be an abelian compact group and $1 \leq p < \infty$. Then $L^p(G) \times C(G)$ is a BSE Banach $L^1(G) \bowtie L^1(G)$-module.*

Proof. Since $L^1(G)$ is a BSE Banach algebra [23] and every BSE Banach algebra is a BSE Banach module over itself [24], $L^p(G)$ and $C(G)$ are BSE Banach $L^1(G)$-modules [24, Theorem 3.3]. Then by Theorem 2.4 the proof holds. □

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References

[1] F. Abtahi, Z. Kamali and M. Toutounchi, *The Bochner-Schoenberg-Eberlein property for vector-valued Lipschitz algebras*, J. Math. Anal. Appl. **479** (1), 1172-1181, 2019.

[2] N. Alizadeh, S. Ostadbashi, A. Ebadian and A. Jabbari, *The BochnerSchoenbergE-berlain property of extensions of Banach algebras and Banach modules*, Bull. Aust. Math. Soc. **105** (1), 134-145, 2022.

[3] P.A. Dabhi, *Multipliers of perturued Cartesian product with an application to BSE-property*, Acta Math. Hungar. **149** (1), 58-66, 2016.

[4] P.A. Dabhi and R.S. Upadhyay, *The Semigroup Algebra* $\ell^1(\mathbb{Z}^2, \max)$ *is a Bochner-SchoenbergEberlein (BSE) Algebra*, Mediterr. J. Math. **16** (1), 12, 2019.

[5] M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: The basic properties*, J. Algebra Appl. **6**, 443-459, 2007.

[6] M. Daws, *Multipliers, self-induced and dual Banach algebras*, Dissert. Math. **470**, 1-62, 2010.

[7] A. Ebadian and A. Jabbari, *Biprojectivity and biflatness of amalgamated duplication of Banach algebras*, J. Algebra Appl. **19** (7), 2050132, 2020.

[8] A. Ebadian and A. Jabbari, C^**-algebras defined by amalgamated duplication of* C^**-algebras*, J. Algebra Appl. **20** (2), 2150019, 2021.

[9] A. Ebadian and A. Jabbari, *The BochnerSchoenbergEberlein property for amalgamated duplication of Banach algebras*, J. Algebra Appl. **21** (8), 2250155, 2022.

[10] M. Essmaili, A. Rejali and A. Salehi Marzijarani, *Biprojectivity of generalized module extension and second dual of Banach algebras*, J. Algebra Appl. **21** (4), 2250070, 2022.

[11] M. Fozouni and M. Nemati, *BSE-property for some certain Segal and Banach algebras*, Mediterr. J. Math. **16** (2), 38, 2019.

[12] J. Inoue and S.E. Takahasi, *Segal algebras in commutative Banach algebras*, Rocky Mountain J. Math. **44**, 539-589, 2014.

[13] H. Javanshiri and M. Nemati, *Amalgamated duplication of the Banach algebra* $\mathfrak{A}$ *along a* $\mathfrak{A}$*-bimodule* $\mathcal{A}$, J. Algebra Appl. **17** (9), 1850169-1-1850169-21, 2018.

[14] Z. Kamali and F. Abtahi, *The BochnerSchoenbergEberlein property for vector-valued* ℓ^p*-spaces*, Mediterr. J. Math. **17** (3), 94, 2020.

[15] Z. Kamali and M. Lashkarizadeh Bami, *Bochner-Schoenberg-Eberlein property for abstract Segal algebras*, Proc. Jpn. Acad. (Ser A) **89**, 107-110, 2013.

[16] Z. Kamali and M. Lashkarizadeh Bami, *The multiplier algebra and BSE property of the direct sum of Banach algebras*, Bull. Aust. Math. Soc. **88**, 250-258, 2013.

[17] Z. Kamali and M. Lashkarizadeh, *The BochnerSchoenbergEberlein property for* $L^1(\mathbb{R}^+)$, J. Fourier Anal. Appl. **20** (2), 225-233, 2014.

[18] Z. Kamali and M. Lashkarizadeh, *A characterization of the* L^∞*-representation algebra* $R(S)$ *of a foundation semigroup and its application to BSE algebras*, Proc. Jpn. Acad. Ser. A Math. Sci. **92** (5), 59-63, 2016.

[19] Z. Kamali and M. Lashkarizadeh, *The BochnerSchoenbergEberlein property for totally ordered semigroup algebras*, J. Fourier Anal. Appl. **22** (6), 1225-1234, 2016.

[20] E. Kaniuth, *The Bochner-Schoenberg-Eberlein property and spectral synthesis for certain Banach algebra products*, Canad. J. Math. **67**, 827-847, 2015.

[21] E. Kaniuth and A. Ülger, *The BochnerSchoenbergEberlein property for commutative Banach algebras, especially Fourier and FourierStieltjes algebras*, Trans. Amer. Math. Soc. **362**, 4331-4356, 2010.

[22] R. Larsen, *An Introduction to the Theorey of Multipliers*, Springer, New York, 1971.

[23] W. Rudin, *Fourier Analysis on Groups*, Interscience, New York, 1962.

[24] S.-E. Takahasi, *BSE Banach modules and multipliers*, J. Funct. Anal. **125**, 67-68, 1994.
[25] S.-E. Takahasi and O. Hatori, *Commutative Banach algebras which satisfy a BochnerSchoenbergEberlein-type theorem*, Proc. Amer. Math. Soc. **110**, 149158, 1990.
[26] S.-E. Takahasi and O. Hatori, *Commutative Banach algebras and BSE-inequalities*, Math. Japonica **37**, 47-52, 1992.