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Aprroximation of Endpoints for Generalized α-Nonexpansive Multivalued Mappings in Hyperbolic Spaces

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Abstract. In this paper, we establish that the sequence of the new iteration converges to endpoints of generalized α-nonexpansive multivalued mappings in 2-uniformly convex hyperbolic space. We present some strong and Δ-convergence theorems for such operator in a hyperbolic metric space. The results presented in this paper extend and improve some recent results in the literature.

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1. Introduction

Fixed point theory contributes significantly to the theory of nonlinear functional analysis. The fixed points of a nonlinear mapping under suitable set of control conditions is constituted with metric fixed point theory. So, fixed point problems associated with a class of mappings in a suitable nonlinear structure have been studied. The metric spaces don't have a such convex structure. Therefore, there is need to introduce convex structure in the metric space. We work in hyperbolic spaces presented by Kohlenbach [7]. Let Y be a metric space. Let K be a nonempty subset of Y and $\varkappa \in Y$. The radius of K relative to $\varkappa$ is defined by

$$r := R(\varkappa, K) := \sup\{d(\varkappa, \omega) : \omega \in K\}.$$

The diameter of K is defined by

$$diam(K) := \sup\{d(\varkappa, \omega) : \varkappa, \omega \in K\}.$$

We show by $CB(K)$ the set of all nonempty closed bounded subets of K. Then, the Hausdorff distance H between $\grave{A}$ and $\tilde{N}$ is defined

$$H\left(\grave{A}, \tilde{N}\right) := \max\left\{\sup_{a\in\grave{A}} d\left(a, \tilde{N}\right), \sup_{b\in\tilde{N}} d(b, \grave{A})\right\} \text{ for all } \grave{A}, \tilde{N} \in CB(K).$$

A mapping $\Gamma : K \to CB(K)$ is said to be multivalued nonexpansive if $H(\Gamma(\varkappa), \Gamma(\omega)) \leq d(\varkappa, \omega)$ for all $\varkappa, \omega \in K$ and is said to be multivalued quasi-nonexpansive if for each $\varkappa \in K$ and $a \in F(\Gamma)$ if $H(\Gamma(\varkappa), \Gamma(a)) \leq d(\varkappa, a)$.

A point $a \in K$ is called an endpoint (strict fixed point) of Γ if $\Gamma(a) = \{a\}$.We show the sets of endpoints of Γ by $End(\Gamma)$ and the sets of fixed points of Γ by $F(\Gamma)$. Notice that for each mapping Γ, $End(\Gamma) \subseteq F(\Gamma)$. Many researchers have given the results with existence of endpoints for mappings in Banach spaces [13, 15, 17–20, 24]. Panyanak [14],

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Kudtha and Panyanak [8], Laokul and Panyanak [9], Ullah et al. [23], Abdeljawad et al. [1], Ullah et al. [25] have used different iteration process to approximating endpoints of multivalued nonexpansive mappings.

Lemma 1.1 ([14]). *Let $\Gamma : K \to CB(K)$ be a multivalued mapping. The following statements hold:*
(i) $End(\Gamma) \subseteq F(\Gamma)$,
(ii) $\varkappa \in F(\Gamma)$ if and only if $dist(\varkappa, \Gamma(\varkappa)) = 0$,
(iii) $\varkappa \in End(\Gamma)$ if and only if $R(\varkappa, \Gamma(\varkappa)) = 0$.

Suzuki [21] introduced generalized nonexpansive mappings which is named condition (C). In 2017, Pant and Shukla [12] presented the class of generalized α- nonexpansive mappings.

Abkar and Eslamian [2] modified Suzuki's condition to incorporate multivalued mappings. They called these mappings generalized multivalued nonexpansive mappings or mappings satisfying the condition (C). In 2019, Igbal et al. [5] expended the new class of α-nonexpansive mapping to the multivalued generalized α-nonexpansive mappings.

Definition 1.2 ([2]). A multivaled mapping $\Gamma : K \to CB(K)$ is said to satisfy the Condition (C) if for all $\varkappa, \omega \in K$ the following condition holds:

$$\frac{1}{2} d(\varkappa, \Gamma(\varkappa)) \leq d(\varkappa, \omega) \Rightarrow H(\Gamma(\varkappa), \Gamma(\omega)) \leq d(\varkappa, \omega).$$

Definition 1.3 ([5]). A mapping $\Gamma : K \to CB(K)$ is said to be a generalized α-nonexpansive multivalued mapping if there exists an $\alpha \in [0, 1)$ such that for each $\varkappa, \omega \in K$;

$$\frac{1}{2} d(\varkappa, \Gamma(\varkappa)) \leq d(\varkappa, \omega) \Rightarrow H(\Gamma(\varkappa), \Gamma(\omega)) \leq \alpha d(\varkappa, \Gamma(\omega)) + \alpha d(\omega, \Gamma(\varkappa)) + (1 - 2\alpha) \|\varkappa - \omega\|. \tag{1.1}$$

Proposition 1.4 ([16]). *Let $\Gamma : K \to CB(K)$ be a multivalued mapping. Then, the followings hold.*

i: *If Γ satisfies condition (C), then Γ is a generalized $\alpha-$ nonexpansive multivalued mapping for some $\alpha \in [0, 1)$.*
ii: *If Γ is a generalized $\alpha-$nonexpansive mapping and $F(\Gamma) \neq \varnothing$, then Γ is quasi-nonexpansive.*

Proposition 1.5 ([16]). *Let $\Gamma : K \to CB(K)$ be a mapping fulfilling (1.1). For $\omega, t \in K$,*

(1) *$H(\Gamma(\omega), \Gamma(z)) \leq \|z - \omega\|$, $\forall z \in \Gamma(\omega)$.*
(2) *Either $\frac{1}{2} d(\omega, \Gamma(\omega)) \leq \|\omega - t\|$ or $\frac{1}{2} d(z, \Gamma(z)) \leq \|z - t\|$ for $z \in \Gamma(\omega)$.*
(3) *Either $H(\Gamma(\omega), \Gamma(t)) \leq \alpha d(\omega, \Gamma(t)) + \alpha d(t, \Gamma(\omega)) + (1 - 2\alpha) \|\omega - t\|$ or $H(\Gamma(z), \Gamma(t)) \leq \alpha d(z, \Gamma(t)) + \alpha d(t, \Gamma(z)) + (1 - 2\alpha) \|z - t\|$, $\forall z \in \Gamma(\omega)$.*

Lemma 1.6 ([5]). *Let $\Gamma : K \to CB(K)$ be a mapping fulfilling (1.1). For $\omega, t \in K$ and $z \in \Gamma(\omega)$, we have*

$$d(\omega, \Gamma(t)) \leq \left(\frac{3 + \alpha}{1 - \alpha}\right) d(\omega, \Gamma(\omega)) + \|\omega - t\|.$$

Definition 1.7 ([9]). Let (Y, d) be a metric space, then (Y, d, W) will be hyperbolic metric space if $W : Y^2 \times [0, 1] \to Y$ fulfilling (i) $d(\nu, W(\varkappa, \omega, \varphi)) \leq (1 - \varphi) d(\nu, \varkappa) + \varphi d(\nu, \omega)$;
(ii) $d(W(\varkappa, \omega, \varphi), W(\varkappa, \omega, \gamma)) = |\varphi - \gamma| d(\varkappa, \omega)$;
(iii) $W(\varkappa, \omega, \varphi) = W(\omega, \varkappa, 1 - \varphi)$;
(iv) $d(W(\varkappa, \nu, \varphi), W(\omega, w, \varphi)) \leq \varphi d(\varkappa, \omega) + (1 - \varphi) d(\nu, w)$
for all $\varkappa, \omega, \nu, w \in Y$ and $\varphi, \gamma \in [0, 1]$.
If $\varkappa, \omega \in Y$ and $\varphi \in [0, 1]$, we use the notation $(1 - \varphi) \varkappa \oplus \varphi \omega$ for $W(\varkappa, \omega, \varphi)$. It follows from (i) that

$$\begin{aligned} d(\varkappa, (1 - \varphi) \varkappa \oplus \varphi \omega) &= \varphi d(\varkappa, \omega), \\ d(\omega, (1 - \varphi) \varkappa \oplus \varphi \omega) &= (1 - \varphi) d(\varkappa, \omega). \end{aligned}$$

A subset K of a hyperbolic space Y is convex if $W(\varkappa, \omega, \varphi) \in K$ for all $x, \omega \in K$ and $\varphi \in [0, 1]$. The class of hyperbolic spaces contains normed spaces and their convex subsets as subclasses and CAT(0) spaces form a very special subclass of the class of hyperbolic spaces with unique geodesic paths. CAT(0) spaces are uniformly convex hyperbolic spaces with modulus of uniform convexity. Uniformly convex hyperbolic spaces are a natural generalization of both uniformly convex Banach spaces and CAT(0) spaces.

Definition 1.8 ([10]). Let Y be a hyperbolic metric space. Then, Y is uniformly convex if for any $t, \varkappa, \omega \in Y$, for every $l > 0$ and for each $\varepsilon > 0$

$$\delta(l, \varepsilon) = \inf\left\{1 - \frac{1}{l} d\left(W\left(\varkappa, \omega, \frac{1}{2}\right), t\right) : d(\varkappa, t) \le l, d(\omega, t) \le l \text{ and } d(\varkappa, \omega) \ge \varepsilon l\right\} > 0.$$

Definition 1.9 ([9]). Let (Y, d) be a uniformly convex hyperbolic space. For all $k \in (0, \infty)$ and $\varepsilon \in (0, 2]$, we define

$$\Psi(k, \varepsilon) := \inf\left\{\frac{1}{2} d^2(y, z) + \frac{1}{2} d^2(\omega, z) - d^2\left(W\left(y, w, \frac{1}{2}\right), z\right)\right\},$$

such that $d(y, z) \le k$, $d(\omega, z) \le k$ and $d(y, \omega) \ge k\varepsilon$ for all $y, \omega, z \in Y$. Then, (Y, d) is 2- uniformly convex if

$$c_M := \inf\left\{\frac{\Psi(k, \varepsilon)}{k^2 \varepsilon^2} : k \in (0, \infty), \varepsilon \in (0, 2]\right\} > 0.$$

Lemma 1.10 ([9]). *Let (Y, d) be a 2- uniformly convex hyperbolic space. Then,*

$$d^2((1-k)\varkappa \oplus k\omega, z) \le (1-k) d^2(\varkappa, z) + k d^2(\omega, z) - 4c_M k(1-k) d^2(\varkappa, \omega),$$

for all $\varkappa, \omega, z \in Y$ and $k \in [0, 1]$.

Definition 1.11. Let $\{\varkappa_n\}$ be any bounded sequence in Y and $\varnothing \ne K \subseteq Y$. An asymptotic radius of $\{\varkappa_n\}$ relative to K is defined by $r(K, \{\varkappa_n\}) = \inf\left\{\limsup_{n\to\infty} d(\varkappa_n, \varkappa) : \varkappa \in K\right\}$ and an asymptotic center of $\{\varkappa_n\}$ relative to K is defined by $A = A(K, \{\varkappa_n\}) = \left\{\varkappa \in K : \limsup_{n\to\infty} d(\varkappa_n, \varkappa) = r(K, \{\varkappa_n\})\right\}$.

Lemma 1.12 ([4]). *Let K be a nonempty closed convex subset of Y and $\{x_n\}$ is a bounded sequence in Y. If $A(K, \{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(K, \{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Definition 1.13 ([11]). If every subsequence $\{\varkappa_{n_i}\}$ of $\{\varkappa_n\} \subseteq Y$ has an unique asymptotic center $\varkappa \in Y$, then we say $\{\varkappa_n\}$ $\Delta-$converges to $\varkappa$. It is writen as $\Delta - \lim \varkappa_n = \varkappa$.

Definition 1.14 ([3]). A sequence $\{x_m\}$ in Y is called a Fejěr monotone with respect to K if $\|\varkappa_{m+1} - b\| \le \|\varkappa_m - b\|$ for all $b \in K$ and $m \in \mathbb{N}$.

Lemma 1.15 ([15]). *Let $\{\alpha_n\}$,$\{\gamma_n\} \in [0, 1)$ be such that $lim_{n\to\infty}\gamma_n = 0$ and $\sum \alpha_n \gamma_n = \infty$. Let $\{\delta_n\}$ be a positive real sequence such that $\sum \alpha_n \gamma_n (1 - \gamma_n) \delta_n < \infty$. Then $\{\delta_n\}$ has a subsequence which converges to zero.*

The purpose of this study is to extend the strong and $\Delta-$ convergence results of endpoints for generalized α-nonexpansive multivalued mappings from the class of uniformly convex Banach spaces and CAT(0) spaces to a wider class of 2-uniformly convex hyperbolic space.We use here the new iterative process (1.2), which was introduced by Kaplan in [6].

Let D be a nonempty convex subset of Y, $\Gamma : D \to CB(D)$ and $\{\alpha_n\}$,$\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[e, f] \subset [0, 1]$.

Define a sequence $\{\varkappa_n\}$ as follows:

$$z_n = (1 - \gamma_n)\varkappa_n \oplus \gamma_n v_n, \ n \in \mathbb{N}, \tag{1.2}$$

where $v_n \in \Gamma(\varkappa_n)$ such that $d(\varkappa_n, v_n) = R(\varkappa_n, \Gamma(\varkappa_n))$,

$$y_n = (1 - \beta_n) v_n \oplus \beta_n w_n,$$

where $w_n \in \Gamma(z_n)$ such that $d(z_n, w_n) = R(z_n, \Gamma(z_n))$,

$$x_{n+1} = (1 - \alpha_n) v_n \oplus \alpha_n u_n,$$

where $u_n \in \Gamma(y_n)$ such that $d(y_n, u_n) = R(y_n, \Gamma(y_n))$.

2. Main Results

Lemma 2.1. *Let Y be a complete 2-uniformly convex hyperbolic space, D be a nonempty closed convex and bounded subset of Y. Assume that $\Gamma : D \to CB(D)$ is a generalized α-nonexpansive multivalued mapping and $End(\Gamma) \neq \emptyset$. Let $\{x_n\}$ be sequence defined by (1.2). Then, $\{\varkappa_n\}$ is Fejěr monotone according to $End(\Gamma)$.*

Proof. Let $a \in End(\Gamma)$. As Γ is generalized α-nonexpansive multivalued mapping, by Proposition 1.4 (ii) for each $n \in \mathbb{N}$ we have

$$\frac{1}{2} d(a, \Gamma(a)) = 0 \leq d(a, \varkappa_n).$$

Then,

$$d(a, \Gamma(\varkappa_n)) \leq H(\Gamma(\varkappa_n), \Gamma(a)) \leq \alpha d(\varkappa_n, \Gamma(a)) + \alpha d(a, \Gamma(\varkappa_n)) + (1-2\alpha) d(\varkappa_n, a).$$

This implies that

$$(1-\alpha) d(\Gamma(\varkappa_n), a) \leq (1-\alpha) d(\varkappa_n, a).$$

Since $(1-\alpha) > 0$, then we have

$$H(\Gamma(\varkappa_n), \Gamma(a)) \leq d(\varkappa_n, a).$$

Similarly, for any $a \in End(\Gamma)$, we have

$$\begin{aligned} H(\Gamma(y_n), \Gamma(a)) &\leq d(y_n, a), \\ H(\Gamma(z_n), \Gamma(a)) &\leq d(z_n, a). \end{aligned}$$

Now by (1.2), we have

$$\begin{aligned} d(z_n, a) &= d((1-\gamma_n)\varkappa_n \oplus \gamma_n v_n, a) \\ &\leq (1-\gamma_n) d(\varkappa_n, a) + \gamma_n d(v_n, a) \\ &\leq (1-\gamma_n) d(\varkappa_n, a) + \gamma_n d(v_n, \Gamma(a)) \\ &\leq (1-\gamma_n) d(\varkappa_n, a) + \gamma_n H(\Gamma(\varkappa_n), \Gamma(a)) \\ &\leq (1-\gamma_n) d(\varkappa_n, a) + \gamma_n d(\varkappa_n, a) = d(\varkappa_n, a), \end{aligned} \tag{2.1}$$

and with (2.1), we have

$$\begin{aligned} d(y_n, a) &= d((1-\beta_n) v_n \oplus \beta_n w_n, a) \\ &\leq (1-\beta_n) d(v_n, a) + \beta_n d(w_n, a) \\ &\leq (1-\beta_n) d(v_n, \Gamma(a)) + \beta_n d(w_n, \Gamma(a)) \\ &\leq (1-\beta_n) H(\Gamma(\varkappa_n), \Gamma(a)) + \beta_n H(\Gamma(z_n), \Gamma(a)) \\ &\leq (1-\beta_n) d(\varkappa_n, a) + \beta_n d(z_n, a) \\ &\leq d(\varkappa_n, a). \end{aligned} \tag{2.2}$$

In view of (2.2), we have

$$\begin{aligned} d(\varkappa_{n+1}, a) &= d((1-\alpha_n) v_n \oplus \alpha_n u_n, a) \\ &\leq (1-\alpha_n) d(v_n, a) + \alpha_n d(u_n, a) \\ &\leq (1-\alpha_n) dist(v_n, \Gamma(a)) + \alpha_n dist(u_n, \Gamma(a)) \\ &\leq (1-\alpha_n) H(\Gamma(\varkappa_n), \Gamma(a)) + \alpha_n H\left(\check{T}(y_n), \Gamma(a)\right) \\ &\leq (1-\alpha_n) d(\varkappa_n, p) + \alpha_n d(y_n, a) \\ &\leq d(\varkappa_n, a). \end{aligned}$$

From thus, $d(\varkappa_n, a)$ is nonincreasing sequence, which implies $\lim_{n\to\infty} d(\varkappa_n, a)$ exists for every $a \in End(\Gamma)$. □

Theorem 2.2. *Let Y, D and Γ be as in Lemma 2.1. Let $\{\varkappa_n\}$ be sequence generated by (1.2). Then, $\{\varkappa_n\}$ $\Delta-$converges to an element in $End(\Gamma)$*

Proof. Let $a \in End(\Gamma)$. From Lemma 1.10, we write

$$\begin{aligned} d^2(z_n, a) &\le (1-\gamma_n) d^2(\varkappa_n, a) + \gamma_n d^2(v_n, a) - 4c_M \gamma_n (1-\gamma_n) d^2(\varkappa_n, v_n) \\ &\le (1-\gamma_n) d^2(\varkappa_n, a) + \gamma_n H^2(\Gamma(\varkappa_n), \Gamma(a)) - 4c_M \gamma_n (1-\gamma_n) d^2(\varkappa_n, v_n) \\ &\le d^2(\varkappa_n, a) - 4c_M \gamma_n (1-\gamma_n) d^2(\varkappa_n, v_n), \end{aligned}$$

and

$$\begin{aligned} d^2(y_n, a) &\le (1-\beta_n) d^2(v_n, a) + \beta_n d^2(w_n, a) - 4c_M \beta_n (1-\beta_n) d^2(v_n, w_n) \\ &\le (1-\beta_n) H^2(\Gamma(\varkappa_n), \Gamma(a)) + \beta_n H^2(\Gamma(z_n), \Gamma(a)) - 4c_M \beta_n (1-\beta_n) d^2(v_n, w_n) \\ &\le (1-\beta_n) d^2(\varkappa_n, a) + \beta_n d^2(z_n, a) \\ &\le (1-\beta_n) d^2(\varkappa_n, a) + \beta_n d^2(\varkappa_n, a) - 4c_M \beta_n \gamma_n (1-\gamma_n) d^2(\varkappa_n, v_n) \\ &= d^2(\varkappa_n, a) - 4c_M \beta_n \gamma_n (1-\gamma_n) d^2(\varkappa_n, v_n). \end{aligned}$$

It means that

$$\begin{aligned} d^2(\varkappa_{n+1}, a) &\le (1-\alpha_n) d^2(v_n, a) + \alpha_n d^2(u_n, a) - 4c_M \alpha_n (1-\alpha_n) d^2(v_n, u_n) \\ &\le (1-\alpha_n) H^2(\Gamma(\varkappa_n), \Gamma(a)) + \alpha_n H^2(\Gamma(y_n), \Gamma(a)) - 4c_M \alpha_n (1-\alpha_n) d^2(v_n, u_n) \\ &\le (1-\alpha_n) d^2(\varkappa_n, a) + \alpha_n d^2(y_n, a) \\ &\le (1-\alpha_n) d^2(\varkappa_n, a) + \alpha_n d^2(\varkappa_n, a) - 4c_M \alpha_n \beta_n \gamma_n (1-\gamma_n) d^2(\varkappa_n, v_n) \\ &= d^2(\varkappa_n, a) - 4c_M \alpha_n \beta_n \gamma_n (1-\gamma_n) d^2(\varkappa_n, v_n) \\ 4c_M \alpha_n \beta_n \gamma_n (1-\gamma_n) d^2(\varkappa_n, v_n) &\le d^2(\varkappa_n, a) - d^2(\varkappa_{n+1}, a). \end{aligned}$$

Since $c_M > 0$ and $0 < e \le f_n \le f < 1$, by Lemma 1.15, it follows that

$$\sum_{n=1}^{\infty} e^3 (1-f) d^2(\varkappa_n, v_n) \le \sum_{n=1}^{\infty} \alpha_n \beta_n \gamma_n (1-\gamma_n) d^2(\varkappa_n, v_n) < \infty. \tag{2.3}$$

Thus, $\lim_{n\to\infty} d^2(x_n, v_n) = 0$, and hence

$$\lim_{n\to\infty} R(\varkappa_n, \Gamma(\varkappa_n)) = \lim_{n\to\infty} d(\varkappa_n, v_n) = 0. \tag{2.4}$$

By Lemma 2.1, $\{d(\varkappa_n, a)\}$ converges for all $a \in End(\Gamma)$.

To prove that $\{\varkappa_n\}$ $\Delta-$ converges to an element in $End(\Gamma)$, it is sufficient to demonstrate that $\{\varkappa_n\}$ has a unique asymptotic center in $End(\Gamma)$. For this one, we assume that there are subsequences in $\{\varkappa_{n_i}\}$ and $\{\varkappa_{m_i}\}$ of $\{\varkappa_n\}$ with $A(\varkappa_{n_i}) = \varkappa_1$ and $A(\varkappa_{m_i}) = \varkappa_2$. Since $diam(\Gamma \varkappa_{n_i}) = 0$, it follows that $\varkappa_1 \in End(\Gamma)$. Similarly, we can get $\varkappa_2 \in End(\Gamma)$. Now, to prove $\varkappa_1 = \varkappa_2$.

On contrary, suppose that $\varkappa_1 \ne \varkappa_2$.

$$\begin{aligned} \lim_{n\to\infty} d(\varkappa_n, \varkappa_1) &= \lim_{i\to\infty} d(\varkappa_{n_i}, \varkappa_1) < \lim_{i\to\infty} d(\varkappa_{n_i}, \varkappa_2) \\ &= \lim_{n\to\infty} d(\varkappa_n, \varkappa_2) = \lim_{i\to\infty} d(\varkappa_{m_i}, \varkappa_2) \\ &< \lim_{i\to\infty} d(\varkappa_{m_i}, \varkappa_1) = \lim_{n\to\infty} d(\varkappa_n, \varkappa_1), \end{aligned}$$

which is a contradiction. Hence, $\{\varkappa_n\}$ $\Delta-$ converges to an element in $End(\Gamma)$. □

Definition 2.3 ([22]). A mapping $\Gamma : D \to CB(D)$ is semicompact if for any bounded sequence $\{\varkappa_n\}$ satisfying $\lim_{n\to\infty} R(\varkappa_n, \Gamma(\varkappa_n)) = 0$ has a convergent subsequence. A mapping Γ implies condition (J) if there is a nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0, h(r) > 0$ for $r \in (0, \infty)$ such that $R(\varkappa_n, \Gamma(\varkappa_n)) \ge h(d(\varkappa_n, End(\Gamma))$ for each $\varkappa \in D$.

Theorem 2.4. *Let Y, D, Γ and $\{\varkappa_n\}$ be as in Lemma 2.1. If Γ satisfies condition (J), then $\{\varkappa_n\}$ converges strongly to an element in $End(\Gamma)$.*

Proof. From (2.4), we have $\lim_{n\to\infty} R(\varkappa_n, \Gamma(\varkappa_n)) = 0$. As Γ satisfies condition (J), we have $R(\varkappa_n, \Gamma(\varkappa_n)) \ge h(d(\varkappa_n, End(\Gamma))$. So, $\lim_{n\to\infty} d(\varkappa_n, End(\Gamma)) = 0$. From Propsition 1.4(ii) Γ is quasi-nonexpansive, $End(\Gamma)$ is closed. By Lemma 2.1, $\{\varkappa_n\}$ is Fejer monotone according to $End(\Gamma)$. So, $\{\varkappa_n\}$ converges strongly to an element in $End(\Gamma)$. □

Theorem 2.5. *Let Y, D, Γ and $\{\varkappa_n\}$ be as in Lemma 2.1. If Γ is semicompact, then $\{\varkappa_n\}$ converges strongly to an element in $End(\Gamma)$.*

Proof. From (2.3), we get

$$\alpha_n\beta_n\gamma_n(1-\gamma_n)\, d^2(\varkappa_n, v_n) < \infty .$$

From Lemma 1.15, there exist subsequences $\{v_{n_k}\}$ and $\{\varkappa_{n_k}\}$ of $\{v_n\}$ and $\{\varkappa_n\}$ respectively, such that $\lim_{k\to\infty} d^2(\varkappa_{n_k}, v_{n_k}) = 0$. So

$$\lim_{k\to\infty} R(x_{n_k}, \Gamma(x_{n_k})) = \lim_{k\to\infty} d(x_{n_k}, v_{n_k}) = 0. \tag{2.5}$$

Since Γ is semicompact, one can find a strongly convergent sequence $\{\varkappa_{n_k}\}$ of $\{\varkappa_n\}$ with the strong limit, i.e., b. We shall show that $b \in End(\Gamma)$. From Lemma 1.6 ,

$$\begin{aligned} d(b, \Gamma(b)) &\leq d(b, \varkappa_{n_k}) + d(\varkappa_{n_k}, \Gamma(b)) \\ &\leq d(b, \varkappa_{n_k}) + \left(\frac{3+\alpha}{1-\alpha}\right) d(\varkappa_{n_k}, \Gamma(\varkappa_{n_k})) + d(\varkappa_{n_k}, b) \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

Hence, $b \in \Gamma(b)$. From Proposition 1.4 (ii),

$$H(\Gamma(\varkappa_{n_k}), \Gamma(b)) \leq d(b, \varkappa_{n_k}) \to 0 \text{ as } k \to \infty. \tag{2.6}$$

Let $a \in \Gamma(b)$ and select $v_{n_k} \in \Gamma(\varkappa_{n_k})$ so that $d(a, v_{n_k}) = d(a, \Gamma(\varkappa_{n_k}))$. By (2.5) and (2.6) we obtain

$$\begin{aligned} d(b, p) &\leq d(b, \varkappa_{n_k}) + d(\varkappa_{n_k}, v_{n_k}) + d(v_{n_k}, a) \\ &= d(b, \varkappa_{n_k}) + d(\varkappa_{n_k}, v_{n_k}) + d(\Gamma(\varkappa_{n_k}), a) \\ &\leq d(b, \varkappa_{n_k}) + R(\varkappa_{n_k}, \Gamma(\varkappa_{n_k})) + H(\Gamma(\varkappa_{n_k}), \Gamma(b)) \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

Hence, $a = b$ for all $a \in \Gamma(b)$, that is $b = \Gamma(b)$. So $b \in End(\Gamma)$. From Lemma 2.1, $\lim_{n\to\infty} d(\varkappa_n, b)$ exists and for that reason b is the strong limit of $\{\varkappa_n\}$. □

3. Conclusion

We proved the results of endpoint convergence for generalized α-multivalued nonexpansive mappings in a hyperbolic metric space. Our results extend the endpoints of multivalued Suzuki mappings in Kaplan [6](Theorem 1, Theorem 2, Theorem 3) to a wider class of uniformly convex hyperbolic spaces, which is more general than Banach spaces, CAT(0) spaces and some CAT(κF) spaces. Also, the class of multivalued generalized α-nonexpansive mappings is in larger than that the class of generalized nonexpansive multivalued mappings properly includes the class of nonexpansive multivalued mappings.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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