MÜHENDİS BEYİNLER DERGİSİ | JOURNAL OF ENGINEER BRAINS

YIL | YEAR 2016

Bilim, Mühendislik ve Teknoloji Yayınları

ISSN: 2528 - 9802

Sayı | No: 1
Cilt | Volume: 1

Yüksek Mertebeden Lineer Diferansiyel Fark Denklemlerinin Residüel Hata Tahminiyle Çözümü için Boubaker Polinom Yaklaşımı

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Özet: *Bu çalışmanın temel amacı başlangıç-sınır koşulları altında fonksiyonel argümentli yüksek mertebeden lineer diferansiyel-fark denklemlerinin çözümü için Boubaker polinomlarını uygulamaktır. Kullandığımız teknik, aslında sıralama noktaları ile birlikte kesilmiş Boubaker serisine ve bunların matris gösterimlerine dayandırılır. Ayrıca, Ortalama-Değer Teoremini ve rezidüel fonksiyonu kullanarak, etkili bir hata tahmin tekniği önerilir; metodun etkinliğini ve uygulanabilirliğini göstermek için bazı açıklayıcı örnekler sunulur.*

Anahtar Kelime: *Boubaker polinomları ve serileri, sıralama noktaları, diferansiyel-fark denklemleri, matris metodu.*

Boubaker Polynomial Approach for Solving High-Order Linear Differential-Difference Equations with Residual Error Estimation

Abstract: *The main aim of this study is to apply the Boubaker polynomials for the solution of high-order linear differential-difference equations with functional arguments under the initial-boundary conditions. The technique we have used is essentially based on the truncated Boubaker series and its matrix representations together with collocation points. Also, by using the Mean-Volue Theorem and residual function, an efficient error estimation technique is proposed and some illustrative examples are presented to demonstrate the validity and applicability of the method.*

Keywords: *Boubaker polynomials and series, collocation points, differential-difference equations, matrix method*

1.Introduction

Differential-difference equations [1-10], which are a class of functional differential-equations, have been treated as models of some physical phenomena. When a mathematical model is developed for a physical system, it is usually assumed that all of the variables, such as space and time, are continuous. This assumption leads to a realistic and justified approximation of the real variables of the system. However, for some of the physical systems, these continuous variable assumptions can't be made. Then differential-difference equations have played an important role modeling problems that appear in various branches of science; e.g., mechanical engineering, condensed matter, biophysics mathematical statistic and control theory. In recent years, the studies of differential-difference equations are developed very rapidly and intensively. It is well known that linear differential-difference equations have been considered by many authors, and have been used in the applications of difference models to problems in biology, physics and engineering.

Recently, a number of different methods associated with the solution of higher-order differential-difference equations, which are the inverse scattering method [11], Hirota's bilinear form method [12], Tanh-method [13], Jacobian elliptic function method [14], numerical techniques [15,16], Taylor polynomial methods [17,18] and Chebyshev methods [8,16], have been given.

In this study, the basic ideas of the mentioned studies are developed to obtain the approximate solutions oh high-order linear differential-difference equation with functional arguments (advanced, neutral or delayed) and variable coefficients in the form

$$\sum_{k=0}^{m} \boldsymbol{f}_k(x) y^{(k)}(x) + \sum_{j=0}^{J} \boldsymbol{P}_j(x) y^{(j)}(\alpha x+\beta) = g(x), \qquad J \le m \tag{1}$$

under the mixed conditions

$$\sum_{k=0}^{m-1} \left(a_{lk} y^{(k)}(a) + b_{lk} y^{(k)}(b)\right) = \lambda_l, \qquad l = 0,1,\dots,m-1 \tag{2}$$

Here $f_k(x), P_j(x)$ and $g(x)$ are known functions defined on the interval $a \le x \le b$; α, β, a_{lk}, b_{lk} and λ_l are appropriate constants; $y(x)$ is an unknown function to be determined. The aim of this study is to get the solution of the problem (1) – (2) as the truncated Boubaker series defined by

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n B_n(x)\,, \quad N \ge m, \;\; a \le x \le b \tag{3}$$

where $a_n, n = 0,1,\dots,N$ are unknown coefficients; $B_n(x); n = 0,1,2,\dots,N$, denote the Boubaker polynomials defined by [11,13]

$$B_n(x) = \sum_{p=0}^{[n/2]} (-1)^p \frac{(n-4p)}{(n-p)} \binom{n-p}{p} x^{n-2p} \tag{4}$$

or recursively

$$B_n(x) = xB_{n-1}(x) - B_{n-2}(x) \quad ; \quad n \ge 2$$

with $B_0(x) = 1$ and $B_1(x) = x$.

On the other hand, by using (4), the first four Boubaker polynomials are given by

$$B_0(x) = 1, \qquad B_1(x) = x, \qquad B_2(x) = x^2+2, \qquad B_3(x) = x^3 + x, \dots$$

and the Boubaker polynomials $B_n(x)$ are solution of the following differential equation:

$$(x^2-1)(3nx^2+n-2)B_n''(x) + 3x(nx^2+3n-2)B_n'(x) - n(3n^2x^2+n^2-6n+8)\, B_n(x) = 0$$

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2. Fundamental Matrix Relations

We first consider the solution $y(x)$ of Eq. (1) defined by the truncated Boubaker series (3), which is given in the form

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n B_n(x)\,, \quad N \geq m, \;\; a \leq x \leq b \;.$$

Then we can convert the finite series (3) to the matrix form as, for $n = 0,1,2, \dots, N$

$$y(x) \cong y_N(x) = \boldsymbol{B}(x)\boldsymbol{A} \tag{5}$$

So that

$$\boldsymbol{B}(x) = [B_0(x) \quad B_1(x) \quad \dots \quad B_N(x)]$$

$$\boldsymbol{A} = [a_0 \quad a_1 \quad \dots \quad a_N]^T.$$

On the other hand, by using the relation (4), the matrix $y(x)$ is obtained as

$$B(x) = \boldsymbol{X}(x)\boldsymbol{H} \tag{6}$$

where

$$\boldsymbol{X}(x) = [1 \quad x \quad x^2 \quad \dots \quad x^N]$$

and if N is odd,

$$\mathbf{H}^T = \begin{bmatrix} \phi_{0,0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \phi_{1,0} & 0 & \cdots & 0 & 0 \\ \phi_{2,1} & 0 & \phi_{2,0} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{\mathrm{N-1},\frac{\mathrm{N-1}}{2}} & 0 & \phi_{\mathrm{N-1},\frac{\mathrm{N-3}}{2}} & \cdots & \phi_{\mathrm{N-1},0} & 0 \\ 0 & \phi_{\mathrm{N},\frac{\mathrm{N-1}}{2}} & 0 & \cdots & 0 & \phi_{\mathrm{N},0} \end{bmatrix}$$

and if N is even,

$$\mathbf{H}^T = \begin{bmatrix} \phi_{0,0} & 0 & 0 & \ldots & 0 & 0 \\ 0 & \phi_{1,0} & 0 & \ldots & 0 & 0 \\ \phi_{2,1} & 0 & \phi_{2,0} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \phi_{N-1,\frac{N-2}{2}} & 0 & \ldots & \phi_{N-1,0} & 0 \\ \phi_{N,\frac{N}{2}} & 0 & \phi_{N,\frac{N-2}{2}} & \ldots & 0 & \phi_{N,0} \end{bmatrix}$$

where

$$B_n(x) = \sum_{p=0}^{[n/2]} \phi_{n,p} x^{n-2p} \quad , \qquad n = 0,1,\dots,N, \qquad p = 0,1,\dots,\left\lfloor\frac{n}{2}\right\rfloor,$$

$$\phi_{n,p} = \left[\frac{(n-4p)}{(n-p)}\binom{n-p}{p}\right](-1)^p .$$

Also, it is clearly seen form (6) that the relation between the matrix $X(x)$ and derivative $B^{(k)}(x)$ is

$$B'(x) = \boldsymbol{X}'(x)\boldsymbol{H}$$

$$= \boldsymbol{X}(x)\boldsymbol{M}\boldsymbol{H}$$

and that repeating the process

$$B^{(k)}(x) = \boldsymbol{X}(x)\boldsymbol{M}^k\boldsymbol{H}, \quad k = 0,1,2,\dots,m \tag{7}$$

where

$$\mathbf{M}^0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & N \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

From the matrix relations (5), (6) and (7), it follows that

$$y(x) \cong y_N(x) = \boldsymbol{B}(x)\boldsymbol{A}$$

$$= \boldsymbol{X}(x)\boldsymbol{H}\boldsymbol{A}$$

and

$$y^{(k)} \cong y_N^{(k)} = \boldsymbol{B}^{(k)}(x)\boldsymbol{A}$$

$$= \boldsymbol{X}^{(k)}(x)\boldsymbol{H}\boldsymbol{A}$$

$$= \boldsymbol{X}(x)\boldsymbol{M}^k\boldsymbol{H}\boldsymbol{A}, \quad k = 0,1,\dots \tag{8}$$

By substituting $x \to \alpha x + \beta$ into the relation (8), we get, $j = 0,1,2,\dots$

$$y^{(j)}(\alpha x + \beta) = \boldsymbol{X}(\alpha x + \beta)\boldsymbol{M}^j\boldsymbol{H}\boldsymbol{A}$$

$$= \boldsymbol{X}(x)\boldsymbol{D}(\alpha,\beta)\boldsymbol{M}^j\boldsymbol{H}\boldsymbol{A} \tag{9}$$

so that, for $\alpha \neq 0$,

$$\mathbf{D}(\alpha,\beta) = \begin{bmatrix} \binom{0}{0}\alpha^0\beta^0 & \binom{1}{0}\alpha^0\beta^1 & \binom{2}{0}\alpha^0\beta^2 & \cdots & \binom{N}{0}\alpha^0\beta^N \\ 0 & \binom{1}{1}\alpha^1\beta^0 & \binom{2}{1}\alpha^1\beta^1 & \cdots & \binom{N}{1}\alpha^1\beta^{N-1} \\ 0 & 0 & \binom{2}{2}\alpha^2\beta^0 & \cdots & \binom{N}{2}\alpha^2\beta^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}\alpha^N\beta^0 \end{bmatrix}.$$

3. Boubaker Collocation Method

For constructing the fundamental matrix equation, we first consider the collocation points defined by, for $i = 0,1,\dots,N$,

$$x_i = a + \frac{b-a}{N}i, \quad \text{(Standard)}$$

or (10)

$$x_i = \frac{b+a}{2} - \frac{b-a}{2}cos\left(\frac{\pi i}{N}\right) \quad \text{(Chebyshev-Lobatto).}$$

Then, by using the collocation points (10) into (1), we have the system of the equations

$$\sum_{k=0}^{m-1} \boldsymbol{f}_k(x_i)y^{(k)}(x_i) + \sum_{j=0}^{J} P_j(x_i)y^{(j)}(\alpha x_i + \beta) = g(x_i)$$

or briefly the corresponding matrix equation

$$\sum_{k=0}^{m} \boldsymbol{F}_k \boldsymbol{Y}^{(k)} + \sum_{j=0}^{J} \boldsymbol{P}_j \boldsymbol{Y}^{(j)}_{(\alpha,\beta)} = \boldsymbol{G} \tag{11}$$

where

$$\boldsymbol{F}_k = diag[f_k(x_0) \quad f_k(x_1) \quad \dots \quad f_k(x_N)]$$

$$\boldsymbol{P}_j = diag\left[p_j(x_0) \quad p_j(x_1) \quad \dots \quad p_j(x_N)\right]$$

$$\mathbf{Y}^{(k)} = \begin{bmatrix} y^{(k)}(x_0) \\ y^{(k)}(x_1) \\ \vdots \\ y^{(k)}(x_N) \end{bmatrix}, \quad \mathbf{Y}^{(j)}_{(\alpha,\beta)} = \begin{bmatrix} y^{(j)}_{(\alpha x_0+\beta)} \\ y^{(j)}_{(\alpha x_1+\beta)} \\ \vdots \\ y^{(j)}_{(\alpha x_N+\beta)} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}.$$

On the other hand, by substituting the collocation points (10) into (8) and (9), we obtain the matrix relations

$$\mathbf{Y}^{(k)} = \begin{bmatrix} y^{(k)}(x_0) \\ y^{(k)}(x_1) \\ \vdots \\ y^{(k)}(x_N) \end{bmatrix} = \begin{bmatrix} X(x_0)\mathrm{M}^k HA \\ X(x_1)\mathrm{M}^k HA \\ \vdots \\ X(x_N)\mathrm{M}^k HA \end{bmatrix} = XM^k HA$$

$$\mathbf{Y}^{(j)}_{(\alpha,\beta)} = \begin{bmatrix} y^{(j)}(x_0) \\ y^{(j)}(x_1) \\ \vdots \\ y^{(j)}(x_N) \end{bmatrix} = \begin{bmatrix} X(x_0)\mathrm{D}(\alpha,\beta)\mathrm{M}^j HA \\ X(x_1)\mathrm{D}(\alpha,\beta)\mathrm{M}^j HA \\ \vdots \\ X(x_N)\mathrm{D}(\alpha,\beta)\mathrm{M}^j HA \end{bmatrix} = X\,\mathrm{D}(\alpha,\beta)\mathrm{M}^j HA$$

so that

$$\mathbf{X} = \begin{bmatrix} X(x_0) \\ X(x_1) \\ \vdots \\ X(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ 1 & x_2 & x_2^2 & \cdots & x_2^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{bmatrix}.$$

Therefore the fundamental matrix equation for Eq. (1) becomes

$$\left[\sum_{k=0}^{m} \boldsymbol{F}_k \boldsymbol{X} \boldsymbol{M}^k + \sum_{j=0}^{J} \boldsymbol{P}_j \boldsymbol{X} \boldsymbol{D}(\alpha,\beta) \boldsymbol{M}^j\right] \boldsymbol{H} \boldsymbol{A} = \boldsymbol{G} \qquad (12)$$

Now we can find the fundamental matrix form for the conditions (2), by using the relation (8), as follows:

$$\sum_{k=0}^{m-1} \left(a_{lk} \boldsymbol{X}(a) + b_{lk} \boldsymbol{X}(b)\right) \boldsymbol{M}^k \boldsymbol{H} \boldsymbol{A} = \lambda_l, \quad l = 0,1,\dots,m-1 \qquad (13)$$

Then we can write the fundamental matrix equations (12) and (13) corresponding to Eq. (1) and the conditions (2), respectively, as follows:

$$\boldsymbol{W} \boldsymbol{A} = \boldsymbol{G} \text{ or } [\boldsymbol{W}; \boldsymbol{G}] \qquad (14)$$

and

$$\boldsymbol{U}_l \boldsymbol{A} = \lambda_l \text{ or } [U_l; \lambda_l], \quad l = 0,1,\dots,m-1 \qquad (15)$$

where

$$\boldsymbol{W} = [w_{pq}] = \left\{\sum_{k=0}^{m} \boldsymbol{F}_k \boldsymbol{X} \boldsymbol{M}^k + \sum_{j=0}^{J} \boldsymbol{P}_j \boldsymbol{X} \boldsymbol{D}(\alpha,\beta) \boldsymbol{M}^j\right\} \boldsymbol{H}$$

and

$$\boldsymbol{U}_l = [u_{l0} \quad u_{l1} \quad \dots \quad u_{lN}] = \sum_{k=0}^{m-1} \left(a_{lk} \boldsymbol{X}(a) + b_{lk} \boldsymbol{X}(b)\right) \boldsymbol{M}^k \boldsymbol{H} \boldsymbol{A} = \lambda_l, \quad l = 0,1,\dots,m-1$$

Consequently, the obtain the solution of $Eq.(1)$ under the conditions (2), by replacing the row matrices (15) by the last (or any) m rows of the augmented matrix (14), we have the required matrix

$$[\widetilde{\boldsymbol{W}}; \widetilde{\boldsymbol{G}}] \text{ or } \widetilde{\boldsymbol{W}} \boldsymbol{A} = \widetilde{\boldsymbol{G}} \qquad (16)$$

If rank $\widetilde{W} = rank[\widetilde{\boldsymbol{W}}; \widetilde{\boldsymbol{G}}] = N+1$, then we can write $A = (\widetilde{\boldsymbol{W}})^{-1} \widetilde{\boldsymbol{G}}$. Thus the matrix $\boldsymbol{A}$ (thereby the coefficients $a_0, a_1, \dots, a_N$) is uniquely determined. Also, $Eq.(1)$ under the conditions (2) has a unique solution. This solution is given by the truncated Boubaker series (3).

4. Accuracy of Solutions and Residual Error Estimation

We can easily check the accuracy of the obtained solutions as follows. Since the truncated Boubaker series (3) is approximate solution of (1), when the function $y_N(x)$ and its derivatives

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are substituted in Eq. (1), the resulting equation must be satisfied approximately; that is, for $x = x_r \in [a,b], r = 0,1, \ldots$

$$R_N(x_r) = \sum_{k=0}^{m} \boldsymbol{f}_k(x_r) y^{(k)}(x_r) + \sum_{j=0}^{J} \boldsymbol{P}_j(x_r) y^{(j)}(\alpha x_r + \beta) - g(x_r) \cong 0$$

or

$$R_N(x_r) \leq 10^{-k_r}, \quad (k_r \text{ is any positive integer}).$$

If max $10^{-k_r} = 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $R_N(x_r)$ at each of the points becomes smaller than the prescribed 10^{-k} [8,9,19].

On the other hand, by means of the residual function defined by $R_N(x)$ and the mean value of the function $|R_N(x)|$ on the interval $[a,b]$, the accuracy of the solution can be controlled and the error can be estimated. If $R_N(x) \to 0$ when N is sufficiently large enough, then the error decreases. Also, by using the Mean-Value Theorem, we can estimate the upper bound of the mean error, $\bar{R}_N$ as follows:

$$\left| \int_a^b R_N(x)\, dx \right| \leq \int_a^b |R_N(x)|\, dx$$

and

$$\int_a^b R_N(x)\, dx = (b-a) R_N(c), \qquad a \leq c \leq b$$

$$\Rightarrow \left| \int_a^b R_N(x)\, dx \right| = (b-a)|R_N(c)|$$

$$\Rightarrow (b-a)|R_N(c)| \leq \int_a^b |R_N(x)|\, dx$$

$$|R_N(c)| \leq \frac{\int_a^b |R_N(x)|\, dx}{b-a} = \bar{R}_N$$

5. Numerical Examples

The method of this study is useful in finding the solution of higher order linear differential-difference equations in terms of Boubaker polynomials. We illustrate the numerical solution with the following examples.

Example 1. [11,18] Let us first consider the boundary value problem

$$y''(x) - xy'(x-1) + y(x-2) = -x^2 - 2x + 5$$

$$y(0) = -1, y'(-1) = -2.$$

with exact solutions $y(x) = x^2 - 1$

We assume that the problem has a Boubaker polynomial solution in the form

$$y(x) = \sum_{n=0}^{4} a_n B_n(x).$$

Here, $N = 4, P_0(x) = 0, P_1(x) = 0, \ P_2(x) = 1, Q_0(x) = 1, Q_1(x) = -x,$

$$\alpha_0 = 1, \beta_0 = -2, \alpha_1 = 1, \beta_1 = -1, - \leq x \leq 0, g(x) = -x^2 - 2x + 5$$

and the collocation points are

$$x_i = a + \left(\frac{b-a}{N}\right) i\,, \ \ i = 0,1,2,3,4 \ \Rightarrow \left\{x_0 = -1, \ x_1 = -\frac{3}{4}, \ x_2 = -\frac{1}{2}, \ x_3 = -\frac{1}{4}, x_4 = 0\right\}.$$

From Eq. (12), the matrix representation of the equation is

$$\{\mathbf{F_0XH} + \mathbf{F_1XMH} + \mathbf{F_2XM^2H} + \mathbf{P_0XDH} + \mathbf{P_1XDMH}\}\mathbf{A} = \mathbf{G},$$

such that

$$\mathbf{F}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ \mathbf{P}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ \mathbf{P}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3/4 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} 6 \\ 95/16 \\ 23/4 \\ 87/16 \\ 5 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & -3/4 & 9/16 & -27/64 & 81/256 \\ 1 & -1/2 & 1/4 & -1/8 & 1/16 \\ 1 & -1/4 & 1/16 & -1/64 & 1/256 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathbf{M}^2 = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D}(1,-2) = \begin{bmatrix} 1 & -2 & 4 & -8 & 16 \\ 0 & 1 & -4 & 12 & -32 \\ 0 & 0 & 1 & -6 & 24 \\ 0 & 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ \mathbf{D}(1,-1) = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 3 & -4 \\ 0 & 0 & 1 & -3 & 6 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ \mathbf{H} = \begin{bmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For the fundamental equation and the conditions $y(0) = -1$ and $y'(-1) = -2$, the augmented matrices are obtained as

$$[\mathbf{W};\mathbf{G}]=\begin{bmatrix} 1 & -2 & 9 & -23 & 59 & ; & 6 \\ 1 & -2 & 143/16 & -653/32 & 11741/256 & ; & 95/16 \\ 1 & -2 & 35/4 & -69/4 & 533/16 & ; & 23/4 \\ 1 & -2 & 135/6 & -439/32 & 5741/256 & ; & 87/16 \\ 1 & -2 & 8 & -10 & 14 & ; & 5 \end{bmatrix}$$

and

$$[u_0;\lambda_0]=[1\quad 0\quad 2\quad 0\quad -2;\ -1],$$
$$[u_1;\lambda_1]=[0\quad 1\quad -2\quad 4\quad -4;\ -2],$$

By replacing the condition matrices by the two rows of the matrix, we have the required augmented matrix

$$[\mathbf{W};\mathbf{G}]=\begin{bmatrix} 1 & -2 & 9 & -23 & 59 & ; & 6 \\ 1 & 0 & 2 & 0 & -2 & ; & -1 \\ 0 & 1 & -2 & 4 & -4 & ; & -2 \\ 1 & -2 & 135/6 & -439/32 & 5741/256 & ; & 87/16 \\ 1 & -2 & 8 & -10 & 14 & ; & 5 \end{bmatrix}.$$

This system has the solution

$$\mathbf{A}=[-3\quad 0\quad 1\quad 0\quad 0]^{\mathrm{T}}.$$

Therefore, we obtain the approximate solution as

$$y_4(x)=\sum_{n=0}^{4}a_nB_n(x)=a_0B_0(x)+a_1B_1(x)+a_2B_2(x)+a_3B_3(x)+a_4B_4(x)\rightarrow y_4(x)=x^2-1$$

which is the exact solution.

Example 2. [16] Let us consider the multi-pantograph problem

$$y'(x)=\frac{1}{2}e^{\frac{x}{2}}y\left(\frac{x}{2}\right)+\frac{1}{2}y(x)$$
$$y(0)=1$$

with the exact solution $y(x)=\mathrm{e}^{x}$. To find the Boubaker polynomial solution of the problem above, we first take $N=5$, where

$$F_0(x)=-\frac{1}{2},\quad F_1=1,\quad P_0=-\frac{1}{2}e^{\frac{x}{2}},\quad 0\le x\le 1\quad g(x)=0,\quad \alpha=\frac{1}{2},\quad \beta=0.$$

Hence, the collocation points are

$$x_i=a+\left(\frac{b-a}{N}\right)i,\quad i=0,1,2,3,4,5\ \Longrightarrow\ \left\{x_0=0,\ x_1=\frac{1}{5},\ x_2=\frac{2}{5},\ x_3=\frac{3}{5},x_4=\frac{4}{5},x_5=1\right\}$$

and the matrix form of the problem is defined by

$$\{\mathbf{F_0XH}+\mathbf{F_1XMH}+\mathbf{P_0XD}(\alpha,\beta)\mathbf{H}\}\mathbf{A}=\mathbf{G}.$$

Also, the augmented matrices of the system and conditions are obtained as follows;

$$[\mathrm{W;G}]=\begin{bmatrix} -1.0000 & 1.0000 & -2.0000 & 1.0000 & 2.0000 & -3.0000 & ; & 0 \\ -1.0526 & 0.8447 & -1.7307 & 0.9602 & 2.1363 & -2.6418 & ; & 0 \\ -1.1107 & 0.6779 & -1.5258 & 1.1210 & 2.4636 & -2.3540 & ; & 0 \\ -1.1749 & 0.4975 & -1.3906 & 1.4513 & 3.1436 & -1.8389 & ; & 0 \\ -1.2459 & 0.3016 & -1.3312 & 1.9179 & 4.3159 & -0.6446 & ; & 0 \\ -1.3244 & 0.0878 & -1.3548 & 2.4848 & 6.0972 & 1.8138 & ; & 0 \end{bmatrix}$$

$$[u_0;\lambda_0]=[1 \quad 0 \quad 2 \quad 0 \quad -2 \quad 0; \;\; 1]$$

$$[\widetilde{\mathrm{W;G}}]=\begin{bmatrix} -1.0000 & 1.0000 & -2.0000 & 1.0000 & 2.0000 & -3.0000 & ; & 0 \\ 1.0000 & 0 & 2.0000 & 0 & -2.0000 & 0 & ; & 1 \\ -1.1107 & 0.6779 & -1.5258 & 1.1210 & 2.4636 & -2.3540 & ; & 0 \\ -1.1749 & 0.4975 & -1.3906 & 1.4513 & 3.1436 & -1.8389 & ; & 0 \\ -1.2459 & 0.3016 & -1.3312 & 1.9179 & 4.3159 & -0.6446 & ; & 0 \\ -1.3244 & 0.0878 & -1.3548 & 2.4848 & 6.0972 & 1.8138 & ; & 0 \end{bmatrix}$$

Solving the augmented system, we find the coefficients of $\mathbf{A}$ as

$$\mathbf{A}=\begin{bmatrix}0.0681 & 0.8574 & 0.4986 & 0.1869 & 0.0327 & 0.0148\end{bmatrix}^{\mathrm{T}}.$$

Therefore, we have the solution

$$y_5(x)=1+\mathrm{x}+0.4986x^2+0.1721x^3+0.0270x^4+0.0148x^5.$$

In addition, the approximate solutions for $N=3$, $N=4$, and $N=5$ are compared with the exact solutions in Table 1, Figure 1 and Figure 2.

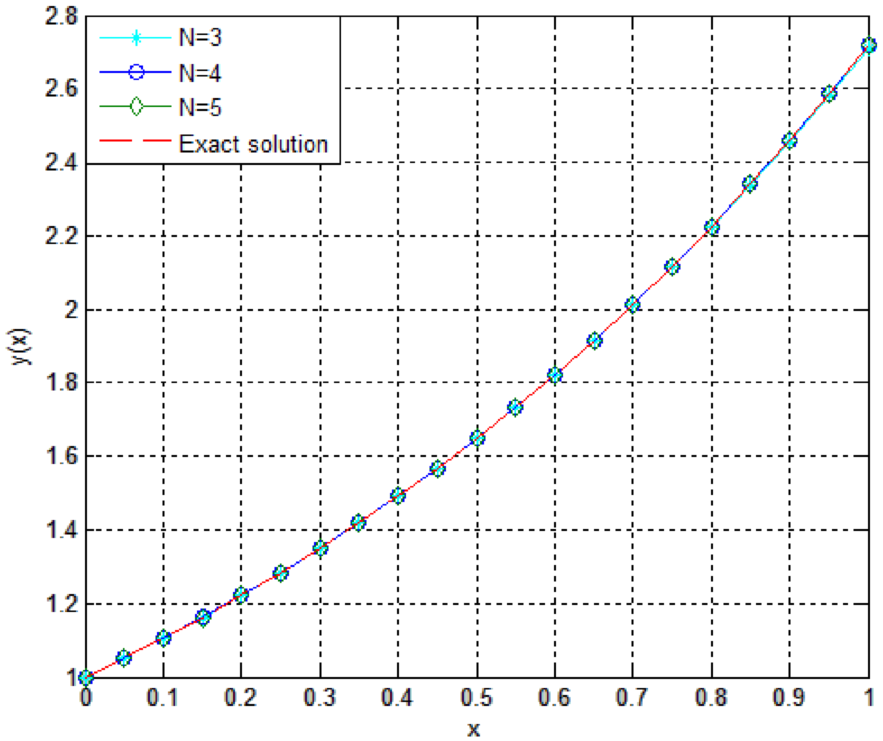

Figure 1. *Graphics of the exact solution and numerical solutions of Example 2 for* $N=3,4,5$

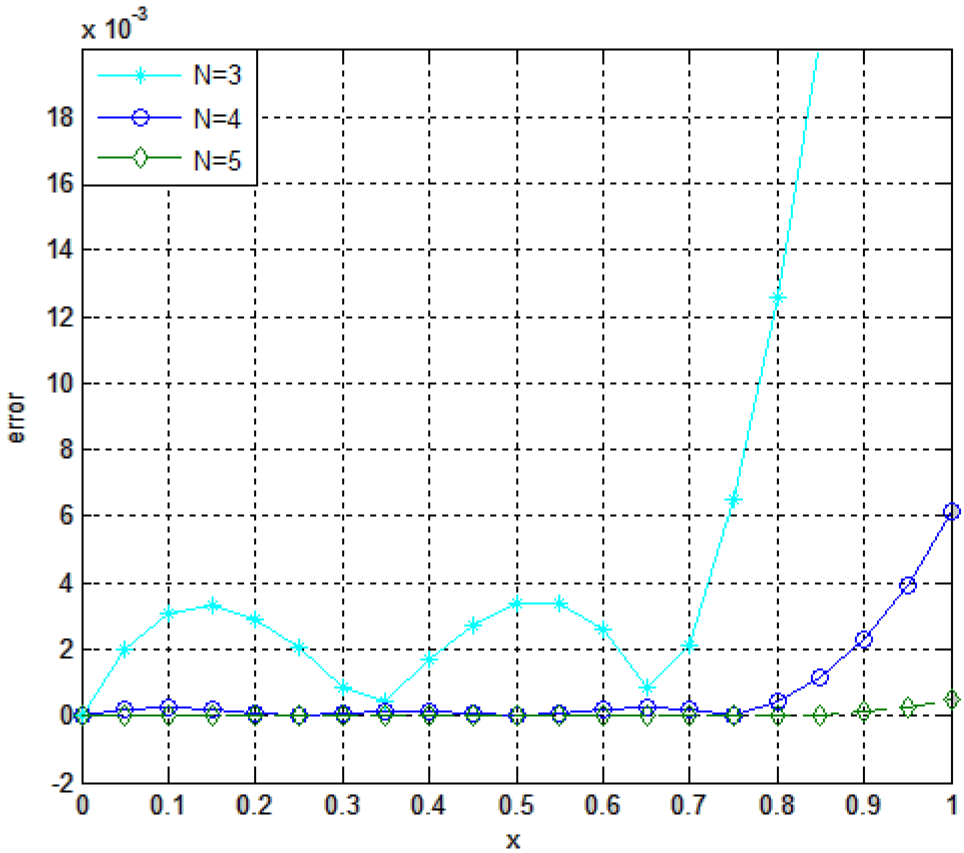

Figure 2. *Graphics of Residual Error Functions of Example 2 for* $N = 3, 4, 5$

Table 1. *Numerical results of Example 2*

x_i	$y(x_i)$	$y_3(x_i)$	$\|e_3(x_i)\|$	$y_4(x_i)$	$\|e_4(x_i)\|$	$y_5(x_i)$	$\|e_5(x_i)\|$
0.0	1.0	1.0	0.0	1.0	0.0	1.0	0.0
0.1	1.105170918	1.104981763	1.89155e-04	1.105187735	1.6817e-05	1.105169791	1.127e-06
0.2	1.221402758	1.220869635	5.33123e-04	1.221439736	3.6978e-05	1.221400844	1.914e-06
0.3	1.349858808	1.349077492	7.81316e-04	1.349899384	4.0576e-05	1.34985711	1.698e-06
0.4	1.491824698	1.491019210	8.05488e-04	1.491856987	3.2289e-05	1.491823152	1.546e-06
0.5	1.648721271	1.648108663	6.12608 e-04	1.648749773	2.8502e-05	1.648719174	2.097e-06
0.6	1.8221188	1.821759727	3.59073e-04	1.822161892	4.3092e-05	1.822116063	2.737e-06
0.7	2.013752707	2.013386277	3.6643e-04	2.013824418	7.1711e-05	2.013750422	2.285e-06
0.8	2.225540928	2.224402189	1.138739e-03	2.225615347	7.4419e-05	2.225539606	1.322e-06
0.9	2.459603111	2.456221339	3.381772e-03	2.459559598	4.3513e-05	2.459596762	6.349e-06
1.0	2.718281828	2.7102576	8.024228e-03	2.717829011	4.52809e-04	2.718245858	3.5962e-05

The residual error for $N = 3, 4, 5$;

$$\bar{R}_3 = \int_0^1 \frac{|R_3(x)|}{1-0} dx = 8.575113393 \text{x} 10^{-3},$$

$$\bar{R}_4 = \int_0^1 \frac{|R_4(x)|}{1-0} dx = 6.302235207 \text{x} 10^{-4},$$

$$\bar{R}_5 = \int_0^1 \frac{|R_5(x)|}{1-0} dx = 3.842434519 \text{x} 10^{-5}.$$

Example 3. Our last example is given by [18], for $-\frac{\pi}{2} \leq x, \leq 0$

$$y'''(\mathrm{x}) - \cos(\mathrm{x})\, y'(\mathrm{x}) - \sin(\mathrm{x})\, y'\left(x - \frac{\pi}{2}\right) + \sqrt{2}\mathrm{y}\left(x - \frac{\pi}{4}\right) = \sin(x) - 2\cos(x) - 1$$

with the exact solution $y(x) = \sin x$ and with conditions

$$y(0) = 0, y'(0) = 1, y''(0) = 0$$

where

$$F_0 = 0,\ F_1 = -\cos(x),\ F_2 = 0,\ F_3 = 1,\ P_0 = \sqrt{2},\ P_1 = -\sin(x),\ \alpha_0 = 1, \beta_0 = -\frac{\pi}{4}, \alpha_1 = 1, \beta_1 = -\frac{\pi}{2},$$

and the fundamental matrix form of the given differential-difference equation becomes

$$\{\mathbf{F_0XH} + \mathbf{F_1XMH} + \mathbf{F_2XM^2H} + \mathbf{F_3XM^3H} + \mathbf{P_0XDH} + \mathbf{P_1XDMH}\}\mathbf{A} = \mathbf{G}$$

Now, let us find the solutions of this problem taking $N = 4$, $N = 5$ and $N = 7$. The comparison of the solutions is given in Table 2, Figure 3 and Figure 4.

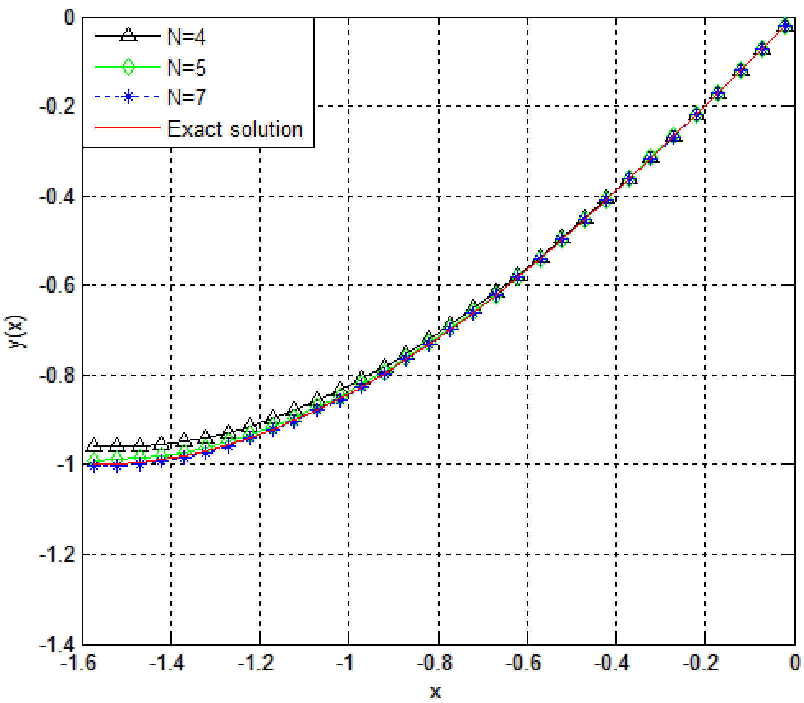

Figure 3. *Graphics of the exact solution and the numerical solutions to Example 3 for* $N = 4, 5, 7$

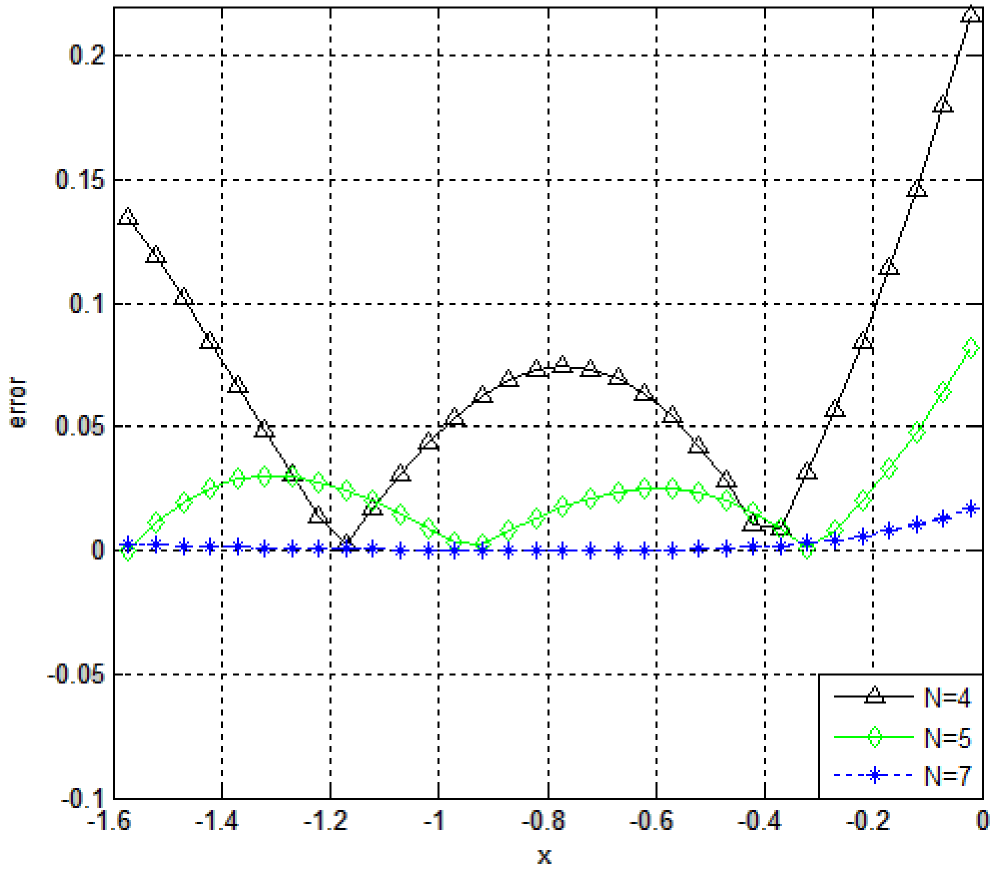

Figure 4. *Graphics of Residual Error Functions of Example 3 for* $N = 3, 4, 5$

Table 2. *Numerical results of Example 3*

x_i	$y(x_i)$	$y_4(x_i)$	$\lvert e_4(x_i)\rvert$	$y_5(x_i)$	$\lvert e_5(x_i)\rvert$	$y_7(x_i)$	$\lvert e_7(x_i)\rvert$
$\pi/2$	-1.0	-0.9589265665	4.10734335e-02	-0.9897292856	1.02707144 e-02	-1. 003932979	3.932979e-03
$\pi/3$	-0.8660254038	-0.8468051665	1.92202373e-02	-0.8607567010	5.2687028e-03	-0.8675589895	1.5335857e-03
$\pi/4$	-0.7071067812	-0.6968292175	1.02775637e-02	-0.7040225957	3.0841855e-03	-0.7078760680	7.692868e-04
$\pi/6$	-0.5	-0.4961625224	3.8374776e-03	-0.498738576	1.261424e-03	-0.5002806396	2.806396e-04
0	0.0	-1.0e-11	1.0e-11	1. 159016994e-10	1 .159016994e-10	-4. 3388e-08	-4. 3388e-08

The residual error for $N = 3, 4, 5$;

$$\bar{R}_4 = \int_0^1 \frac{|R_4(x)|}{\left|0-\left(-\frac{\pi}{2}\right)\right|} dx = 6.740208465 \text{x} 10^{-2},$$

$$\bar{R}_5 = \int_0^1 \frac{|R_5(x)|}{\left|0-\left(-\frac{\pi}{2}\right)\right|} dx = 2.232083286 \text{x} 10^{-2},$$

$$\bar{R}_7 = \int_0^1 \frac{|R_7(x)|}{\left|0-\left(-\frac{\pi}{2}\right)\right|} dx = 2.516258961 \text{x} 10^{-3}.$$

6. Conclusions and Discussions

A new technique based on the Boubaker polynomials to numerically solve the high-order linear differential-difference equations with functional arguments is presented. High-order linear differential-difference equations are usually difficult to solve analytically. Then, numerical methods are required to obtain the approximate solutions. For this reason, the present method has been proposed for approximate solution and also analytical solution.

On the other hand, from Table 1, it may be observed that the errors found for different N show close agreement for various values of x_i. Tables and Figures indicate that as *N* increases, the errors decrease more rapidly; hence for better results, the large number *N* is recommended. Another considerable advantage of the method is that Boubaker coefficients of the solution are found very easily by using the computer programs. On the other hand, *N*th order approximation gives the exact solution when the solution is polynomial of degree equal to or less than *N*. If the solution is not polynomial, Boubaker series approximation converges to the exact solution as *N* increases.

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