http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 72, Number 4, Pages 1110–1125 (2023)
DOI:10.31801/cfsuasmas.1262668
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: March 9, 2023; Accepted: June 22, 2023

NEW PROOFS OF FEJER'S AND DISCRETE HERMITE-HADAMARD INEQUALITIES WITH APPLICATIONS

Cagla SEKIN[1], Mehmet Emin TAMAR[2] and Ilham A. ALIYEV[3]

[1,3]Department of Mathematics, Akdeniz University, Antalya, TÜRKİYE
[2]Department of Mathematics, Abdullah Gul University, Kayseri, TÜRKİYE

Abstract. New proofs of the classical Fejer inequality and discrete Hermite-Hadamard inequality (HH) are presented and several applications are given, including (HH)-type inequalities for the functions, whose derivatives have inflection points. Morever, some estimates from below and above for the first moments of functions $f : [a, b] \to \mathbb{R}$ about the midpoint $c = (a+b)/2$ are obtained and the reverse Hardy inequality for convex functions $f : (0, \infty) \to (0, \infty)$ is established.

1. Introduction

The famous Hermite-Hadamard inequality (HH) asserts that the integral mean value of a convex function $f : [a, b] \to \mathbb{R}$ can be estimated above and below by its values at the points a, b and $(a + b)/2$. More precisely,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{HH}$$

Equality holds only for functions of the form $f(x) = cx + d$. Following Niculescu and Persson [17], we denote the right and left sides of (HH) by (RHH) and (LHH), respectively.

(HH) has many generalizations, extensions and refinements. There is an extensive literature in this area, such as books by Niculescu and Persson [18]; Mitrinovic,

2020 *Mathematics Subject Classification.* Primary 26D10; Secondary 26D15, 26A51.
Keywords. Fejer inequality, convex functions, discrete Hermite-Hadamard inequality, Jensen inequality, Hardy inequality.

1 caglasekin@akdeniz.edu.tr; 0000-0001-7176-5164;
2 mehmetemin.tamar@agu.edu.tr; 0000-0001-8933-8769
3 ialiev@akdeniz.edu.tr-Corresponding author; 0000-0003-2353-7700.

Pecaric and Fink [16]; Dragomir and Pearce [6] and papers [1–5, 7–12, 15, 17, 19, 21, 22], which are a small part of the relevant references.

The content of this article is organized as follows.

In Section 2 we give two new proofs of (HH). We first present a short proof of Fejer's inequality, from which (HH) follows immediately. The second proof includes a discrete version of (HH), which, in our opinion, is of independent interest. As an application, we give an estimation from below and above of the integral of the convex function $f : [0, \infty) \to (0, \infty)$ via the series $\sum_1^\infty f(k)$ and $\sum_1^\infty f\left(k - \frac{1}{2}\right)$.

In Section 3, we give some new inequalities arising as a combination of (HH) with Hardy's inequality and iterated Hölder's inequality. For example, as a consequence we prove that, if $f : (0, \infty) \to (0, \infty)$ is convex and $f \in L_p(0, \infty)$, $\forall p > 1$, then

$$\lim_{p\to\infty} \frac{\left\| \frac{1}{x} \int_0^x f \right\|_p}{\|f\|_p} = 1.$$

Moreover, we obtain a reverse Hardy inequality for some family of convex functions on $(0, \infty)$.

Section 4 is devoted to the (HH)-type inequalities for the functions whose first derivatives have an inflection point. As a particular case, we show that if f' is concave on $\left[a, \frac{a+b}{2}\right]$ and convex on $\left[\frac{a+b}{2}, b\right]$, then

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \le \frac{b-a}{12} \left(f'(b) - f'(a)\right).$$

In the last section we prove various inequalities for functions having convex first or second order derivatives. According to the authors' knowledge, there are some inequalities for functions whose *absolute values* of the derivatives are convex, see, e.g. [2, 5, 19]. In Theorems 7, 8, and 9 the convexity condition is imposed on the derivatives themselves, not on their absolute values. One of the interesting particular results obtained in this section is as follows.

Given $f : [a, b] \to \mathbb{R}$, let f' be convex. Then

$$\int_{\frac{a+b}{2}}^b f(x)dx - \int_a^{\frac{a+b}{2}} f(x)dx \le \frac{b-a}{4}(f(b) - f(a)).$$

Another new result in this section is the estimation from below and above of the first moment about the midpoint $c = (a + b)/2$ of a function $f : [a, b] \to \mathbb{R}$, i.e. the integral $M_f = \int_a^b \left(x - \frac{a+b}{2}\right) f(x)dx$, when f' is convex.

2. New Proofs of Fejer's Inequality and Discrete (HH)

At first, we give an auxiliary inequality that is satisfied by convex functions.

Lemma 1. *(cf. [11] and Lemma 1.3 in [15]) Let f be a convex function on $[a, b]$. Then*

$$f(a) + f(b) \ge f(a + b - x) + f(x), \ (\forall x \in [a, b]). \tag{1}$$

By making use of (1) we give here a short proof of the (HH) "without pulling the pen on the paper". More precisely, we give a short proof of a generalization of Hermite-Hadamard's inequality, which is named as the Fejer inequality and asserts that if f is convex on $[a,b]$ and the function $g:[a,b]\to[0,\infty)$ is integrable and symmetric with respect to the midpoint $\frac{a+b}{2}$, i.e. $g(a+b-x)=g(x)$, $(\forall x\in[a,b])$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{2}$$

For $g=1$, (2) turns into (HH).

To prove this inequality we will use of (1) and the following easily verifiable equality:

$$\int_a^b f(x)g(x)dx = \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]g(x)dx. \tag{3}$$

Now, we give a short proof of (2):

We have

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx &= \int_a^b f\left(\frac{a+b}{2}\right)g(x)dx = \int_a^b f\left(\frac{x+a+b-x}{2}\right)g(x)dx \\ &\le \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]g(x)dx \overset{(3)}{=} \int_a^b f(x)g(x)dx \\ &\overset{(3)}{=} \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]g(x)dx \\ &\overset{(1)}{\le} \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \end{aligned}$$

which is nothing but Fejer's inequality (2).

Remark 1. *Although the (HH) has several proofs, as far as we know the first simple proof was given by Azbetia [3]; (see, also Niculescu and Persson [17], p. 664). Another simple proof and refinement was given by El Farissi [8].*

The inequality given in the following theorem is a discrete version of (HH), and classical (HH) can be obtained by passing to limit in this inequality.

Theorem 1. *If $f:[a,b]\to\mathbb{R}$ is convex and $x_k = a + k\frac{b-a}{n}$, $(k=1,2,\cdots,n)$, then*

$$f\left(\frac{\left(1-\frac{1}{n}\right)a+\left(1+\frac{1}{n}\right)b}{2}\right) \le \frac{1}{n}\sum_{k=1}^{n} f(x_k) \le \frac{1}{2}\left[f(a)\left(1-\frac{1}{n}\right)+f(b)\left(1+\frac{1}{n}\right)\right]. \tag{4}$$

Proof. Let $x_k = a + k\frac{b-a}{n}$, $(k=1,2,\cdots,n)$. Then writing x_k as

$$x_k = \frac{b-x_k}{b-a}a + \frac{x_k-a}{b-a}b$$

and using

$$f(x_k) \le \frac{b - x_k}{b-a} f(a) + \frac{x_k - a}{b-a} f(b),$$

one has

$$\begin{aligned} \sum_{k=1}^{n} f(x_k) &\le \frac{f(a)}{b-a} \sum_{k=1}^{n} (b - x_k) + \frac{f(b)}{b-a} \sum_{k=1}^{n} (x_k - a) \\ &= \frac{1}{2} [f(a)(n-1) + f(b)(n+1)], \end{aligned}$$

and therefore,

$$\frac{1}{n} \sum_{k=1}^{n} f(x_k) \le \frac{1}{2} \left[f(a) \left(1 - \frac{1}{n} \right) + f(b) \left(1 + \frac{1}{n} \right) \right]. \tag{5}$$

On the other hand, the Jensen inequality yields

$$\frac{1}{n} \sum_{k=1}^{n} f(x_k) \ge f\left(\frac{1}{n} \sum_{k=1}^{n} x_k \right) = f\left(\frac{\left(1 - \frac{1}{n}\right) a + \left(1 + \frac{1}{n}\right) b}{2} \right). \tag{6}$$

By combining (5) and (6) we obtain (4). □

Corollary 1. *After taking limit as $n \to \infty$ in (4) and using the fact that the convex function is continuous (maybe except the end-points a and b), we obtain (HH).*

The following two theorems are the simple consequences of (HH).

Theorem 2 (a "refinement" of (RHH))**.** *Let $f : [a, b] \to \mathbb{R}$ be convex. Then*

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)dx &\le \frac{1}{b-a} \int_a^b f(x) \left[\ln \frac{(b-a)^2}{(b-x)(x-a)} - 1 \right] dx \\ &\le \frac{f(a) + f(b)}{2} \end{aligned} \tag{7}$$

Proof. For any $x \in (a, b]$ one has

$$f\left(\frac{a+x}{2} \right) \le \frac{1}{x-a} \int_a^x f(t)dt \le \frac{f(a) + f(x)}{2}.$$

Integrating over (a, b) we have

$$\int_a^b f\left(\frac{a+x}{2} \right) dx \le \int_a^b \frac{1}{x-a} \left(\int_a^x f(t)dt \right) dx \le \int_a^b \frac{f(a) + f(x)}{2} dx. \tag{8}$$

After simple calculations, (8) leads to

$$2 \int_a^{\frac{a+b}{2}} f(x)dx \le \int_a^b f(x) \ln \frac{b-a}{x-a} dx \le \frac{1}{2} \left[f(a)(b-a) + \int_a^b f(x)dx \right]. \tag{9}$$

Similarly, integrating the inequality

$$f\left(\frac{x+b}{2}\right) \le \frac{1}{b-x}\int_x^b f(t)dt \le \frac{f(x)+f(b)}{2}$$

over (a,b) we get

$$\int_a^b f\left(\frac{x+b}{2}\right)dx \le \int_a^b \frac{1}{b-x}\left(\int_x^b f(t)dt\right)dx \le \int_a^b \frac{f(x)+f(b)}{2}dx$$

which leads to

$$2\int_{\frac{a+b}{2}}^b f(x)dx \le \int_a^b f(x)\ln\frac{b-a}{b-x}dx \le \frac{1}{2}\left[f(b)(b-a)+\int_a^b f(x)dx\right]. \qquad (10)$$

After summing up (9) and (10) we obtain (7). □

Theorem 3. *Let $f:[0,\infty)\to(0,\infty)$ be a strictly convex function and $\sum_{k=1}^{\infty} f(k) < \infty$. Then*

$$\sum_{k=1}^{\infty} f\left(k-\frac{1}{2}\right) < \int_0^\infty f(x)dx < \frac{1}{2}f(0)+\sum_{k=1}^{\infty} f(k). \qquad (11)$$

Proof. For, $0 \le a < b < \infty$, denote $x_0 = a$ and $x_k = a + k\frac{b-a}{n}$, $(k = 1, 2, \cdots, n)$. Since f is strictly convex, we have

$$f\left(\frac{x_{k-1}+x_k}{2}\right) < \frac{1}{x_k - x_{k-1}}\int_{x_{k-1}}^{x_k} f(x)dx < \frac{f(x_{k-1})+f(x_k)}{2}, \ (k=1,2,\cdots,n).$$

Taking into account the formulas

$$x_k - x_{k-1} = \frac{b-a}{n} \text{ and } \frac{x_{k-1}+x_k}{2} = a + \left(k-\frac{1}{2}\right)\frac{b-a}{n}$$

and summing the inequalities above we obtain

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{n}f\left(a+\left(k-\frac{1}{2}\right)\frac{b-a}{n}\right) &< \frac{1}{b-a}\int_a^b f(x)dx \\ &< \frac{1}{n}\left[\frac{f(a)+f(b)}{2}+\sum_{k=1}^{n-1} f\left(a+k\frac{b-a}{n}\right)\right].\end{aligned}$$

Setting now $a = 0$, $b = n$ we have

$$\sum_{k=1}^{n} f\left(k-\frac{1}{2}\right) < \int_0^n f(x)dx < \frac{f(0)+f(n)}{2} + \sum_{k=1}^{n-1} f(k).$$

Taking limit as $n \to \infty$ and using $\lim_{n\to\infty} f(n) = 0$ we obtain the desired formula (11). □

Remark 2. *Since $f : [0,\infty) \to (0,\infty)$ is convex and $\lim_{n\to\infty} f(n)$ is finite (actually, zero), then f is monotonically decreasing and therefore the comparison of the areas under graphics gives the following well-known inequalities*

$$\sum_{k=1}^{\infty} f(k) < \int_0^{\infty} f(x)dx < f(0) + \sum_{k=1}^{\infty} f(k). \tag{12}$$

It is clear that, the inequalities (11) are better than (12).

Example 1. *If $f(x) = e^{-x}$ then from (11) we have*

$$\frac{\sqrt{e}}{e-1} < 1 < \frac{1}{2} + \frac{1}{e-1} \text{ and therefore, } \sqrt{e} < e - 1 < \frac{1}{2}(e+1),$$

whereas the formula (12) gives the rougher estimate $1 < e - 1 < e$.

3. Some inequalities arising as a combination of (HH) with the other inequalities

Theorem 4. *Let $1 < p < \infty$ and $\alpha p > 1$. Let further, $f : (0,\infty) \to (0,\infty)$ be convex and such that*

$$\left\|x^{1-\alpha} f(x)\right\|_p \equiv \left(\int_0^{\infty} \left(x^{1-\alpha} f(x)\right)^p dx\right)^{1/p} < \infty.$$

Then

$$2^{1-\alpha+\frac{1}{p}} \le \frac{\left\|x^{-\alpha} \int_0^x f\right\|_p}{\left\|x^{1-\alpha} f(x)\right\|_p} \le \frac{1}{\alpha - 1/p}. \tag{13}$$

Corollary 2. *(a) If $\alpha = 1$, then*

$$2^{\frac{1}{p}} \le \frac{\left\|\frac{1}{x}\int_0^x f\right\|_p}{\|f\|_p} \le \frac{1}{1-1/p}. \tag{14}$$

(b) Let, in addition, $f \in L_p(0,\infty)$, $(\forall p > 1)$. Then by taking the limit in (14) as $p \to \infty$ one has

$$\lim_{p\to\infty} \frac{\left\|\frac{1}{x}\int_0^x f\right\|_p}{\|f\|_p} = 1. \tag{15}$$

Proof of Theorem 4. We will use the classical weighted Hardy inequality, which asserts that

$$\left(\int_0^{\infty} \left|x^{-\alpha} \int_0^x f(t)dt\right|^p dx\right)^{1/p} \le c \left(\int_0^{\infty} \left|x^{1-\alpha} f(x)\right|^p dx\right)^{1/p}, \tag{16}$$

where $c = \frac{p}{\alpha p - 1}$, $1 < p < \infty$, $\alpha p > 1$.

Now, by (LHH) we have

$$f\left(\frac{x}{2}\right) < \frac{1}{x}\int_0^x f(t)dt \Rightarrow x^{1-\alpha} f\left(\frac{x}{2}\right) < x^{-\alpha} \int_0^x f(t)dt,$$

and therefore

$$\int_0^\infty \left(x^{1-\alpha} f\left(\frac{x}{2}\right)\right)^p dx \quad \leq \quad \int_0^\infty \left(x^{-\alpha}\int_0^x f(t)dt\right)^p dx$$
$$\overset{(16)}{\leq} \quad \left(\frac{p}{\alpha p-1}\right)^p \int_0^\infty (x^{1-\alpha}f(x))^p dx. \tag{17}$$

Since

$$\int_0^\infty \left(x^{1-\alpha} f\left(\frac{x}{2}\right)\right)^p dx = 2^{p(1-\alpha)+1}\int_0^\infty \left(x^{1-\alpha}f(x)\right)^p dx,$$

we have from (17) the desired result (13) and its consequences (14) and (15). □

Remark 3. *The left hand side of (13) shows that under the conditions of Theorem 4 the following reverse Hardy's inequality is valid:*

$$\left\| x^{-\alpha}\int_0^x f \right\|_p \geq 2^{1-\alpha+\frac{1}{p}} \left\| x^{1-\alpha}f(x)\right\|_p .$$

Example 2. *a) Let $k>0$ and $f(x)=e^{-kx}$. Then (15) yields*

$$\lim_{p\to\infty}\left(\int_0^\infty \left(\frac{1-e^{-kx}}{x}\right)^p dx\right)^{1/p} = k.$$

b) If $f(x)=\frac{1}{x+1}$, $(0<x<\infty)$, then from (15) we have

$$\lim_{p\to\infty}\left(\int_0^\infty \frac{\ln^p(x+1)}{x^p}dx\right)^{1/p} = 1.$$

In the next theorem we will make use of a combination of (RHH)

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

and the inequality

$$\left(\int_a^b \left(\prod_{k=1}^n u_k(x)\right)dx\right)^n \leq \prod_{k=1}^n \left(\int_a^b u_k^n(x)dx\right), \tag{18}$$

where $u_1\geq 0,\cdots,u_n\geq 0$.

Recall that the inequality (18) is a special case of the iterated Hölder inequality.

We need also the following

Lemma 2. *If $u:[a,b]\to(0,\infty)$ is convex, then u^n is convex as well for any $n\in\mathbb{N}$.*

This Lemma is actually a special case of the following more general proposition: If $u:[a,b]\to(0,\infty)$ is convex and $f:(0,\infty)\to(0,\infty)$ is increasing and convex, then the composition $f\circ u:[a,b]\to(0,\infty)$ is convex as well. Here, we get $f(t)=t^n$, $(0<t<\infty)$. Note that Lemma 2 can also be proved by induction.

Remark 4. *The convexity of the functions* $u_1 \geq 0, u_2 \geq 0, \cdots, u_n \geq 0$ *does not guarantee the convexity of their product* $u_1 u_2 \cdots u_n$. *Indeed, for example, although the functions* $u_1(x) = x^2, u_2(x) = x^2, \cdots, u_{n-1}(x) = x^2$ *and* $u_n(x) = (2-x)^{2n-2}$, $(n \geq 2)$ *are convex on* $[0,2]$, *their product* $u(x) = x^{2n-2}(2-x)^{2n-2}$ *is not convex because of* $u''(1) = 4(2n-2)(1-n) < 0$.

Theorem 5. *For given* $n \geq 2$, *let the functions* $u_1 \geq 0, u_2 \geq 0, \cdots, u_n \geq 0$ *be convex on* $[a,b]$. *Then*

$$\frac{1}{b-a}\int_a^b \left(\prod_{k=1}^n u_k(x)\right) dx \leq \frac{1}{2}\prod_{k=1}^n (u_k^n(a) + u_k^n(b))^{\frac{1}{n}}. \tag{19}$$

Proof. Since u_k, $(k = 1, 2, \cdots, n)$ is convex on $[a,b]$, then u_k^n is also convex by Lemma 2. Then the (RHH) yields

$$\frac{1}{b-a}\int_a^b u_k^n(x)dx \leq \frac{1}{2}\left[u_k^n(a) + u_k^n(b)\right], \ (k = 1, 2, \cdots, n).$$

By multiplying these inequalities we have

$$\frac{1}{(b-a)^n}\prod_{k=1}^n \left(\int_a^b u_k^n(x)dx\right) \leq \frac{1}{2^n}\prod_{k=1}^n (u_k^n(a) + u_k^n(b)). \tag{20}$$

Here, by making use of the inequality (18), we get

$$\frac{1}{(b-a)^n}\left(\int_a^b \left(\prod_{k=1}^n u_k(x)\right) dx\right)^n \leq \frac{1}{2^n}\prod_{k=1}^n (u_k^n(a) + u_k^n(b)),$$

from which the inequality (19) follows. □

Remark 5. *For* $n = 2$, *the inequality (19) was proved by Amrahov [1]. Another generalization of Amrahov's result for the product of two functions was noted by D. A. Ion [12]:*

If $u \geq 0, v \geq 0$ *are convex and* $\frac{1}{p} + \frac{1}{q} = 1$, $(1 < p, q < \infty)$, *then*

$$\frac{1}{b-a}\int_a^b u(t)v(t)dt \leq \frac{1}{2}\left(u^p(a) + u^p(b)\right)^{1/p}\left(u^q(a) + u^q(b)\right)^{1/q}.$$

It should also be mentioned that, in the same paper [12] Ion gives some generalization of Amrahov's result for the product of two functions in Orlicz spaces.

4. (RHH)-TYPE INEQUALITY FOR THE FUNCTIONS WHOSE DERIVATIVES HAVE AN INFLECTION POINT

Theorem 6. *Given* $c \in [a,b]$ *and* $f : [a,b] \to \mathbb{R}$, *let the derivative* f' *be concave on* $[a,c]$ *and convex on* $[c,b]$. *Then*

$$\left[\frac{c-a}{b-a}f(a) + \frac{b-c}{b-a}f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx$$

$$\leq \quad \frac{1}{3}\left[\frac{(b-c)^2}{b-a}f'(b)-\frac{(c-a)^2}{b-a}f'(a)+\left(\frac{a+b}{2}-c\right)f'(c)\right]. \tag{21}$$

Corollary 3. *In case of* $c=\frac{a+b}{2}$ *we have*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{b-a}{12}(f'(b)-f'(a))$$

Proof of Theorem 6. Integration by parts yields

$$\frac{c-a}{b-a}f(a)+\frac{b-c}{b-a}f(b)-\frac{1}{b-a}\int_a^b f(x)dx=\frac{1}{b-a}\int_a^b (x-c)f'(x)dx$$

$$\begin{aligned} &= \frac{1}{b-a}\int_a^c (x-c)f'(x)dx+\frac{1}{b-a}\int_c^b (x-c)f'(x)dx \\ &\equiv A+B. \end{aligned}$$

By changing variables as $x=(1-\lambda)a+\lambda c$, $(0<\lambda<1)$ in A and $x=(1-\lambda)c+\lambda b$ in B and applying Jensen's inequality, we have

$$\begin{aligned} A &\equiv \frac{1}{b-a}\int_a^c (x-c)f'(x)dx=\frac{(a-c)^2}{b-a}\int_0^1(\lambda-1)f'((1-\lambda)a+\lambda c)d\lambda \\ &\leq \frac{(a-c)^2}{b-a}\int_0^1(\lambda-1)\left[(1-\lambda)f'(a)+\lambda f'(c)\right]d\lambda \\ &= -\frac{(a-c)^2}{6\,(b-a)}\left[2f'(a)+f'(c)\right]; \end{aligned} \tag{22}$$

$$\begin{aligned} B &\equiv \frac{1}{b-a}\int_c^b (x-c)f'(x)dx=\frac{(b-c)^2}{(b-a)}\int_0^1 \lambda f'((1-\lambda)c+\lambda b)d\lambda \\ &\leq \frac{(b-c)^2}{(b-a)}\int_0^1\left(\lambda(1-\lambda)f'(c)+\lambda^2 f'(b)\right)d\lambda \\ &= \frac{(b-c)^2}{6(b-a)}[f'(c)+2f'(b)]. \end{aligned} \tag{23}$$

It follows from (22) and (23) that

$$A+B\leq \frac{1}{3}f'(b)\frac{(b-c)^2}{b-a}-\frac{1}{3}f'(a)\frac{(a-c)^2}{b-a}+\frac{1}{6}f'(c)(a+b-2c),$$

which completes the proof. □

Remark 6. *A simple calculation shows that the equality in (21) holds for the functions* $f(x)=k(x-c)^2+m$, $(k,m\in\mathbb{R})$.

Remark 7. *In the "critical" cases $c = a$ or $c = b$, i.e. in the cases when f' is convex or concave on $[a, b]$ we have from (21)*

$$f(b) - \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{b-a}{6}[f'(a) + 2f'(b)]$$

and

$$f(a) - \frac{1}{b-a} \int_a^b f(x)dx \leq -\frac{b-a}{6}[2f'(a) + f'(b)],$$

respectively.

5. Various inequalities for functions having convex first or second order derivatives

The first moment of a function f about the center point $c = (a+b)/2$ is defined by $M_f = \int_a^b \left(x - \frac{a+b}{2}\right) f(x)dx$. In the following theorem we obtain some estimation from above and below for M_f, when f' is convex.

Theorem 7. *Suppose that the derivative f' of the function $f : [a, b] \to \mathbb{R}$ is convex. Then the first moment of f about the center point $c = (a+b)/2$ satisfies the following inequality*

$$A \leq \int_a^b \left(x - \frac{a+b}{2}\right) f(x)dx \leq B, \tag{24}$$

where

$$A = \frac{(a-b)^2}{8}(f(b) - f(a)) - \frac{(b-a)^3}{48}(f'(a) + f'(b))$$

and

$$B = \frac{(b-a)^3}{24}(f'(a) + f'(b)).$$

Proof. Integration by parts leads to

$$\begin{aligned} \int_a^b (x-a)(b-x)f'(x)dx &= \int_a^b (x-a)(b-x)df(x) \\ &= 2\int_a^b \left(x - \frac{a+b}{2}\right) f(x)dx. \end{aligned}$$

Hence,

$$\int_a^b \left(x - \frac{a+b}{2}\right) f(x)dx = \frac{1}{2}\int_a^b (x-a)(b-x)f'(x)dx$$

(set $x = (1-t)a + tb$, $(x-a)(b-x) = (b-a)^2 t(1-t)$, $0 \leq t \leq 1$)

$$\begin{aligned} &= \frac{1}{2}(b-a)^3 \int_0^1 t(1-t)f'((1-t)a + tb)dt \\ &\leq \frac{1}{2}(b-a)^3 \int_0^1 t(1-t)[f'(a)(1-t) + f'(b)t]dt \end{aligned}$$

$$= \frac{(b-a)^3}{24}(f'(a)+f'(b)).$$

This proved the right hand side of (24).

Further, again using integration by parts we have

$$\int_a^b \left(x-\frac{a+b}{2}\right)^2 f'(x)dx = \frac{(b-a)^2}{4}(f(b)-f(a)) - 2\int_a^b \left(x-\frac{a+b}{2}\right) f(x)dx,$$

and therefore,

$$\int_a^b \left(x-\frac{a+b}{2}\right) f(x)dx = \frac{(b-a)^2}{8}(f(b)-f(a)) - \frac{1}{2}\int_a^b \left(x-\frac{a+b}{2}\right)^2 f'(x)dx. \tag{25}$$

Furhermore, setting $x = (1-t)a + tb$, $\left(x-\frac{a+b}{2}\right)^2 = (b-a)^2\left(t-\frac{1}{2}\right)^2$ and $dx = (b-a)dt$, $(0 \le t \le 1)$, we get

$$\begin{aligned}
\int_a^b \left(x-\frac{a+b}{2}\right)^2 f'(x)dx &= (b-a)^3 \int_0^1 \left(t-\frac{1}{2}\right)^2 f'((1-t)a+tb)dt \\
&\le (b-a)^3 \int_0^1 \left(t-\frac{1}{2}\right)^2 [(1-t)f'(a)+tf'(b)]dt \\
&= (b-a)^3 \left[f'(a)\int_0^1 \left(t-\frac{1}{2}\right)^2 (1-t)dt + f'(b)\int_0^1 t\left(t-\frac{1}{2}\right)^2 dt\right] \\
&= \frac{(b-a)^3}{24}(f'(a)+f'(b)).
\end{aligned}$$

Taking into account this in (25) we obtain the left hand side of inequality (24).

The proof is complete. □

A straightforward calculation shows that the equality in both sides of (24) is attained for $f(x) = k(x^2-(a+b)x)+n$, where k and n are arbitrary real numbers.

Theorem 8. *Given $f : [a,b] \to \mathbb{R}$, let f'' be convex. Then the following inequality holds*

$$A \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \le B, \tag{26}$$

where

$$A = \frac{b-a}{8}(f'(b)-f'(a)) - \frac{(b-a)^2}{48}(f''(a)+f''(b))$$

and

$$B = \frac{(b-a)^2}{24}(f''(a)+f''(b)).$$

Proof. Integration by parts twice gives

$$\int_a^b (x-a)(b-x)f''(x)dx = (b-a)(f(a)+f(b)) - 2\int_a^b f(x)dx.$$

Hence,

$$\begin{aligned}
\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx &= \frac{1}{2(b-a)}\int_a^b (x-a)(b-x)f''(x)dx \\
\text{(Set } x &= (1-t)a+tb,\ 0\le t\le 1) \\
&= \frac{(b-a)^2}{2}\int_0^1 t(1-t)f''((1-t)a+tb)dt \\
&\le \frac{(b-a)^2}{2}\int_0^1 t(1-t)[(1-t)f''(a)+tf''(b)]dt \\
&= \frac{(b-a)^2}{24}(f''(a)+f''(b)).
\end{aligned}$$

The right hand side of (26) is proved.

Straightforward calculations show that, integration by parts twice yields

$$\begin{aligned}
&\int_a^b \left(x-\frac{a+b}{2}\right)^2 f''(x)dx \\
&= \left(\frac{b-a}{2}\right)^2 (f'(b)-f'(a)) - 2(b-a)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \\
&= \frac{b-a}{8}(f'(b)-f'(a)) - \frac{1}{2(b-a)}\int_a^b \left(x-\frac{a+b}{2}\right)^2 f''(x)dx. \qquad (27)
\end{aligned}$$

Setting $x=(1-t)a+tb$, $(0\le t\le 1)$ and using the convexity of f'', we have

$$\begin{aligned}
&\int_a^b \left(x-\frac{a+b}{2}\right)^2 f''(x)dx = (b-a)^3\int_0^1 \left(t-\frac{1}{2}\right)^2 f''((1-t)a+tb)dt \\
&\le (b-a)^3\left[f''(a)\int_0^1 \left(t-\frac{1}{2}\right)^2(1-t)dt + f''(b)\int_0^1 \left(t-\frac{1}{2}\right)^2 tdt\right] \\
&= \frac{(b-a)^3}{24}(f''(a)+f''(b)).
\end{aligned}$$

By making use of this in (27) we obtain the left hand side of inequality (26).

The proof is complete. □

It is easy to verify that the equality in both sides of (24) is attained for the functions $f(x)=k(2x^3-3(a+b)x^2)+mx+n$, with arbitrary real numbers k, m and n.

Remark 8. *In the literature there are results of the type (24) and (26) under the condition of the convexity of* $|f'|$ *or* $|f''|$ *(see, e.g. [2, 5, 19]). As far as we know, the conditions and assertions of the theorems 7 and 8 completely differ from those known in the literature.*

In the following theorem we give some estimations for the mean value of a function f whose first derivative is convex.

Theorem 9. *Let* $f : [a, b] \to \mathbb{R}$ *be differentiable and its derivative* f' *be convex. Then*

(a)

$$N \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M, \tag{28}$$

where

$$N = \frac{1}{3}(f(a) + 2f(b)) - \frac{1}{6}f'(b)(b-a)$$

and

$$M = \frac{1}{3}(f(b) + 2f(a)) + \frac{1}{6}f'(a)(b-a);$$

(b)

$$\mathcal{N} \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \mathcal{M}, \tag{29}$$

where

$$\mathcal{N} = f(a) + 2f\left(\frac{a+b}{2}\right) - \frac{4}{b-a}\int_a^{\frac{a+b}{2}} f(x)dx$$

and

$$\mathcal{M} = f(b) + 2f\left(\frac{a+b}{2}\right) - \frac{4}{b-a}\int_{\frac{a+b}{2}}^b f(x)dx.$$

Corollary 4.

$$\int_{\frac{a+b}{2}}^b f(x)dx - \int_a^{\frac{a+b}{2}} f(x)dx \leq \frac{1}{4}(b-a)(f(b) - f(a)). \tag{30}$$

Proof of Theorem 9. Since f' is convex, (HH) leads to

$$f'\left(\frac{a+x}{2}\right) \leq \frac{1}{x-a}(f(x) - f(a)) \leq \frac{f'(a) + f'(x)}{2}; \tag{31}$$

$$f'\left(\frac{x+b}{2}\right) \leq \frac{1}{b-x}(f(b) - f(x)) \leq \frac{f'(x) + f'(b)}{2}. \tag{32}$$

Multiplying the inequalities (31) by $(x-a)$ and integrating over $[a, b]$, after simple calculations we obtain

$$2(b-a)f\left(\frac{a+b}{2}\right) - 4\int_a^{\frac{a+b}{2}} f(x)dx \leq \int_a^b f(x)dx - f(a)(b-a)$$

$$\leq \frac{1}{4}f'(a)(b-a)^2 + \frac{1}{2}(b-a)f(b) - \frac{1}{2}\int_a^b f(x)dx.$$

The above inequalities can be written as two seperate inequalities:

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{3}(2f(a)+f(b)) + \frac{1}{6}f'(a)(b-a) \tag{33}$$

and

$$\frac{1}{b-a}\int_a^b f(x)dx + \frac{4}{b-a}\int_a^{\frac{a+b}{2}} f(x)dx \geq f(a) + 2f\left(\frac{a+b}{2}\right). \tag{34}$$

In a similar way, multiplying inequalities (32) by $(b-x)$ and integrating over $[a,b]$, after some calculations we have the following two inequalities:

$$\frac{1}{b-a}\int_a^b f(x)dx \geq \frac{1}{3}(f(a)+2f(b)) - \frac{1}{6}f'(b)(b-a) \tag{35}$$

and

$$\frac{1}{b-a}\int_a^b f(x)dx + \frac{4}{b-a}\int_{\frac{a+b}{2}}^b f(x)dx \leq f(b) + 2f\left(\frac{a+b}{2}\right). \tag{36}$$

Now, the inequalities (33) and (35) yields (28) and the inequalities (34) and (36) yields (29). The Corollary follows by subtracting (34) from (36).

The proof is complete. □

Example 3. *For $f(x) = \ln x$, $0 < a < x < b < \infty$, the inequality (30) yields*

$$a^{\frac{3a+b}{4(a+b)}} \cdot b^{\frac{a+3b}{4(a+b)}} \leq \frac{a+b}{2}. \tag{37}$$

Since $\alpha + \beta = 1$ for $\alpha = \frac{3a+b}{4(a+b)}$ and $\beta = \frac{a+3b}{4(a+b)}$, then by the generalized AM-GM inequality we have

$$a^\alpha \cdot b^\beta < \alpha \cdot a + \beta \cdot b = \frac{3a+b}{4(a+b)} \cdot a + \frac{a+3b}{4(a+b)} \cdot b. \tag{38}$$

A simple calculation shows that

$$\frac{a+b}{2} < \frac{3a+b}{4(a+b)} \cdot a + \frac{a+3b}{4(a+b)} \cdot b,$$

and therefore, the inequality (37) is better than (38).

Remark 9. *In our opinion, most of the results in this article can also be examined for non-classical convexity types (abstract convexity, s-convexity, p-convexity, m-convexity, etc.). Necessary information about the mentioned convexity types can be found, for example, in [4, 5, 7, 13, 14, 20]*

Author Contribution Statements The authors contributed equally. All authors read and approved the final copy of the paper.

Declaration of Competing Interests The authors declare that they have no competing financial interests and personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Amrahov, S. E., A note on Hadamard inequalities for the product of the convex functions, *International J. of Research and Reviews in Applied Sciences*, 5(2) (2010), 168-170.

[2] Alomari, M., Darus, M. and Dragomir, S. S., New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are quasi-convex, *Tamkang J. Math.*, 41(4) (2010), 353-359.

[3] Azpetia, A. G., Convex functions and the Hadamard inequality, *Revista Colombiana Mat.*, 28 (1994), 7-12.

[4] Bakula, M. K., Özdemir, M. E. and Pecaric, J., Hadamard-type inequalities for m-convex and (α, m)-convex functions. *J. Ineq. Pure Appl. Math.*, 9(4) (2008), Article 96.

[5] Chen, F., Liu, X., On Hermite-Hadamard type inequalities for functions whose second derivatives absolute values are s-convex, *ISRN Applied Mathematics*, (2014), 1-4, DOI 10.1155/2014/829158.

[6] Dragomir, S. S., Pearce, C. E. M., Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.

[7] Dragomir, S. S., On some new inequalities of Hermite-Hadamard type for m-convex functions,*Tamkang J. Math.*, 3(1) (2002).

[8] El Farissi, A., Simple proof and refinement of Hermite-Hadamard inequality. *J. Math. Inequal*, 4(3) (2010), 365-369.

[9] Fink, A. M., A best possible Hadamard inequality, *Math. Inequal Appl.*, 1 (1998), 223-230.

[10] Florea, A., Niculescu, C. P., A Hermite-Hadamard inequality for convex-concave symmetric functions, *Bull. Soc. Sci. Math. Roum.*, 50(98) No:2 (2007), 149-156.

[11] Hwang, D. Y., Tseng, K. L. and Yang, G. S., Some Hadamard's inequalities for co-ordinated convex functions in a rectangle from the plane, *Taiwanese J. Math.*, 11(1) (2007), 63-73, DOI 10.11650/twjm/1500404635.

[12] Ion, D. A., On an inequality due to Amrahov, *Annals of University of Craiova, Math. Comp. Sci. Ser.*, 38(1) (2011), 92-95, DOI 10.52846/ami.v38i1.396.

[13] Kemali, S., Yesilce, I., Adilov, G., B-Convexity, B-1-Convexity and Their Comparison, *Numerical Functional Analysis and Optimization*, 36(2) (2015), 133-146, 10.1080/01630563.2014.970641.

[14] Kemali, S., Sezer, S., Tınaztepe, G., Adilov, G., s-Convex functions in the third sense, *Korean Journal of Mathematics*, 29(3) (2021), 593-602, DOI 10.11568/kjm.2021.29.3.593.

[15] Mercer, A. M. D., A variant of Jensen's inequality, *J. Ineq. Pure and Appl. Math.*, 4(4) (2003), Article 73.

[16] Mitrinovic, D. S., Pecaric, J. E. and Fink, A. M., Classical and New Inequalities in Analysis, Kluwer Academic, Dordrecht, 1993.

[17] Niculescu C. P., Persson, L. E., Old and new on the Hermite-Hadamard inequality, *Real Anal. Exchange*, 29 (2003/2004), 663-686.

[18] Niculescu C. P., Persson, L. E., Convex Functions and Their Applications, A Contemporary Approach, CMS Books in Mathematics, V.23, Springer-Verlag, 2006.

[19] Sarikaya, M. Z., Saglam, A. and Yildirim, H., New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex or quasi-convex, *International J. of Open Problems in Comp. Sci. and Math.*, 5(3) (2012), 1-14.

[20] Sezer, S., Eken, Z., Tınaztepe, G., Adilov, G., p-Convex functions and some of their properties,*Numerical Functional Analysis and Optimization*, 42(4) (2021), 443-459, DOI 10.1080/01630563.2021.1884876.

[21] Tseng, K. L., Hwang S. R. and Dragomir, S. S., On some new inequalities of Hermite-Hadamard-Fejer type involving convex functions, *Demons. Math.*, 40(1) (2007), 51-64, DOI 10.1515/dema-2007-0108.

[22] Qi, F., Yang, Z. L., Generalizations and refinements of Hermite-Hadamard's inequality, *Rocky Mountain J. Math.*, 35 (2005), 235-251.