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Coherent hybrid block method for approximating fourth-order ordinary differential equations

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Abstract — Conventionally, the most used method of solving fourth-order initial value problems of ordinary differential is to first reduce to a system of first-order differential equations. This approach affects the effectiveness and convergence of the numerical method due to the transformation. This paper comprises the derivation, analysis, and implementation of a new hybrid block method for direct solution of fourth-order equations. The method is derived by collocation and interpolation of an assumed basis function. The basic properties of the block method, including zero stability, error constants, consistency, order, and convergence, were analyzed. From the analysis, the block method derived was found to be zero-stable, consistent, and convergent. Errors were computed for the proposed method, and they were proven to produce approximations that agree with exact solutions and as such this shows improvement with those of existing works.

Keywords: *Hybrid methods, block method, linear stability, errors*

Subject Classification (2020): 65L05, 65L70

1. Introduction

Differential Equations are among the essential tools used in producing models in engineering, mathematics, physics, aeronautics, elasticity, astronomy, dynamics, biology, chemistry, medicine, environmental sciences, social sciences, and banking. We study several differential equations in calculus to get closed-form solutions, but not all differential equations possess finite solutions. It is not easy to get even if they possess closed-form solutions. In such situations, depending on the need, numerical solutions of the differential equations are also sought [1,2]. In general, equations arising from modeling physical phenomena do not have analytical or exact solutions. Only a few can be solved analytically; hence, developing numerical methods becomes necessary. Numerical methods play a key role in providing approximate solutions to differential equations due to the difficulty in obtaining the exact solutions. Many numerical techniques, as found in [3-5], have been developed and implemented. Implementing the numerical method in the predictor-corrector approach has some setbacks, including lengthy computational time due to more function evaluations needed per step and computational burden which may affect the method's accuracy in terms of error [6]. Numerical methods are necessary tools that provide solutions despite the complexities of problems. This study seeks to derive Hybrid Block Linear

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Multi-Step Method for the direct solution of general fourth-order initial value problem of ordinary differential equations of the form:

$$y^{iv}(x) = f(x, y', y'', y''')$$
$$y(x_0) = \phi_0, \quad y'(x_0) = \phi_1, \quad y''(x_0) = \phi_2, \text{ and } y'''(x_0) = \phi_3 \tag{1.1}$$

where $f: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ is continuous, x_0 is the initial point, $y \in \mathbb{R}$ is an n-dimensional vector, x is a scalar variable and a set of equally spaced points on the integration interval defined by; $x_0 < x_1 < x_2 < \cdots < x_n < \cdots < x_{n+k} < x_N$ with a specified positive integer step number $k > 0$. According to Awoyemi [1], Continuous Linear Multi-step Methods (CLMMs) have greater advantages over the discrete methods since they give better error estimates and provide simplified form, allowing easy solution approximation at all interior points of the integration interval. Block methods for approximating the numerical solution have been proposed by scholars for the solution of initial value problems using different polynomial trial functions ranging from power series, Lagrange polynomial, and Chebychev polynomial. Among these methods are in [4,6]. In particular, fourth-order differential equations arise in many physical problems, such as in ship dynamics, deflection of beams, control theory, and mechanics. Therefore, fourth-order equations have attracted significant interest from researchers. Thereby, theoretical and numerical studies dealing with (1.1) have recently appeared in the literature.

2. Specification and Derivation of the Numerical Scheme

We consider a power series of a single variable x as an approximate solution to (1.1) as

$$y(x) = \sum_{j=0}^{t+c-1} \alpha_j x^j \tag{2.1}$$

where $\alpha_j \in \Re$, $j \in \{0,1,\dots,t+c-1\}$, $y \in C^m$, t is the interpolation points, and c is the collocation points. The derivatives of (2.1) are given as:

$$y'(x) = \sum_{j=0}^{t+c-1} j\alpha_j x^{j-1} \tag{2.2}$$

$$y''(x) = \sum_{j=0}^{t+c-1} j(j-1)\alpha_j x^{j-2} \tag{2.3}$$

$$y'''(x) = \sum_{j=0}^{t+c-1} j(j-1)(j-2)\alpha_j x^{j-3} \tag{2.4}$$

$$y^{(iv)}(x) = \sum_{j=0}^{t+c-1} j(j-1)(j-2)(j-3)\alpha_j x^{j-4} \tag{2.5}$$

From (1.1) and (2.2), we have

$$f(x, y, y', y'', y''') = \sum_{j=0}^{t+c-1} j(j-1)(j-2)(j-3)\alpha_j x^{j-4} \tag{2.6}$$

such that $\alpha_j \in \mathbb{R}$ are parameters to be determined, $c \in \left\{0, \frac{1}{10}, 1, \frac{11}{10}, \frac{19}{10}, 2\right\}$ are points of collocation, and $t \in \left\{0, \frac{1}{10}, \frac{11}{10}, \frac{19}{10}\right\}$ are points of interpolation. Collocating (2.2) at $x = x_{n+i}$ such that $i \in \left\{0, \frac{1}{10}, 1, \frac{11}{10}, \frac{19}{10}, 2\right\}$ interpolating (2.2) as well at $x = x_{n+i}$ such that $i \in \left\{0, \frac{1}{10}, \frac{11}{10}, \frac{19}{10}\right\}$ and evaluating at the end point $x = x_{n+i}$ such that $i = 2$ gives a system of nonlinear equations that were solved and then substituted into (2.1), which yields a continuous linear multi-step method in the form:

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^4 \left(\sum_{j=0}^{k} \beta_j(x) f_{n+j} + \beta_w(x) f_{n+w} + \beta_v(x) f_{n+v} + \beta_u(x) f_{n+u} \right) \tag{2.7}$$

where the numerical solution of the initial value problem is approximated to be equivalent to the true solution $y(x)$, $w = \frac{1}{10}$, $v = \frac{11}{10}$, $u = \frac{19}{10}$, α_j, and β_j are constants, $y_{n+j} = y(x_n + jh)$, and

$$f_{n+j} = f(x_n + jh, y'_n + jh, y''_n + jh, y'''_n + jh)$$

Substituting the obtained coefficient and evaluating the resulting method (2.7) produces the continuous linear multi-step method of the form:

$$y(x) = a_0 y_0 + a_{\frac{1}{10}} y_{\frac{1}{10}} + a_{\frac{11}{10}} y_{\frac{11}{10}} + a_{\frac{19}{10}} y_{\frac{19}{10}} + \beta_0 f_0 + \beta_{\frac{1}{10}} y_{\frac{1}{10}} + \beta_1 f_1 + \beta_{\frac{11}{10}} y_{\frac{11}{10}} + \beta_{\frac{19}{10}} y_{\frac{19}{10}} + \beta_2 f_2 \tag{2.8}$$

Evaluating (2.8) at the non-interpolation point (evaluating point) $x = x_{n+1}$ and $x = x_{n+2}$ gives, respectively, the discrete schemes below:

$$\begin{aligned} y_{n+1} = & -\frac{81}{209} y_n + \frac{1}{2} y_{n+\frac{1}{10}} + \frac{81}{88} y_{n+\frac{11}{10}} - \frac{5}{152} y_{n+\frac{19}{10}} - \frac{58941}{76000000} h^4 f_n + \frac{368459}{273600000} h^4 f_{n+\frac{1}{10}} \\ & + \frac{213557}{36000000} h^4 f_{n+1} - \frac{21501}{6400000} h^4 f_{n+\frac{11}{10}} + \frac{145063}{218880000} h^4 f_{n+\frac{19}{10}} - \frac{8199}{19000000} h^4 f_{n+2} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} y_{n+2} = & -\frac{9}{11} y_n + y_{n+\frac{1}{10}} - \frac{19}{44} y_{n+\frac{11}{10}} + \frac{5}{4} y_{n+\frac{19}{10}} - \frac{8963}{4000000} h^4 f_n + \frac{81283}{21600000} h^4 f_{n+\frac{1}{10}} \\ & + \frac{255683}{27000000} h^4 f_{n+1} + \frac{45391}{28800000} h^4 f_{n+\frac{11}{10}} + \frac{70571}{17280000} h^4 f_{n+\frac{19}{10}} - \frac{86477}{36000000} h^4 f_{n+2} \end{aligned} \tag{2.10}$$

Finding the first derivative of (2.8) and evaluating all the collocating points $x = x_n$, $x = x_{n+\frac{1}{10}}$, $x = x_{n+1}$, $x = x_{n+\frac{11}{10}}$, $x = x_{n+\frac{19}{10}}$, and $x = x_{n+2}$ and let $r_{n+j} = y'_{n+j}$ such that $j \in \left\{0, \frac{1}{10}, 1, \frac{11}{10}, \frac{19}{10}, 2\right\}$ gives the following discrete schemes:

$$\begin{aligned} r_n = \frac{1}{34128864000000h} \Big(& -390277440000000 y_n + 396274032000000 y_{n+\frac{1}{10}} - 7368732000000 y_{n+\frac{11}{10}} \\ & + 1372140000000 y_{n+\frac{19}{10}} + 110860639956 h^4 f_n - 218693892340 h^4 f_{n+\frac{1}{10}} - 593503880992 h^4 f_{n+1} \\ & + 426421129365 h^4 f_{n+\frac{11}{10}} - 66216430225 h^4 f_{n+\frac{19}{10}} + 43926910236 h^4 f_{n+2} \Big) \end{aligned} \tag{2.11}$$

$$r_{n+\frac{1}{10}} = -\frac{1}{94802400000h}\Big(816480000000y_n - 800553600000y_{n+\frac{1}{10}} - 19391400000y_{n+\frac{11}{10}} +3465000000y_{n+\frac{19}{10}} + 265118103h^4f_n - 506277640h^4f_{n+\frac{1}{10}} - 1452235246h^4f_{n+1} +1036749915h^4f_{n+\frac{11}{10}} -161328475h^4f_{n+\frac{19}{10}} + 106955343h^4f_{n+2}\Big) \quad (2.12)$$

$$r_{n+1} = \frac{1}{1264032000000h}\Big(4898880000000y_n - 6390384000000y_{n+\frac{1}{10}} + 1163484000000y_{n+\frac{11}{10}} +328020000000y_{n+\frac{19}{10}} + 9440623548h^4f_n - 16443847420h^4f_{n+\frac{1}{10}} -71846823136h^4f_{n+1} +43138143495h^4f_{n+\frac{11}{10}} - 7747315675h^4f_{n+\frac{19}{10}} + 5064247188h^4f_{n+2}\Big) \quad (2.13)$$

$$r_{n+\frac{11}{10}} = \frac{1}{71101800000h}\Big(272160000000y_n - 347608800000y_{n+\frac{1}{10}} + 46862550000y_{n+\frac{11}{10}} +28586250000y_{n+\frac{19}{10}} + 563697981h^4f_n - 976068280h^4f_{n+\frac{1}{10}} - 4303171642h^4f_{n+1} +2284734705h^4f_{n+\frac{11}{10}} - 501523825h^4f_{n+\frac{19}{10}} + 325265061h^4f_{n+2}\Big) \quad (2.14)$$

$$r_{n+\frac{19}{10}} = -\frac{1}{118503000000h}\Big(816480000000y_n - 1000692000000y_{n+\frac{1}{10}} + 460545750000y_{n+\frac{11}{10}} -276333750000y_{n+\frac{19}{10}} + 2172127671h^4f_n -473460525h^4f_{n+\frac{11}{10}} - 3589256275h^4f_{n+\frac{19}{10}} + 2155609071h^4f_{n+2}\Big) \quad (2.15)$$

$$r_{n+2} = -\frac{1}{11376288000000h}\Big(108319680000000y_n - 132091344000000y_{n+\frac{1}{10}} +54166644000000y_{n+\frac{11}{10}} - 30394980000000y_{n+\frac{19}{10}} + 303775046388h^4f_n -508773529220h^4f_{n+\frac{1}{10}} - 1179138420416h^4f_{n+1} - 330876154455h^4f_{n+\frac{11}{10}} -598023427925h^4f_{n+\frac{19}{10}} + 345412673628h^4f_{n+2}\Big) \quad (2.16)$$

Finding the second derivative of (2.8) and evaluating all the collocating points $x = x_n$, $x = x_{n+\frac{1}{10}}$, $x = x_{n+1}$, $x = x_{n+\frac{11}{10}}$, $x = x_{n+\frac{11}{10}}$, and $x = x_{n+2}$ and let $s_{n+j} = y''_{n+j}$ such that $j \in \left\{0, \frac{1}{10}, 1, \frac{11}{10}, \frac{19}{10}, 2\right\}$ gives the following discrete schemes:

$$s_n = -\frac{1}{284407200000h^2}\Big(-8436960000000y_n + 9480240000000y_{n+\frac{1}{10}} - 1292760000000y_{n+\frac{11}{10}} +249480000000y_{n+\frac{19}{10}} + 20917549323h^4f_n - 42632273200h^4f_{n+\frac{1}{10}} - 110678017846h^4f_{n+1} +79914551475h^4f_{n+\frac{11}{10}} - 12388620475h^4f_{n+\frac{19}{10}} + 8222376723h^4f_{n+2}\Big) \quad (2.17)$$

$$s_{n+\frac{1}{10}} = -\frac{1}{213305400000h^2}\Big(-5715360000000y_n + 6399162000000y_{n+\frac{1}{10}} - 824134500000y_{n+\frac{11}{10}}$$
$$+140332500000y_{n+\frac{19}{10}} + 9941764959h^4f_n - 18346830920h^4f_{n+\frac{1}{10}} - 56496792638h^4f_{n+1} \quad (2.18)$$
$$+39989691495h^4f_{n+\frac{11}{10}} - 6239749175h^4f_{n+\frac{19}{10}} + 4133232279h^4f_{n+2}\Big)$$

$$s_{n+1} = \frac{1}{853221600000h^2}\Big(816480000000y_n - 1939140000000y_{n+\frac{11}{10}} + 1122660000000y_{n+\frac{19}{10}}$$
$$+6842497761h^4f_n - 11187319000h^4f_{n+\frac{1}{10}} - 63806750722h^4f_{n+1} \quad (2.19)$$
$$+7077591675h^4f_{n+\frac{11}{10}} - 9746416075h^4f_{n+\frac{19}{10}} + 6117758361h^4f_{n+2}\Big)$$

$$s_{n+\frac{11}{10}} = \frac{1}{426610800000h^2}\Big(-816480000000y_n + 1422036000000y_{n+\frac{1}{10}} - 1260441000000y_{n+\frac{11}{10}}$$
$$+654885000000y_{n+\frac{19}{10}} + 500908617h^4f_n - 542479960h^4f_{n+\frac{1}{10}} \quad (2.20)$$
$$+842757806h^4f_{n+1} - 20192751315h^4f_{n+\frac{11}{10}} - 3020740525h^4f_{n+\frac{19}{10}} + 1792783377h^4f_{n+2}\Big)$$

$$s_{n+\frac{19}{10}} = -\frac{1}{426610800000h^2}\Big(10614240000000y_n - 12798324000000y_{n+\frac{1}{10}} + 3587409000000y_{n+\frac{11}{10}}$$
$$-1403325000000y_{n+\frac{19}{10}} + 32936727627h^4f_n - 54572970760h^4f_{n+\frac{1}{10}} - 83172019414h^4f_{n+1} \quad (2.21)$$
$$-87056974665h^4f_{n+\frac{11}{10}} - 80228037175h^4f_{n+\frac{19}{10}} + 45278532387h^4f_{n+2}\Big)$$

$$s_{n+2} = -\frac{1}{853221600000h^2}\Big(23677920000000y_n - 28440720000000y_{n+\frac{1}{10}} + 7756560000000y_{n+\frac{11}{10}}$$
$$-2993760000000y_{n+\frac{19}{10}} + 76992556509h^4f_n - 126928542400h^4f_{n+\frac{1}{10}} - 146034445018h^4f_{n+1} \quad (2.22)$$
$$-253898837775h^4f_{n+\frac{11}{10}} - 220541306425h^4f_{n+\frac{19}{10}} + 116527553109h^4f_{n+2}\Big)$$

Finding the third derivative of (2.8) and evaluating all the collocating points $x = x_n$, $x = x_{n+\frac{1}{10}}$, $x = x_{n+1}$, $x = x_{n+\frac{11}{10}}$, $x = x_{n+\frac{10}{10}}$, and $x = x_{n+2}$ and let $t_{n+j} = y''_{n+j}$ such that $j \in \left\{0, \frac{1}{10}, 1, \frac{11}{10}, \frac{19}{10}, 2\right\}$ gives the following discrete schemes:

$$t_n = \frac{1}{85322160000h^3}\Big(-2449440000000y_n + 2844072000000y_{n+\frac{1}{10}} - 581742000000y^{10}_{n+11}$$
$$+187110000000y_{n+\frac{19}{10}} + 20341524573h^4f_n - 56174813860h^4f_{n+\frac{1}{10}} - 105707500426h^4f_{n+1} \quad (2.23)$$
$$+79470524610h^4f_{n+\frac{11}{10}} - 12149227750h^4f_{n+\frac{19}{10}} + 8094818853h^4f_{n+2}\Big)$$

$$t_{n+\frac{1}{10}} = \frac{1}{85322160000h^3}\Big(-2449440000000y_n + 2844072000000y_{n+\frac{1}{10}} - 581742000000y_{n+\frac{11}{10}} + 187110000000y_{n+\frac{19}{10}} + 24211867707h^4f_n - 51427648360h^4f_{n+\frac{1}{10}} - 106372925494h^4f_{n+1} + 80086291335h^4f_{n+\frac{11}{10}} - 12261672775h^4f_{n+\frac{19}{10}} + 8171629587h^4f_{n+2}\Big) \quad (2.24)$$

$$t_{n+1} = -\frac{1}{85322160000h^3}\Big(2449440000000y_n - 2844072000000y_{n+\frac{1}{10}} + 581742000000y_{n+\frac{11}{10}} - 187110000000y_{n+\frac{10}{10}} + 5853875427h^4f_n - 10122186140h^4f_{n+\frac{1}{10}} \quad (2.25)$$

$$t_{n+\frac{11}{10}} = -\frac{1}{85322160000h^3}\Big(2449440000000y_n - 2844072000000y_{n+\frac{1}{10}} + 581742000000y_{n+\frac{11}{10}} - 187110000000y_{n+\frac{19}{10}} + 5827792293h^4f_n - 10083031640h^4f_{n+\frac{1}{10}} - 66904794506h^4f_{n+1} + 44593228665h^4f_{n+\frac{11}{10}} - 3677327225h^4f_{n+\frac{19}{10}} + 2514430413h^4f_{n+2}\Big) \quad (2.26)$$

$$t_{n+\frac{19}{10}} = -\frac{1}{85322160000h^3}\Big(2449440000000y_n - 2844072000000y_{n+\frac{1}{10}} + 581742000000y_{n+1\frac{11}{10}} - 187110000000y_{n+\frac{19}{10}} + 11143839141h^4f_n - 17818020440h^4f_{n+\frac{1}{10}} + 20572961398h^4f_{n+1} - 80086291335h^4f_{n+\frac{11}{10}} - 56983996025h^4f_{n+\frac{19}{10}} + 27184077261h^4f_{n+2}\Big) \quad (2.27)$$

$$t_{n+2} = -\frac{1}{85322160000h^3}\Big(2449440000000y_n - 2844072000000y_{n+\frac{1}{10}} + 581742000000y_{n+\frac{11}{10}} + 187110000000y_{n+\frac{10}{10}} + 11092955427h^4f_n - 17745186140h^4f_{n+\frac{1}{10}} + 20034220426h^4f_{n+1} - 79470524610h^4f_{n+\frac{11}{10}} - 61770772250h^4f_{n+\frac{10}{10}} + 23339661147h^4f_{n+2}\Big) \quad (2.28)$$

Solving the resulting system for the unknown variables y_{n+j}, r_{n+j}, s_{n+j}, and t_{n+j} such that $j \in \left\{\frac{1}{10}, 1, \frac{11}{10}, \frac{19}{10}, 2\right\}$ gives the discrete schemes which are combined to form the required method below:

$$y_{n+\frac{1}{10}} = \frac{1637}{51710400000}h^4f_{n+2} - \frac{4927}{106375680000}h^4f_{n+\frac{19}{10}} + \frac{301157}{1197504000000}h^4f_{n+\frac{11}{10}} - \frac{949}{3499200000}h^4f_{n+1} + \frac{4774487}{4653936000000}h^4f_{n+\frac{1}{10}} + \frac{100337}{31600800000}h^4f_n + \frac{1}{6000}h^3t_n + \frac{1}{200}h^2s_n + \frac{1}{10}hr_n + y_n \quad (2.29)$$

$$y_{n+1} = -\frac{71}{23085}h^4f_{n+2} + \frac{1325}{290871}h^4f_{n+\frac{19}{10}} - \frac{1145}{37422}h^4f_{n+\frac{11}{10}} + \frac{4315}{122472}h^4f_{n+1} + \frac{3535}{83106}h^4f_{n+\frac{1}{10}} - \frac{2759}{395010}h^4f_n + \frac{1}{6}h^3t_n + \frac{1}{2}h^2s_n + hr_n + y_n \quad (2.30)$$

$$y_{n+\frac{11}{10}} = -\frac{2570535011}{517104000000}h^4f_{n+2}+\frac{288764443}{39191040000}h^4f_{n+\frac{19}{10}}-\frac{5483098423}{108864000000}h^4f_{n+\frac{11}{10}}+\frac{511981129}{8748000000}h^4f_{n+1} + \frac{290459826877}{4653936000000}h^4f_{n+\frac{1}{10}}-\frac{5168273}{432000000}h^4f_n+\frac{1331}{6000}h^3t_n+\frac{121}{200}h^2s_n+\frac{11}{10}hr_n+y_n \tag{2.31}$$

$$y_{n+\frac{19}{10}} = -\frac{1141481639}{19440000000}h^4f_{n+2}+\frac{17212667359}{195955200000}h^4f_{n+\frac{19}{10}}-\frac{696137901157}{1197504000000}h^4f_{n+\frac{11}{10}}+\frac{228690027541}{306180000000}h^4f_{n+1} + \frac{121063908407}{244944000000}h^4f_{n+\frac{1}{10}}-\frac{3467972131}{23760000000}h^4f_n+\frac{6859}{6000}h^3t_n+\frac{361}{200}h^2s_n+\frac{19}{10}hr_n+y_n \tag{2.32}$$

$$y_{n+2} = -\frac{11902}{161595}h^4f_{n+2}+\frac{4600}{41553}h^4f_{n+\frac{19}{10}}-\frac{1880}{2673}h^4f_{n+\frac{11}{10}}+\frac{70256}{76545}h^4f_{n+1}+\frac{172960}{290871}h^4f_{n+\frac{1}{10}} - \frac{35456}{197505}h^4f_n+\frac{4}{3}h^3t_n+2h^2s_n+2hr_n+y_n \tag{2.33}$$

$$r_{n+\frac{1}{10}} = \frac{1182947}{10032000000}h^3f_n-\frac{164531}{13608000000}h^3f_{n+1}+\frac{80891}{57456000000}h^3f_{n+2}+\frac{2600291}{51710400000}h^3f_{n+\frac{1}{10}} + \frac{149111}{13305600000}h^3f_{n+\frac{11}{10}}-\frac{85223}{41368320000}h^3f_{n+\frac{19}{10}}+\frac{1}{200}t_nh^2+\frac{1}{10}s_nh+r_n \tag{2.34}$$

$$r_{n+1} = -\frac{14549}{351120}h^3f_n+\frac{6493}{34020}h^3f_{n+1}-\frac{239}{15120}h^3f_{n+2}+\frac{1175}{6804}h^3f_{n+\frac{1}{10}}-\frac{775}{4752}h^3f_{n+\frac{11}{10}} + \frac{12125}{517104}h^3f_{n+\frac{19}{10}}+\frac{1}{2}t_nh^2+s_nh+r_n \tag{2.35}$$

$$r_{n+\frac{11}{10}} = -\frac{124817187}{2128000000}h^3f_n+\frac{3787758469}{13608000000}h^3f_{n+1}-\frac{61038329}{2736000000}h^3f_{n+2}+\frac{11692463651}{51710400000}h^3f_{n+\frac{1}{10}} - \frac{94664713}{403200000}h^3f_{n+\frac{11}{10}}+\frac{1370060857}{41368320000}h^3f_{n+\frac{19}{10}}+\frac{121}{200}t_nh^2+\frac{11}{10}s_nh+r_n \tag{2.36}$$

$$r_{n+\frac{19}{10}} = -\frac{1153759249}{3696000000}h^3f_n+\frac{21767386309}{13608000000}h^3f_{n+1}-\frac{410641471}{3024000000}h^3f_{n+2}+\frac{365811047}{388800000}h^3f_{n+\frac{1}{10}} - \frac{15379050889}{13305600000}h^3f_{n+\frac{11}{10}}+\frac{89914631}{435456000}h^3f_{n+\frac{19}{10}}+\frac{361}{200}t_nh^2+\frac{19}{10}s_nh+r_n \tag{2.37}$$

$$r_{n+2} = -\frac{2631}{7315}h^3f_n+\frac{2212}{1215}h^3f_{n+1}-\frac{979}{5985}h^3f_{n+2}+\frac{34520}{32319}h^3f_{n+\frac{1}{10}}-\frac{890}{693}h^3f_{n+\frac{11}{10}}+\frac{8150}{32319}h^3f_{n+\frac{10}{10}} +2t_nh^2+2s_nh+r_n \tag{2.38}$$

$$s_{n+\frac{1}{10}} = \frac{14317}{4620000}h^2f_n-\frac{3383}{8505000}h^2f_{n+1}+\frac{221}{4788000}h^2f_{n+2}+\frac{2523209}{1292760000}h^2f_{n+\frac{1}{10}}+\frac{40853}{110880000}h^2f_{n+\frac{11}{10}} - \frac{3677}{54432000}h^2f_{n+\frac{19}{10}}+\frac{1}{10}t_nh+s_n \tag{2.39}$$

$$s_{n+1} = -\frac{5507}{35112}h^2f_n+\frac{5273}{6804}h^2f_{n+1}-\frac{563}{9576}h^2f_{n+2}+\frac{16010}{32319}h^2f_{n+\frac{1}{10}}-\frac{445}{693}h^2f_{n+\frac{11}{10}}+\frac{2825}{32319}h^2f_{n+\frac{19}{10}} +t_nh+s_n \tag{2.40}$$

$$s_{n+\frac{11}{10}} = -\frac{199529}{1064000}h^2f_n + \frac{33191147}{34020000}h^2f_{n+1} - \frac{3411353}{47880000}h^2f_{n+2} + \frac{740819959}{1292760000}h^2f_{n+\frac{1}{10}} - \frac{7972327}{10080000}h^2f_{n+\frac{11}{10}} + \frac{109617167}{1034208000}h^2f_{n+\frac{19}{10}} + \frac{11}{10}t_nh + s_n \quad (2.41)$$

$$s_{n+\frac{19}{10}} = -\frac{21096479}{46200000}h^2f_n + \frac{367374899}{170100000}h^2f_{n+1} - \frac{3244307}{12600000}h^2f_{n+2} + \frac{83618069}{68040000}h^2f_{n+\frac{1}{10}} - \frac{142440853}{110880000}h^2f_{n+\frac{11}{10}} + \frac{22591741}{54432000}h^2f_{n+\frac{10}{10}} + \frac{19}{10}t_nh + s_n \quad (2.42)$$

$$s_{n+2} = -\frac{38}{77}h^2f_n + \frac{3844}{1701}h^2f_{n+1} - \frac{356}{1197}h^2f_{n+2} + \frac{42520}{32319}h^2f_{n+\frac{1}{10}} + \frac{850}{1701}h^2f_{n+\frac{19}{10}} + s_n - \frac{890}{693}h^2f_{n+\frac{11}{10}} + 2t_nh \quad (2.43)$$

$$t_{n+\frac{1}{10}} = \frac{1137667}{25080000}hf_n - \frac{37903}{4860000}hf_{n+1} + \frac{18473}{20520000}hf_{n+2} + \frac{41101}{738720}hf_{n+\frac{1}{10}} + \frac{6859}{950400}hf_{n+\frac{11}{10}} - \frac{19471}{14774400}hf_{n+\frac{19}{10}} + t_n \quad (2.44)$$

$$t_{n+1} = -\frac{35}{114}hf_n + \frac{959}{486}hf_{n+1} - \frac{64}{513}hf_{n+2} + \frac{7175}{9234}hf_{n+\frac{1}{10}} - \frac{325}{216}hf_{n+\frac{11}{10}} + \frac{6875}{36936}hf_{n+\frac{19}{10}} + t_n \quad (2.45)$$

$$t_{n+\frac{11}{10}} = -\frac{233101}{760000}hf_n + \frac{9832097}{4860000}hf_{n+1} - \frac{94501}{760000}hf_{n+2} + \frac{573661}{738720}hf_{n+\frac{1}{10}} - \frac{4653}{3200}hf_{n+\frac{11}{10}} + \frac{2740529}{14774400}hf_{n+\frac{19}{10}} + t_n \quad (2.46)$$

$$t_{n+\frac{19}{10}} = -\frac{487103}{1320000}hf_n + \frac{4849313}{4860000}hf_{n+1} - \frac{446557}{1080000}hf_{n+2} + \frac{168587}{194400}hf_{n+\frac{1}{10}} + \frac{6859}{950400}hf_{n+\frac{11}{10}} + \frac{630059}{777600}hf_{n+\frac{19}{10}} + t_n \quad (2.47)$$

$$t_{n+2} = -\frac{7}{19}hf_n + \frac{244}{243}hf_{n+1} - \frac{7}{19}hf_{n+2} + \frac{4000}{4617}hf_{n+\frac{1}{10}} + \frac{4000}{4617}hf_{n+\frac{19}{10}} + t_n \quad (2.48)$$

2.1. Order and Error Constant of the Block Methods

The linear operator associated with the methods is defined as:

$$L[y(x),h] = \sum_{j=0}^{k} \left[\alpha_j y(x+jh) - h^4\beta_j y^{iv}(x+jh)\right] \quad (2.49)$$

where the function $y(x)$ is assumed to have continuous derivatives of sufficiently high order. Therefore, expanding (2.49) in the Taylor series about the point x to obtain the expression

$$L[y(x),h] = C_0y(x) + hC_1y'(x) + h^2C_2y''(x) + \cdots + h^{p+4}C_{p+4}y^{p+4}(x) \quad (2.50)$$

and

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=0}^{k} j\alpha_j$$

$$C_2 = \frac{1}{2!}\sum_{j=0}^{k} j^2\alpha_j$$

$$\vdots$$

$$C_q = \frac{1}{q!}\left(\sum_{j=0}^{k} j^q q(q-1)(q-2)(q-3)\alpha_j \sum_{j=1}^{k} \beta_j j^{q-4}\right), \quad q \in \{0,1,2,3,\cdots,p+4\} \tag{2.51}$$

In the sense of (2.49), we say that the methods are of order p and error constant C_{p+4} if

$$C_0 = C_1 = C_2 = C_3 = \cdots = C_p = C_{p+1} = C_{p+2} = C_{p+3} = 0, C_{p+4} \neq 0$$

Considering (2.30)

$$y_{n+1} = -\frac{71}{23085}h^4 f_{n+2} + \frac{1325}{290871}h^4 f_{n+\frac{19}{10}} - \frac{1145}{37422}h^4 f_{n+\frac{11}{10}} + \frac{4315}{122472}h^4 f_{n+1} + \frac{3535}{83106}h^4 f_{n+\frac{1}{10}} - \frac{2759}{395010}h^4 f_n + \frac{1}{6}h^3 t_n + \frac{1}{2}h^2 s_n + hr_n + y_n$$

and expanding in Taylor's series about x_n, and collecting the coefficient of the like powers of h gives where $D = \frac{d}{dx}$

$$-y_n(x+h) = -\frac{487103}{1320000}hf_n + \frac{4849313}{4860000}hf_{n+1} - \frac{446557}{1080000}hf_{n+2} + \frac{168587}{194400}hf_{n+\frac{1}{10}} + \frac{6859}{950400}hf_{n+\frac{11}{10}} + \frac{630059}{777600}hf_{n+\frac{19}{10}} - \frac{1}{720}D^6(y_n)(x)h^6 - \frac{1}{5040}D^7(y_n)(x)h^7 - \frac{1}{40320}D^8(y_n)(x)h^8 - \frac{1}{362880}D^9(y_n)(x)h^9 - \frac{1}{3628800}D^{10}(y_n)(x)h^{10} + O(h^{11})$$

$$\frac{1}{6}D^3(y_n)(x)h^3 = \frac{1}{6}D^3(y_n)(x)h^3$$

$$\frac{1}{2}D^2(y_n)(x)h^2 = \frac{1}{2}D^2(y_n)(x)h^2$$

$$D^1(y_n)(x)h^1 = D^1(y_n)(x)h^1$$

$$\frac{-2759}{395010}D^4(y_{n+0h})(x)h^4 = \frac{-2759}{395010}D^4(y_n)(x)h^4$$

$$\frac{3535}{83106}D^4(y_n)\left(x+\frac{h}{10}\right)h^4 = \frac{3535}{83106}D^4(y_n)(x)h^4 + \frac{707}{166212}D^5(y_n)(x)h^5 + \frac{707}{3324240}D^6(y_n)(x)h^6 + \frac{707}{99727200}D^7(y_n)(x)h^7 + \frac{707}{3989088000}D^8(y_n)(x)h^8 + \frac{707}{199454400000}D^9(y_n)(x)h^9 + \frac{707}{11967264000000}D^{10}(y_n)(x)h^{10} + O(h^{11})$$

$$\frac{4315}{122472}D^4(y_n)(x+h)h^4 = \frac{4315}{122472}D^4(y_n)(x)h^4 + \frac{4315}{122472}D^5(y_n)(x)h^5 + \frac{4315}{244944}D^6(y_n)(x)h^6 + \frac{4315}{734832}D^7(y_n)(x)h^7 + \frac{4315}{2939328}D^8(y_n)(x)h^8 + \frac{863}{2939328}D^9(y_n)(x)h^9 + \frac{863}{17635968}D^{10}(y_n)(x)h^{10} + O(h^{11})$$

$$\frac{-1145}{37422}D^4(y_n)\left(x+\frac{11h}{10}\right)h^4 = \frac{-1145}{37422}D^4(y_n)(x)h^4 + \frac{-229}{6804}D^5(y_n)(x)h^5 + \frac{-2519}{136080}D^6(y_n)(x)h^6 + \frac{-27709}{4082400}D^7(y_n)(x)h^7$$
$$+\frac{-304799}{163296000}D^8(y_n)(x)h^8 + \frac{-3352789}{8164800000}D^9(y_n)(x)h^9 + \frac{-36880679}{489888000000}D^{10}(y_n)(x)h^{10} + O(h^{11})$$

$$\frac{1325}{290871}D^4(y_n)\left(x+\frac{11h}{10}\right)h^4 = \frac{1325}{290871}D^4(y_n)(x)h^4 + \frac{265}{30618}D^5(y_n)(x)h^5 + \frac{1007}{122472}D^6(y_n)(x)h^6 + \frac{19133}{3674160}D^7(y_n)(x)h^7$$
$$+\frac{363527}{146966400}D^8(y_n)(x)h^8 + \frac{6907013}{7348320000}D^9(y_n)(x)h^9 + \frac{131233247}{440899200000}D^{10}(y_n)(x)h^{10} + O(h^{11})$$

$$\frac{-71}{23085}D^4(y_n)\left(x+\frac{11h}{10}\right)h^4 = \frac{-71}{23085}D^4(y_n)(x)h^4 + \frac{-142}{23085}D^5(y_n)(x)h^5 + \frac{-142}{23085}D^6(y_n)(x)h^6 + \frac{-284}{69255}D^7(y_n)(x)h^7$$
$$+\frac{-142}{69255}D^8(y_n)(x)h^8 + \frac{-284}{346275}D^9(y_n)(x)h^9 + \frac{-284}{1038825}D^{10}(y_n)(x)h^{10} + O(h^{11})$$

and

$$y_n(x+oh) = y_n$$

Applying (2.30) and collecting the coefficients of like terms of h, we have

$$C_0 = -1 + 1 = 0$$

$$C_1 = -1 + 1 = 0$$

$$C_2 = -\frac{1}{2} + \frac{1}{2} = 0$$

$$C_3 = -\frac{1}{6} + \frac{1}{6} = 0$$

$$C_4 = \frac{-1}{24} - \frac{2759}{395010} + \frac{3535}{83106} + \frac{4315}{122472} - \frac{1145}{37422} + \frac{1325}{290871} - \frac{71}{23085} = 0$$

$$C_5 = \frac{-1}{120} + \frac{707}{166212} + \frac{4315}{122472} - \frac{229}{6804} + \frac{265}{30618} - \frac{142}{23085} = 0$$

$$C_6 = -\frac{1}{720} + \frac{707}{3324240} + \frac{4315}{244944} - \frac{2519}{136080} + \frac{1007}{122472} - \frac{142}{23085} = 0$$

$$C_7 = -\frac{1}{5040} + \frac{707}{99727200} + \frac{4315}{734832} - \frac{27709}{4082400} + \frac{19133}{3674160} - \frac{284}{69255} = 0$$

$$C_8 = -\frac{1}{40320} + \frac{707}{3989088000} + \frac{4315}{2939328} - \frac{304799}{163296000} + \frac{363527}{146966400} - \frac{142}{69255} = 0$$

$$C_9 = -\frac{1}{362880} + \frac{707}{199454400000} + \frac{863}{2939328} - \frac{3352789}{8164800000} + \frac{6907013}{7348320000} - \frac{284}{346275} = 0$$

and

$$C_{10} = \frac{-12857}{5443200000} = C_{p+4}$$

which implies

$$p + 4 = 10$$

such that

$$p = 6$$

The method is of order $p = 6$ with an error constant $C_{10} = C_{p+4} = \frac{-12857}{5443200000}$. This procedure will be adopted for other schemes derived. The orders and error constants of the schemes are presented in Table 1.

Table 1. Order and error constant of the derived schemes

Schemes	Order	Error Constant
$y_{n+\frac{1}{10}} = \frac{1637}{51710400000}h^4f_{n+2} - \frac{4927}{106375680000}h^4f_{n+\frac{19}{10}} + \frac{30157}{1197504000000}h^4f_{n+\frac{11}{10}} - \frac{949}{3499200000}h^4f_{n+1} + \frac{4774487}{4653936000000}h^4f_{n+\frac{1}{10}} + \frac{100337}{31600800000}h^4f_n + \frac{1}{6000}t_nh^3 + \frac{1}{200}s_nh^2 + \frac{1}{10}r_nh + y_n$	6	$\frac{3228263}{1088640} * 10^{-11}$
$y_{n+\frac{11}{10}} = -\frac{2570535011}{517104000000}h^4f_{n+2} + \frac{288764443}{39191040000}h^4f_{n+\frac{19}{10}} - \frac{5483098423}{108864000000}h^4f_{n+\frac{11}{10}} + \frac{511981129}{8748000000}h^4f_{n+1} + \frac{290495826877}{4653936000000}h^4f_{n+\frac{1}{10}} - \frac{5168273}{432000000}h^4f_n + \frac{1331}{6000}t_nh^3 + \frac{121}{200}s_nh^2 + \frac{11}{10}r_nh + y_n$	6	$\frac{-411546021227}{1088640} * 10^{-11}$
$y_{n+\frac{19}{10}} = -\frac{1141481639}{19440000000}h^4f_{n+2} + \frac{17212667359}{195955200000}h^4f_{n+\frac{19}{10}} - \frac{696137901157}{1197504000000}h^4f_{n+\frac{11}{10}} + \frac{228690027541}{306180000000}h^4f_{n+1} + \frac{121063908407}{244944000000}h^4f_{n+\frac{1}{10}} - \frac{3467972131}{23760000000}h^4f_n + \frac{6859}{6000}t_nh^3 + \frac{361}{200}s_nh^2 + \frac{19}{10}r_nh + y_n$	6	$\frac{-4543267774051}{1088640} * 10^{-11}$
$y_{n+2} = -\frac{11902}{161595}h^4f_{n+2} + \frac{4600}{41553}h^4f_{n+\frac{19}{10}} - \frac{1880}{2673}h^4f_{n+\frac{11}{10}} + \frac{70256}{76545}h^4f_{n+1} + \frac{172960}{290871}h^4f_{n+\frac{1}{10}} - \frac{35456}{197505}h^4f_n + 4/3h^3t_n + 2h^2s_n + 2hr_n + y_n$	6	$\frac{-2189}{42525000}$

2.2. Consistency

A linear multi-step method is said to be consistent if the order $p \geq 1$ and obeys the following axioms:

i. $a_0 + a_1 + a_2 + \cdots + a_k = 0$
ii. $\rho(1) = \rho'(1) = \rho''(1) = 0$
iii. $\rho^{iv}(r) = 4!\,\sigma(r)$

where $\rho(r) = \sum_{j=0}^{k} a_j r^j$ and $\sigma(r) = \sum_{j=0}^{k} \beta_j r^j$. Considering (2.33),

$$y_{n+2} = -\frac{9}{11}y_n + y_{n+\frac{1}{10}} - \frac{19}{44}y_{n+\frac{11}{10}} + \frac{5}{4y_{n+\frac{19}{10}}} - \frac{8963}{4000000}h^4f_n + \frac{81283}{21600000}h^4f_{n+\frac{1}{10}} + \frac{255683}{27000000}h^4f_{n+1} + \frac{45391}{28800000}h^4f_{n+\frac{11}{10}} + \frac{70571}{17280000}h^4f_{n+\frac{19}{10}} - \frac{86477}{36000000}h^4f_{n+2}$$

we have

i.

$$\sum_{j=0}^{4} a_j = a_0 + a_{\frac{1}{10}} + a_1 + a_{\frac{11}{10}} + a_{\frac{19}{10}} + a_2 = \frac{9}{11} + 1 + 0 + \frac{19}{44} - \frac{5}{4} - 1 = 0$$

ii.

$$\rho(r) = \frac{9}{11} + 1r^{\frac{1}{10}} + 0r^1 + \frac{19}{44}r^{\frac{11}{10}} - \frac{5}{4}r^{\frac{19}{10}} - 1r^2$$

$$\rho(1) = \frac{9}{11} + 1(1)^{\frac{1}{10}} + 0(1)^1 + \frac{19}{44}(1)^{\frac{11}{10}} - \frac{5}{4}(1)^{\frac{19}{10}} - 1(1)^2 = 0$$

$$\rho'(r) = 2r - \frac{1}{10}r^{-\frac{9}{10}} + \frac{19}{40}r^{\frac{1}{10}} - \frac{19}{8}r^{\frac{9}{10}}$$

$$\rho'(1) = 2(1) - \frac{1}{10}(1)^{-\frac{9}{10}} + \frac{19}{40}(1)^{\frac{1}{10}} - \frac{19}{8}(1)^{\frac{9}{10}} = 0$$

iii. For the LHS

$$\rho^{iv}(r) = \frac{4959}{10000}r^{-\frac{39}{10}} + \frac{3249}{40000}r^{-\frac{29}{10}} - \frac{1881}{8000}r^{-\frac{21}{10}}$$

and

$$\rho^{iv}(1) = \frac{4959}{10000}(1)^{-\frac{39}{10}} + \frac{3249}{40000}(1)^{-\frac{29}{10}} - \frac{1881}{8000}(1)^{-\frac{21}{10}} = \frac{171}{500}$$

and for the RHS

$$\sigma(r) = \sum_{j=0}^{4} \beta_j r^j = \beta_0 r^0 + \beta_{1/10} r^{\frac{1}{10}} + \beta_1 r^1 + \beta_{\frac{11}{10}} r^{\frac{11}{10}} + \beta_{\frac{19}{10}} r^{\frac{19}{10}} + \beta_2 r^2$$

$$\sigma(r) = \frac{-8963}{4000000}r^0 + \frac{81283}{21600000}r^{\frac{1}{10}} + \frac{255683}{27000000}r^1 + \frac{45391}{28800000}r^{\frac{11}{10}} + \frac{70571}{17280000}r^{\frac{19}{10}} + \frac{-86477}{36000000}r^2$$

$$\sigma(1) = \frac{-8963}{4000000}(1)^0 + \frac{81283}{21600000}(1)^{\frac{1}{10}} + \frac{255683}{27000000}(1)^1 + \frac{45391}{28800000}(1)^{\frac{11}{10}} + \frac{70571}{17280000}(1)^{\frac{19}{10}} + \frac{-86477}{36000000}(1)^2 = \frac{57}{4000}$$

then

$$4!\,\sigma(1) = 24 * \frac{57}{4000} = \frac{171}{500}$$

and

$$LHS = RHS$$

This implies that $\rho^4(r) = 4!\,\sigma(r)$ for the principal root, $r = 1$, and also since the order $p = 6$, hence it satisfies $p \geq 1$. Therefore, the derived schemes are consistent. Hence, the method is consistent.

2.3. Zero Stability of the Blocks

A numerical method is said to be zero-stable if the roots $z \in \{1,2,3,\cdots,N\}$ of the characteristics polynomial $p(z) = \det(zA^0 - A')$, satisfies $|z| \leq 1$, and the roots $|z| = 1$ have multiplicity not exceeding the order of the differential equation, which is 4. Moreover, as $h^\gamma \to 0 p(z) = z^{r-\gamma}(\lambda - 1)$, where γ is the order of the differential equation. From the resulting schemes, we have that

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1/10} \\ y_{n+1} \\ y_{n+11/10} \\ y_{n+19/10} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-1/10} \\ y_{n-1} \\ y_{n-11/10} \\ y_{n-19/10} \\ y_n \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & 0 & 1/10 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 11/10 \\ 0 & 0 & 0 & 0 & 19/10 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} r_{n-1/10} \\ r_{n-1} \\ r_{n-11/10} \\ r_{n-19/10} \\ r_n \end{pmatrix}$$

$$+h^2\begin{pmatrix}0&0&0&0&1/200\\0&0&0&0&1/2\\0&0&0&0&121/200\\0&0&0&0&361/200\\0&0&0&0&2\end{pmatrix}\begin{pmatrix}s_{n-1/10}\\s_{n-1}\\s_{n-11/10}\\s_{n-19/10}\\s_n\end{pmatrix}+h^3\begin{pmatrix}0&0&0&0&1/6000\\0&0&0&0&1/6\\0&0&0&0&1331/6000\\0&0&0&0&6859/6000\\0&0&0&0&4/3\end{pmatrix}\begin{pmatrix}t_{n-1/10}\\t_{n-1}\\t_{n-11/10}\\t_{n-19/10}\\t_n\end{pmatrix}$$

$$+h^4\begin{pmatrix}0&0&0&0&100337/31600800000\\0&0&0&0&-2759/395010\\0&0&0&0&-5168273/432000000\\0&0&0&0&-3467972131/23760000000\\0&0&0&0&-35456/197505\end{pmatrix}\begin{pmatrix}f_{n-1/10}\\f_{n-1}\\f_{n-11/10}\\f_{n-19/10}\\f_n\end{pmatrix}$$

$$+h^4\begin{pmatrix}\frac{4774487}{465393600000}&-\frac{949}{3499200000}&\frac{301157}{1197504000000}&-\frac{4927}{106375680000}&\frac{1637}{51710400000}\\\frac{3535}{83106}&\frac{4315}{122472}&-\frac{1145}{3742}&\frac{1325}{29071}&-\frac{71}{23085}\\\frac{290459826877}{4653936000000}&\frac{511981129}{8748000000}&-\frac{5483098423}{108864000000}&\frac{288764443}{39191040000}&-\frac{2570535011}{517104000000}\\\frac{121063908407}{244944000000}&\frac{228690027541}{306180000000}&-\frac{696137901157}{1197504000000}&\frac{17212667359}{195955200000}&-\frac{1141481639}{19440000000}\\\frac{172960}{290871}&\frac{70256}{76545}&-\frac{1880}{2673}&\frac{4600}{41553}&-\frac{11902}{161595}\end{pmatrix}\begin{pmatrix}f_{n+1/10}\\f_{n+1}\\f_{n+11/10}\\f_{n+19/10}\\f_{n+2}\end{pmatrix}$$

which is expressed in the form

$$A^0Y_N = A'Y_{N-1} + hA''Y'_{N-1} + h^2B'Y''_{N-1} + h^3B''Y'''_{N-1} + h^4(E^0F_N + E'F_{N-1}) \tag{2.52}$$

where

$$Y_N=\begin{pmatrix}y_{n+\frac{1}{10}}\\y_{n+1}\\y_{n+\frac{11}{10}}\\y_{n+\frac{19}{10}}\\y_{n+2}\end{pmatrix},Y_{N-1}=\begin{pmatrix}y_{n-\frac{1}{10}}\\y_{n-1}\\y_{n-\frac{11}{10}}\\y_{n-\frac{19}{10}}\\y_n\end{pmatrix},Y'_{N-1}=\begin{pmatrix}r_{n-\frac{1}{10}}\\r_{n-1}\\r_{n-\frac{11}{10}}\\r_{n-\frac{19}{10}}\\r_n\end{pmatrix},Y''_{N-1}=\begin{pmatrix}s_{n-\frac{1}{10}}\\s_{n-1}\\s_{n-\frac{11}{10}}\\s_{n-\frac{19}{10}}\\s_n\end{pmatrix},Y'''_{N-1}=\begin{pmatrix}t_{n-\frac{1}{10}}\\t_{n-1}\\t_{n-\frac{11}{10}}\\t_{n-\frac{19}{10}}\\t_n\end{pmatrix},$$

$$F_N=\begin{pmatrix}f_{n-\frac{1}{10}}\\f_{n-1}\\f_{n-\frac{11}{10}}\\f_{n-\frac{19}{10}}\\f_n\end{pmatrix},F_{N-1}=\begin{pmatrix}f_{n+\frac{1}{10}}\\f_{n+1}\\f_{n+\frac{11}{10}}\\f_{n+\frac{19}{10}}\\f_{n+2}\end{pmatrix},$$

$$A^0=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},A'=\begin{pmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{pmatrix},A''=\begin{pmatrix}0&0&0&0&\frac{1}{10}\\0&0&0&0&1\\0&0&0&0&\frac{11}{10}\\0&0&0&0&\frac{19}{10}\\0&0&0&0&2\end{pmatrix},$$

$$B'=\begin{pmatrix}0&0&0&0&\frac{1}{200}\\0&0&0&0&\frac{1}{2}\\0&0&0&0&\frac{121}{200}\\0&0&0&0&\frac{361}{200}\\0&0&0&0&2\end{pmatrix},B''=\begin{pmatrix}0&0&0&0&\frac{1}{6000}\\0&0&0&0&\frac{1}{6}\\0&0&0&0&\frac{1331}{6000}\\0&0&0&0&\frac{6859}{6000}\\0&0&0&0&\frac{4}{3}\end{pmatrix},$$

$$E^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 100337/31600800000 \\ 0 & 0 & 0 & 0 & -2759/395010 \\ 0 & 0 & 0 & 0 & -5168273/432000000 \\ 0 & 0 & 0 & 0 & -3467972131/23760000000 \\ 0 & 0 & 0 & 0 & -35456/197505 \end{pmatrix}$$

and

$$E' \begin{pmatrix} \frac{4774487}{4653936000000} & -\frac{949}{3499200000} & \frac{301157}{1197504000000} & -\frac{4927}{106375680000} & \frac{1637}{51710400000} \\ \frac{3535}{83106} & \frac{4315}{122472} & -\frac{1145}{37422} & \frac{1325}{290871} & -\frac{71}{23085} \\ \frac{290459826877}{4653936000000} & \frac{511981129}{8748000000} & -\frac{5483098423}{108864000000} & \frac{288764443}{39191040000} & -\frac{2570535011}{517104000000} \\ \frac{121063908407}{244944000000} & \frac{228690027541}{306180000000} & -\frac{696137901157}{1197504000000} & \frac{17212667359}{195955200000} & -\frac{1141481639}{19440000000} \\ \frac{172960}{290871} & \frac{70256}{76545} & -\frac{1880}{2673} & \frac{4600}{41553} & -\frac{11902}{161595} \end{pmatrix}$$

From the condition that,

$$\begin{aligned} p(z) &= \det(zA^0 - A') \\ &= \det\left[z \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] \\ &= \det\left[\begin{pmatrix} z & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & z \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] \\ &= \det \begin{pmatrix} z & 0 & 0 & 0 & -1 \\ 0 & z & 0 & 0 & -1 \\ 0 & 0 & z & 0 & -1 \\ 0 & 0 & 0 & z & -1 \\ 0 & 0 & 0 & 0 & z-1 \end{pmatrix} \\ &= \begin{vmatrix} z & 0 & 0 & 0 & -1 \\ 0 & z & 0 & 0 & -1 \\ 0 & 0 & z & 0 & -1 \\ 0 & 0 & 0 & z & -1 \\ 0 & 0 & 0 & 0 & z-1 \end{vmatrix} \\ &= z\left(z \begin{vmatrix} z & 0 & -1 \\ 0 & z & -1 \\ 0 & 0 & z-1 \end{vmatrix} - 1 \begin{vmatrix} 0 & z & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{vmatrix} \right) - 1\left(-z \begin{vmatrix} 0 & z & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{vmatrix} \right) \\ &= z\left(z\left(z^2(z-1) \right) \right) \\ &= z^4(z-1) \end{aligned}$$

Hence, $z_{1,2,3,4,5} = 0,0,0,0,1$. Therefore, the new methods are zero-stable since $|z| = 1$, i.e., simple, and the magnitude of other roots are zeros $|z| = 0$.

2.4. Convergence

A linear multi-step method is said to be convergent if it is consistent and zero-stable (or if it satisfies the root conditions). Since our methods are zero-stable and consistent, then the methods are convergent.

3. Numerical Scheme Implementation and Results

3.1. Numerical Experiments

This subsection tests the accuracy of the proposed methods with some numerical problems and compares the results with the existing methods. Error is defined as

$$\text{Error} = \max_{a\le x\le b} |y(x) - y_N(x)|, \; N \ge 1,2,3,\ldots \tag{2.53}$$

where $y(x)$ is the exact solution and $y_N(x)(N \ge 1)$ are the approximate solutions. The following examples are considered:

Example 3.1. Consider $y^{iv} - 4y'' = 0$ with initial conditions $y'''(0) = 16, y''(0) = 0, y'(0) = 3, y(0) = 1$, and $h = 0.003125$. Then, the exact solution is

$$y(x) = e^{2x} - e^{-2x} - x + 1$$

Example 3.2. Consider $y^{iv} = x, t \in [0,1]$ with initial conditions, $y'''(0) = 0, y''(0) = 0, y'(0) = 1, y(0) = 0$, and $h = \frac{1}{10}$. Then, the exact solution is

$$y(x) = \frac{x^5}{120} + x$$

Example 3.3. Consider $y^{iv} = \cos x - \sin x$ with initial conditions $y'''(0) = 7$, $y''(0) = -1$, $y'(0) = -1$, $y(0) = 0$, and $h = \frac{1}{320}$. Then, the exact solution is

$$y(x) = \cos x - \sin x + x^3 - 1$$

Table 2 shows the exact result, the computed result from the proposed schemes using power series (PPSS), and the result in [3]. Table 3 shows errors in the proposed scheme and [3] for Example 3.1.

Table 2. Numerical results for Example 3.1

X	**Exact**	**PPSS**	[3]
0	1	1.00000	1.000000
0.003125	1.0093750813803672792	1.0093750813803672792	1.009375081380367264
0.006250	1.0187506510467529486	1.0187506510467529486	1.018750651046752800
0.009375	1.0281271973042491331	1.0281271973042491331	1.028127197304248000
0.012500	1.0375052084960961721	1.0375052084960961721	1.037505208496096000
0.015625	1.0468851730227585890	1.0468851730227585891	1.046885173022758387
0.018750	1.0562675793610032975	1.0562675793610032975	1.056267579361001125
0.021875	1.0656529160829807860	1.0656529160829807861	1.065652916082977682
0.025000	1.0750416718753100306	1.0750416718753100306	1.075041671875305141
0.028125	1.0844343355581678774	1.0844343355581678774	1.084434335558156293
0.031250	1.0938313961043836435	1.0938313961043836434	1.093831396104360521
0.034375	1.1032333426585396797	1.1032333426585396797	1.1032333426585396797
0.037500	1.1126406645560786435	1.1126406645560786435	1.1126406645560786435
⋮	⋮	⋮	⋮
0.31250	2.0204845289132321645	2.0204845289132321645	2.0204845289132321645
0.62500	3.5788381606016512758	3.5788381606016512756	3.5788381606016512756
1	7.2537208156940375353	7.2537208156940375326	7.2537208156940375326

Table 3. Comparison of errors with numerical results of Example 3.1

n	x	**PPSS Error**	**Error** [3]
1	0.003125	$0.0000\text{E}+00$	$1.5000\text{E}-17$
2	0.006250	$0.0000\text{E}+00$	$1.4900\text{E}-16$
3	0.009375	$0.0000\text{E}+00$	$1.1330\text{E}-15$
4	0.012500	$0.0000\text{E}+00$	$1.7200\text{E}-16$
5	0.015625	$1.0000\text{E}-20$	$2.0200\text{E}-16$
6	0.018750	$0.0000\text{E}+00$	$2.1720\text{E}-15$
7	0.021875	$1.0000\text{E}-20$	$3.1040\text{E}-15$
8	0.025000	$0.0000\text{E}+00$	$4.8900\text{E}-15$
9	0.028125	$0.0000\text{E}+00$	$1.1584\text{E}-14$
10	0.031250	$1.0000\text{E}-20$	$2.3123\text{E}-14$
11	0.034375	$0.0000\text{E}+00$	-
12	0.037500	$0.0000\text{E}+00$	-
13	0.31250	$0.0000\text{E}+00$	-
14	0.62500	$2.1000\text{E}-98$	-
15	1	$2.7000\text{E}-18$	-

Table 4 shows the exact and computed results from the proposed schemes using power series (PPSS) for Example 3.2. Table 5 shows errors in the proposed scheme and [7] for Example 3.2.

Table 4. Numerical results for Example 3.2

x	**Exact Solutions**	**PPSS**
0.1	0.100000083333333333	0.10000008333333333333
0.2	0.200002666666666667	0.20000266666666666667
0.3	0.300020250000000000	0.30002025000000000000
0.4	0.400085333333333333	0.40008533333333333333
0.5	0.500260416666666667	0.50026041666666666667
0.6	0.600648000000000000	0.60064800000000000000
0.7	0.701400583333333333	0.70140058333333333333
0.8	0.802730666666666667	0.80273066666666666667
0.9	0.904920750000000000	0.90492075000000000000
1.0	1.008333333333333333	1.00833333333333333333

Table 5. Comparison of errors with numerical results of Example 3.2

n	[7]	[8]	[9]	**PPSS Error**
1	$1.658\text{E}-13$	$7.000\text{E}-10$	$0.000\text{E}-00$	$0.000\text{E}-00$
2	$3.316\text{E}-12$	$8.999\text{E}-10$	$0.000\text{E}-00$	$0.000\text{E}-00$
3	$7.183\text{E}-12$	$2.999\text{E}-09$	$0.000\text{E}-00$	$0.000\text{E}-00$
4	$6.649\text{E}-11$	$5.100\text{E}-09$	$0.000\text{E}-00$	$0.000\text{E}-00$
5	$9.906\text{E}-11$	$7.799\text{E}-09$	$0.000\text{E}-00$	$0.000\text{E}-00$
6	$3.217\text{E}-11$	$1.180\text{E}-08$	$0.000\text{E}-00$	$0.000\text{E}-00$
7	$2.432\text{E}-10$	$1.240\text{E}-08$	$0.000\text{E}-00$	$0.000\text{E}-00$
8	$3.202\text{E}-10$	$1.410\text{E}-08$	$2.000\text{E}-18$	$0.000\text{E}-00$
9	$2.540\text{E}-10$	$1.880\text{E}-08$	$2.000\text{E}-18$	$0.000\text{E}-00$
10	$2.020\text{E}-10$	$2.600\text{E}-08$	$1.000\text{E}-17$	$0.000\text{E}-00$

Table 6 shows the exact result, the computed result from the proposed schemes using power series (PPSS), and the result in [3]. Table 7 shows errors in the proposed scheme and [3] for Example 3.3.

Table 6. Numerical results for Example 3.3

x	**Exact**	**PPSS**	[3]
0	0	0.00000	-
$\frac{1}{320}$	-0.00312984720468769600	-0.00312984720468769600	-0.00312984720468770183
$\frac{2}{320}$	-0.00626924635577210114	-0.00626924635577210114	-0.00626924635577214781
$\frac{3}{320}$	-0.00941798368752841945	-0.00941798368752841944	-0.00941798368752885697
$\frac{4}{320}$	-0.01257584533946248273	-0.01257584533946248273	-0.0125758453394627160
$\frac{5}{320}$	-0.01574261735661109244	-0.01574261735661109244	-0.0157426173566113317
$\frac{6}{320}$	-0.01891808568984328399	-0.01891808568984328399	-0.0189180856898435642
$\frac{7}{320}$	-0.02210203619616251069	-0.02210203619616251069	-0.0221020361961631824
$\frac{8}{320}$	-0.02529425463900974441	-0.02529425463900974441	-0.025294254639010215
$\frac{9}{320}$	-0.02849452668856748983	-0.02849452668856748983	-0.0284945266885679628
$\frac{10}{320}$	-0.03170263792206470950	-0.03170263792206470950	-0.00312984720468770183

Table 7. Comparison of errors with numerical results of Example 3.3

x	**PPSS Error**	**Error** [3]
0	0	-
$\frac{1}{320}$	$0.0000\text{E}+00$	$5.8350\text{E}-18$
$\frac{2}{320}$	$0.0000\text{E}+00$	$4.6712\text{E}-17$
$\frac{3}{320}$	$1.0000\text{E}-20$	$4.3748\text{E}-16$
$\frac{4}{320}$	$0.0000\text{E}+00$	$2.3340\text{E}-16$
$\frac{5}{320}$	$0.0000\text{E}+00$	$2.3920\text{E}-16$
$\frac{6}{320}$	$0.0000\text{E}+00$	$2.8020\text{E}-16$
$\frac{7}{320}$	$0.0000\text{E}+00$	$6.7177\text{E}-16$
$\frac{8}{320}$	$0.0000\text{E}+00$	$4.6706\text{E}-16$
$\frac{9}{320}$	$0.0000\text{E}+00$	$5.1408\text{E}-16$
$\frac{10}{320}$	$1.0000\text{E}-20$	$5.8350\text{E}-18$

Figures 1-3 present the behavior of the exact solutions compared with the PPSS for Example 3.1-3.3, respectively.

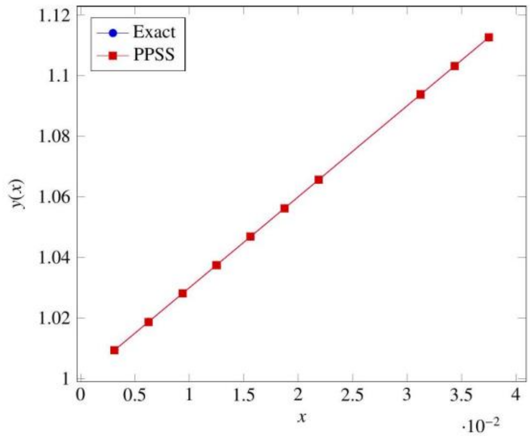

Figure 1. The behavior of the exact solution compared with the PPSS for Example 3.1

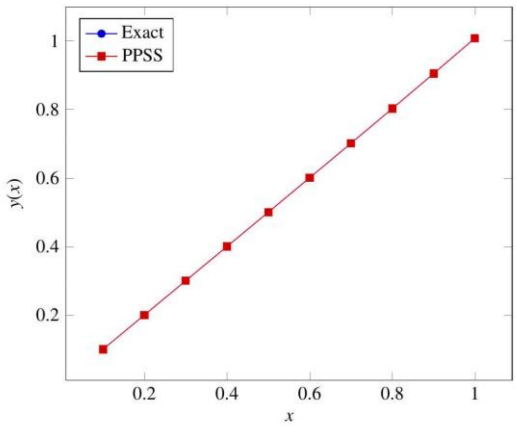

Figure 2. The behavior of the exact solution compared with the PPSS for Example 3.2

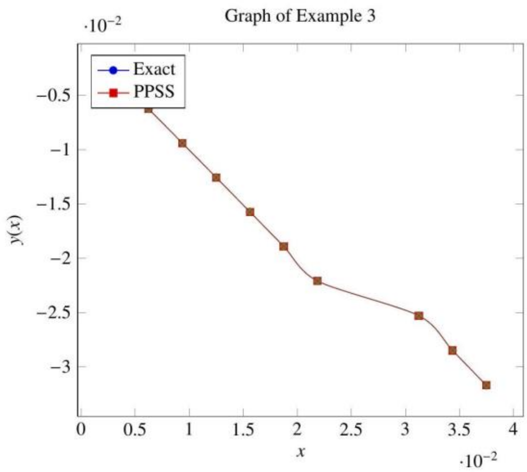

Figure 3. The behavior of the exact solution compared with the PPSS for Example 3.3

4. Conclusion

We have incorporated off-step points for collocation and interpolation to develop a more accurate Two-step Block Hybrid Linear Multi-step method for the numerical solution of initial value problems of fourth-order differential equations. In developing the numerical method with two steps ($k = 2$), we used three off-step points at interpolation and collocation points. The order and error constants were obtained using the method employed by [6]. The Two-step Block Hybrid Linear Multi-step method has six orders of accuracy. Moreover, the zero stability of the Block Hybrid Method was analyzed using the concept of [6], and the method was zero stable. Hence, block hybrid methods are convergent. The method was used to solve some problems adapted from [3], and the computed result and the exact solution were compared with the solution from [3]. Errors were computed, and the proposed method produced approximations closer to the exact solution than the reviewed work. The introduction of three off-step points at both collocation and interpolation has proven to produce more accurate results than the literature results by those who used one off-step point at both collocation and interpolation. Therefore, introducing more off-step points at both collocation and interpolation in developing block methods results in a better approximation.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

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