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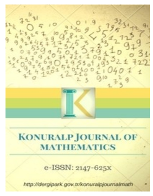

On Unicity of Non-linear Differential Polynomial of Meromorphic Function With Its Shift and q-difference

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Abstract

This article is insisted to studying the problems on sharing value for the non-linear differential polynomial with its shift and q-difference. The results in this paper improve and generalize the recent results due to S. C. Kumar, S. Rajeshwari, T. Bhuvaneshwari [J. Math. Comput. Sci., 12(2022), 1-16.]

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1. Introduction, Definitions and Result

In this article, by a meromorphic function we always mean a meromorphic function in the complex plane and use standard notations and main results of Nevanlinna theory [6, 9, 18]. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic function of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$ as $r \to \infty$, possibly outside of a set of finite linear measure. We say that the meromorphic function α is a small function of f, if $T(r,\alpha) = S(r,f)$.
Let $a \in \mathbb{C} \cup \{\infty\}$. Set $E(a,f) = \{z : f(z) - a = 0\}$, where a zero with multiplicity k is counted k times. If the zeros are counted only once, then we denote the set by $\overline{E}(a,f)$. Let f and g be two nonconstant meromorphic functions. If $E(a,f) = E(a,g)$, then we say that f and g share the value a CM (Counting Multiplicities). If $\overline{E}(a,f) = \overline{E}(a,g)$, then we say that f and g share the value a IM (Ignoring Multiplicities). In addition, we need the following definitions.

Definition 1.1. *[6] A non-linear differential polynomial P[f] of a nonconstant meromorphic function f is defined as*

$$P[f] = \sum_{i=1}^{t} M_i[f]$$

where $M_i[f] = a_i \prod_{j=0}^{k} \left(f^{(j)}\right)^{n_{ij}}$ *with* $n_{i0}, n_{i1}, ..., n_{ik}$ *as nonnegative integers and* $a_i(\not\equiv 0)$ *are meromorphic functions satisfying* $T(r,a_i) = o(T(r,f))$ *as* $r \to \infty$.
The numbers $\overline{d}(P) = max_{1\leq i\leq t} \sum_{j=0}^{k} n_{ij}$ *and* $\underline{d}(P) = min_{1\leq i\leq t} \sum_{j=0}^{k} n_{ij}$ *are respectively called the degree and lower degree of* $P[f]$. *If* $\overline{d}(P) = \underline{d}(P) = d$*(say), then we say that* $P[f]$ *is a homogeneous differential polynomial of degree d. Also we define* $Q = \max_{1\leq i\leq t}\{n_{i0} + n_{i1} + 2n_{i2} + ... + kn_{ik}\}$.

Definition 1.2. *[11] Let* $a \in \mathbb{C} \cup \{\infty\}$. *We denote by* $N(r,a;f)$ *the counting function of simple a-points of* f. *For a positive integer k we denote by* $N_{k)}(r,a;f)$ *the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than k. By* $\overline{N}_{k)}(r,a;f)$ *we denote the corresponding reduced counting function. Analogously we can define* $N_{(k}(r,a;f)$ *and* $\overline{N}_{(k}(r,a;f)$.

Definition 1.3. *[10] Let k be a positive integer or infinity. We denote by* $N_k(r,a;f)$ *the counting function of a-points of* f, *where an a-points of multiplicity m is counted m times if* $m \leq k$ *and k times if* $m > k$. *Then*

$$N_k(r,a;f) = \overline{N}(r,a;f) + \overline{N}_{(2}(r,a;f) + ... + \overline{N}_{(k}(r,a;f).$$

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clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

Definition 1.4. *[8] Let z_0 be a zero of $f-1$ of multiplicity p and a zero of $g-1$ of multiplicity q. We denote by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f where $p=q=1$; by $N_L(r,1;f)$ the counting function of the 1-points of f whose multiplicities are greater than 1-points of g; each point in these counting functions is counted only once.*

A lot of works on meromorphic functions whose differential polynomials generated by them share certain value, small function or fixed points have been investigated by many mathematician across the world. In 1997, L. A. Rubel and C. C. Yang proved the following theorem.

Theorem 1.5. *[15] Let f be a nonconstant entire function. If f and f' share two distinct finite values CM, then $f=f'$.*

The function $f = e^{e^z}\int_0^z e^{-e^t}(1-e^t)dt$ from [3] shows clearly that f and f' share 1 CM but $f \neq f'$. In a special case, we recall a well-known conjecture by Brück:

Conjecture 1.6. *[3] Let f be a nonconstant entire function such that hyper-order $\rho_2(f) = limsup_{r\to\infty}\frac{loglogT(r,f)}{logr}$ is not a positive integer or infinity. If f and f' share the finite value a CM, then $\frac{f'-a}{f-a} = c$, where c is a nonzero constant.*

The conjecture has been verified in the special cases when $a=0$ [3], or when f is of finite order [5], or when $\rho_2(f) < \frac{1}{2}$ [4].
In 2009, J. Heittokangas, R. J. Korhonen, R. Laine, I. Rieppo and J. L. Zhang, considered analogues of Brück's conjecture for meromorphic functions concerning their shifts and proved the following theorem.

Theorem 1.7. *[7]Let f be a meromorphic function of order*

$$\rho(f) = limsup_{r\to\infty}\frac{logT(r,f)}{logr} < 2$$

and let $c \in \mathbb{C}$. If $f(z)$ and $f(z+c)$ share the values $a \in \mathbb{C}$ and ∞ CM, then

$$\frac{f(z+c)-a}{f(z)-a} = \tau,$$

In 2018, Qi, Li and Yang considered the value sharing problem related to $f'(z)$ and $f(z+c)$, where c is a complex number. They obtained the following result.

Theorem 1.8. *[14]Let f be a nonconstant meromorphic function of finite order and $n \geq 9$ be an integer. If $[f'(z)]^n$ and $f^n(z+c)$ share $a(\neq 0)$ and ∞ CM, then $f'(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.*

It is natural to ask whether the f' can be extended to $f^{(k)}$ in Theorem C. Here f^n denotes the n^{th} power of the function f and $f^{(k)}$ stands for the k^{th} derivative of f, where k is a nonnegtive integer. Considering this question, C. Meng and G. Liu proved the following results.

Theorem 1.9. *[13] Let f be a nonconstant meromorphic function of finite order and n be a positive integer. If one of the following conditions is satisfied:*

$$(I)\ [f^{(k)}(z)]^n \text{ and } f^n(z+c) \text{ share } (1,2),(\infty,0) \text{ and } n \geq 2k+8;$$

$$(II)\ [f^{(k)}(z)]^n \text{ and } f^n(z+c) \text{ share } (1,2),(\infty,\infty) \text{ and } n \geq 2k+7;$$

$$(III)\ [f^{(k)}(z)]^n \text{ and } f^n(z+c) \text{ share } (1,0),(\infty,0) \text{ and } n \geq 3k+14;$$

then $f^{(k)}(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.

If they consider entire function instead of meromorphic function, the counting functions related to the poles of $[f^{(k)}(z)]^n$ and $f^n(z+c)$ can be neglected. Arguing similarly as in Theorem D, one can see that k is not related to the coefficient of $N_{k+1}(r,0;f)$. So obtained the following corollary.

Corollary 1.10. *[13] Let f be a nonconstant entire function of finite order and $n \geq 5$ be an integer. If $[f^{(k)}(z)]^n$ and $f^n(z+c)$ share $(1,2)$, then $f^{(k)}(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.*

If the shifts $f(z+c)$ in Theorem C and D are replaced by q-difference $f(qz)$, where $q \in \mathbb{C}\backslash\{0\}$, they obtained:

Theorem 1.11. *[13] Let f be a nonconstant meromorphic function of zero order and n be a positive integer. If one of the following conditions is satisfied:*

$$(I)\ [f^{(k)}(z)]^n \text{ and } f^n(qz) \text{ share } (1,2),(\infty,0) \text{ and } n \geq 2k+8;$$

$$(II)\ [f^{(k)}(z)]^n \text{ and } f^n(qz) \text{ share } (1,2),(\infty,\infty) \text{ and } n \geq 2k+7;$$

$$(III)\ [f^{(k)}(z)]^n \text{ and } f^n(qz) \text{ share } (1,0),(\infty,0) \text{ and } n \geq 3k+14;$$

then $f^{(k)}(z) = tf(qz)$, for a constant t that satisfies $t^n = 1$.

Corollary 1.12. *[13] Let f be a nonconstant entire function of zero order and $n \geq 5$ be an integer. If $[f^{(k)}(z)]^n$ and $f^n(qz)$ share $(1,2)$, then $f^{(k)}(z) = tf(qz)$, for a constant t that satisfies $t^n = 1$.*

In 2022, by considering the difference operator $\Delta_c F$ in Theorem D and E, S. C. Kumar, S. Rajeshwari and T. Bhuvaneshwari obtained the following analogous results.

Theorem 1.13. *[8] Let f be a nonconstant meromorphic function of finite order and n be a positive integer. If one of the following conditions is satisfied:*

$$(I)\ [f^{(k)}(z)]^n \text{ and } \Delta_c F \text{ share } (1,2),(\infty,0) \text{ and } n \geq k+6;$$

$$(II)\ [f^{(k)}(z)]^n \text{ and } \Delta_c F \text{ share } (1,2),(\infty,\infty) \text{ and } n \geq k+5;$$

$$(III)\ [f^{(k)}(z)]^n \text{ and } \Delta_c F \text{ share } (1,0),(\infty,0) \text{ and } n \geq 2k+12;$$

where $\Delta_c F = f^n(z+n) - f^n(z)$ then $f^{(k)} = t f(z+c)$, for a constant t that satisfies $t^n = 1$.
Observing the result of S. C. Kumar, S. Rajeshwari and T. Bhuvaneshwari stated in Theorem F it is natural to ask the following question which is the motive of the present paper.

Question 1.14. *What will happen if we replace $[f^{(k)}(z)]^n$ by the nonlinear differential polynomial $P[f]$, defined in definition 1.1, in Theorem F.*

In the paper, our main concern is to find the possible answer of the above question. We prove the following theorem which extend and improve Theorem F. The following theorem is the main result of the paper.

Theorem 1.15. *Let f be a nonconstant meromorphic function of finite order, n and t be positive integers. If one of the following conditions is satisfied:*

$$(I)\ P[f] \text{ and } \Delta_c F \text{ share } (1,2),(\infty,0) \text{ and } n \geq t\mu+5;$$

$$(II)\ P[f] \text{ and } \Delta_c F \text{ share } (1,2),(\infty,\infty) \text{ and } n \geq t\mu+4;$$

$$(III)\ P[f] \text{ and } \Delta_c F \text{ share } (1,0),(\infty,0) \text{ and } n \geq 2t\mu+10;$$

where $\Delta_c F = f^n(z+n) - f^n(z)$ *and* $\mu = \sum_{j=0}^{k}(1+j)$ *then* $P[f] \equiv \Delta_c F$.

Example 1.16. *Let $f(z) = e^{Az}$, where A is a constant and $z \in \mathbb{C}$. Then one can easily verify that $P[f]$ and $\Delta_c F$ satisfy all the conditions of Theorem 1.1.*

Remark 1.17. *For $t = 1$ and $j = k$ we have $P[f] = (f^{(k)}(z))^{n_{1k}}$ then the Theorem 1.1 reduces to Theorem F for $n_{1k} = n$. Hence Theorem F is the special case of Theorem 1.1. Thus Theorem 1.1 improves and extends Theorem F.*

2. Lemmas

In this section, we state some Lemmas which will be needed in the sequel. We denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

where F and G are nonconstant meromorphic functions defined in the complex plane $\mathbb{C}$.

Lemma 2.1. *[1] Let F, G be two nonconstant meromorphic functions. If F, G share $(1,2)$ and (∞,k), where $0 \leq k \leq \infty$, and $H \not\equiv 0$, then*

$$\begin{aligned} T(r,F) \quad &\leq \quad N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) \\ &\quad + S(r,F) + S(r,G), \end{aligned}$$

where $\overline{N}_*(r,\infty;F,G)$ denotes the reduced counting function of those poles of F whose multiplicities differ from the multiplicities of the corresponding poles of G.

Lemma 2.2. *[17] Let f be a nonconstant meromorphic function, and let $a_1, a_2, ..., a_n$ be finite complex numbers, $a_n \neq 0$. Then*

$$T(r, a_n f^n + ... + a_2 f^2 + a_1 f + a_0) = nT(r,f) + S(r,f).$$

Lemma 2.3. *[12]Let f be a meromorphic function of finite order $\rho(f)$, and let c be a nonzero constant. Then*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\rho(f)-1+\varepsilon}) + O(logr),$$

for an arbitrary $\varepsilon > 0$.

We mention that Lemma 2.3 holds also for $c = 0$ as in the case $T(r, f(z+c)) = T(r, f(z))$.

Lemma 2.4. *[21] Let f be a nonconstant meromorphic function, p, k be positive integers, then*

$$N_p\left(r,0;f^{(k)}\right) \leq k\overline{N}(r,\infty;f)+N_{p+k}(r,0;f)+S(r,f),$$

where $N_p\left(r,0;f^{(k)}\right)$ denotes the counting function of the zeros of $f^{(k)}$ where a zero of multiplicity m is counted m times if $m\leq p$ and p times if $m>p$.

We point out that in Lemma 2.4 one obviously has $\overline{N}\left(r,0;f^{(k)}\right)=N_1\left(r,0;f^{(k)}\right)$.

Lemma 2.5. *[19] Suppose that two nonconstant meromorphic functions F and G share 1 and ∞ IM. Let H be given as above. If $H\not\equiv 0$, then*

$$\begin{aligned} T(r,F)+T(r,G) \leq\ & 3\overline{N}(r,\infty;F)+N_2(r,0;F)+N_2(r,0;G)+N_E^{1)}(r,1;F) \\ & +2N_E^{(2}(r,1;F)+3N_L(r,1;F)+3N_L(r,1;G) \\ & +S(r,F)+S(r,G). \end{aligned}$$

Lemma 2.6. *[20] Let f be a zero order meromorphic function, and $q\in\mathbb{C}\backslash\{0\}$. Then*

$$T(r,f(qz))=(1+o(1))T(r,f(z))$$

and

$$N(r,f(qz))=(1+o(1))N(r,f(z))$$

on a set of lower logarithmic density 1.

Lemma 2.7. *[2] Let f be a zero order meromorphic function, and $q\in\mathbb{C}\backslash\{0\}$. Then*

$$m\left(r,\frac{f(qz)}{f(z)}\right)=S(r,f)$$

on a set of logarithmic density 1.

Lemma 2.8. *[16] Let f be a nonconstant meromorphic function and $P[f]$ be a differential polynomial of f. Then*

$$m\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\overline{d}(P)-\underline{d}(P))m(r,0;f)+S(r,f),$$

$$N\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\overline{d}(P)-\underline{d}(P))N(r,0;f)+Q[\overline{N}(r,\infty;f)+\overline{N}(r,0;f)]+S(r,f)$$

and

$$N(r,0;P[f]) \leq (\overline{d}(P)-\underline{d}(P))m(r,0;f)+Q\overline{N}(r,\infty;f)+N(r,0;f^{\overline{d}(P)})+S(r,f)$$

3. Proof of Theorems

Proof of Theorem 1.15. **Case I.** Let

$$F=\Delta_c F=f^n(z+n)-f^n(z) \text{ and } G=P[f]. \tag{3.1}$$

Then it is easy to verify F and G share $(1,2)$ and $(\infty,0)$. Let H be defined as above. Suppose that $H\not\equiv 0$. It follows from Lemma 2.1 that

$$\begin{aligned} T(r,F) \leq\ & N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G) \\ & +S(r,F)+S(r,G). \end{aligned} \tag{3.2}$$

According to Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} T(r,F) &= nT(r,f(z+\eta))+nT(r,f(z))+S(r,f) \\ &= 2nT(r,f)+O\left(r^{\rho(f)-1+\varepsilon}\right)+S(r,f) \end{aligned} \tag{3.3}$$

It is obvious that

$$\begin{aligned} N_2(r,0;F) &= 2\overline{N}(r,0;f^n(z+\eta)-f^n(z)) \\ &\leq 2\overline{N}(r,0;f(z+\eta))+2\overline{N}(r,0;f(z)) \\ &\leq 4T(r,f)+O\left(r^{\rho(f)-1+\varepsilon}\right)+S(r,f) \end{aligned} \tag{3.4}$$

$$\begin{aligned} \overline{N}(r,\infty;F) &= \overline{N}(r,\infty;f^n(z+\eta)-f^n(z)) \\ &\leq 2T(r,f)+O\left(r^{\rho(f)-1+\varepsilon}\right)+S(r,f) \end{aligned} \tag{3.5}$$

$$\overline{N}_*(r,\infty;F,G) \le \overline{N}(r,\infty;F) \le 2T(r,f) + O\left(r^{\rho(f)-1+\varepsilon}\right) + S(r,f) \tag{3.6}$$

Since $\overline{E}(\infty, f^{(k)}) = \overline{E}(\infty, f)$, we have

$$\overline{N}(r,\infty;G) = \overline{N}(r,\infty;P[f]) = \overline{N}(r,\infty;f) \le T(r,f) \tag{3.7}$$

Using $a_i = 1$, where $i = 1,...,t$, in Definition 1.1 and Lemma 2.4 we have

$$\begin{aligned} N_2(r,0;G) &= N_2(r,0;P[f]) \\ &\le \sum_{i=1}^{t} N_2\left(r,0;\prod_{j=0}^{k}(f^{(j)})^{n_{ij}}\right) \\ &\le \sum_{i=1}^{t} 2\overline{N}\left(r,0;\prod_{j=0}^{k} f^{(j)}\right) \\ &\le 2t\left(\sum_{j=0}^{k} \overline{N}(r,0;f^{(j)})\right) \\ &\le 2t\left(\sum_{j=0}^{k}(1+j)T(r,f)\right) + S(r,f) \\ &= 2t\mu T(r,f) + S(r,f) \end{aligned} \tag{3.8}$$

where $\mu = \sum_{j=0}^{k}(1+j)$
Combining (3.2) - (3.8), we deduce

$$\begin{aligned} T(r,F) \le\ & N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\ & + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G) \end{aligned}$$

i.e.,

$$\begin{aligned} 2nT(r,f) \le\ & 4T(r,f) + 2t\mu T(r,f) + 2T(r,f) + T(r,f) \\ & + 2T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f) \end{aligned}$$

i.e.,

$$(2n - 2t\mu - 9)T(r,f) \le O(r^{\rho(f)-1+\varepsilon}) + S(r,f), \tag{3.9}$$

which contradicts $n \ge \frac{2t\mu+10}{2} \ge t\mu + 5$. Therefore $H \equiv 0$, that is

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both side twice, we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.10}$$

where $A \ne 0$ and B are constants. From (3.10) we have

$$G = \frac{(B-A)F + (A-B-1)}{BF - (B+1)} \tag{3.11}$$

Suppose that $B \ne 0, -1$. From (3.11), we have

$$\overline{N}\left(r, \frac{B+1}{B}; F\right) = \overline{N}(r,\infty;G) \tag{3.12}$$

From the second fundamental theorem and Lemma 2.3, we have

$$\begin{aligned} 2nT(r,f) &= T(r,F) + S(r,f) \\ &\le \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\frac{B+1}{B};F) + S(r,f) \\ &\le \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + S(r,f) \\ &\le 2T(r,f) + 2T(r,f) + T(r,f) + S(r,f) \\ &\le 5T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f) \end{aligned} \tag{3.13}$$

which contradicts $n \ge t\mu + 5$. Suppose that $B = -1$. From (3.11) we have

$$G = \frac{(A+1)F - A}{F} \tag{3.14}$$

If $A \neq -1$, we obtain from (3.14) that

$$\overline{N}\left(r, \frac{A}{A+1}; F\right) = \overline{N}(r,0;G) \tag{3.15}$$

From the second fundamental theorem, Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned} 2nT(r,f) &= T(r,F)+S(r,f) \\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,\frac{A}{A+1};F)+S(r,f) \\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,f) \\ &\leq (t\mu+4)T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f) \end{aligned} \tag{3.16}$$

which contradicts with $n \geq t\mu+5$. Hence $A=-1$. From (3.14) we get $FG=1$, that is

$$[f^n(z+\eta)-f^n(z)]P[f]=1 \tag{3.17}$$

Now from (3.17) and Lemma 2.8, we have

$$\begin{aligned} (2n+\overline{d}(P))T(r,f) &\leq T\left(r, \frac{1}{[f^n(z+\eta)-f^n(z)]f^{\overline{d}(P)}}\right)+S(r,f) \\ &\leq T\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right)+S(r,f) \\ &\leq (\overline{d}(P)-\underline{d}(P))T(r,f)+Q[\overline{N}(r,\infty;f)+\overline{N}(r,0;f)]+S(r,f) \end{aligned}$$

Hence

$$(2n+\underline{d}(P))T(r,f) \leq S(r,f)$$

which is a contradiction.
Suppose that $B=0$. From (3.11), we have

$$G = AF-(A-1) \tag{3.18}$$

If $A \neq 1$, from (3.18) we obtain

$$\overline{N}\left(r, \frac{A-1}{A}; F\right) = \overline{N}(r,0;G) \tag{3.19}$$

From the second fundamental theorem, Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned} 2nT(r,f) &= T(r,F)+S(r,f) \\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,\frac{A-1}{A};F)+S(r,f) \\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,f) \\ &\leq (t\mu+4)T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f) \end{aligned} \tag{3.20}$$

which contradicts with $n \geq t\mu+5$. Thus $A=1$. From (3.18), we have $F=G$.
Hence $f^n(z+\eta)-f^n(z)=P[f]$.

Case II. Let

$$F = \Delta_c F = f^n(z+n)-f^n(z) \ and \ G = P[f].$$

Then it is easy to verify F and G share $(1,2)$ and (∞,∞). Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.1 that

$$\begin{aligned} T(r,F) \leq\ & N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G) \\ & +S(r,F)+S(r,G). \end{aligned} \tag{3.21}$$

According to Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} T(r,F) &= nT(r,f(z+\eta))+nT(r,f(z))+S(r,f) \\ &= 2nT(r,f)+O\left(r^{\rho(f)-1+\varepsilon}\right)+S(r,f) \end{aligned} \tag{3.22}$$

It is obvious that

$$N_2(r,0;F) = 4T(r,f)+O\left(r^{\rho(f)-1+\varepsilon}\right)+S(r,f), \tag{3.23}$$

$$\overline{N}(r,\infty;F) = 2T(r,f)+O\left(r^{\rho(f)-1+\varepsilon}\right)+S(r,f), \tag{3.24}$$

$$\overline{N}(r,\infty;G)=\overline{N}(r,\infty,f)\leq T(r,f), \tag{3.25}$$

$$\overline{N}_*(r,\infty;F,G)=0 \tag{3.26}$$

From (3.8) we have

$$N_2(r,0;G)\leq 2t\mu T(r,f)+S(r,f) \tag{3.27}$$

Combining (3.21)-(3.27), we deduce

$$(2n-2t\mu-7)T(r,f)\leq O\left(r^{\rho(f)-1+\varepsilon}\right)+S(r,f) \tag{3.28}$$

which contradicts with $n\geq t\mu+4$. Therefore $H\equiv 0$. Rest of the result follows from the proof of case I.

Case III. Let

$$F=\Delta_c F=f^n(z+n)-f^n(z) \text{ and } G=P[f].$$

Then it is easy to verify F and G share $(1,0)$ and $(\infty,0)$. Let H be defined as above. Suppose that $H\not\equiv 0$. It follows from Lemma 2.5 that

$$\begin{aligned} T(r,F)+T(r,G) &\leq 3\overline{N}(r,\infty;F)+N_2(r,0;F)+N_2(r,0;G)+N_E^{1)}(r,1;F)\\ &\quad +2N_E^{(2}(r,1;F)+3N_L(r,1;F)+3N_L(r,1;G)\\ &\quad +S(r,F)+S(r,G). \end{aligned} \tag{3.29}$$

Since

$$\begin{aligned} &N_E^{1)}(r,1;F)+2N_E^{(2}(r,1;F)+N_L(r,1;F)+2N_L(r,1;G)\\ &\qquad \leq N(r,1;G)\leq T(r,G)+O(1). \end{aligned} \tag{3.30}$$

we get from (3.29) and (3.30) that

$$\begin{aligned} T(r,F) &\leq 3\overline{N}(r,\infty;F)+N_2(r,0;F)+N_2(r,0;G)+2N_L(r,1;F)\\ &\quad +N_L(r,1;G)+S(r,F)+S(r,G). \end{aligned} \tag{3.31}$$

According to Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} T(r,F) &= nT(r,f(z+\eta))+nT(r,f(z))+S(r,f)\\ &= 2nT(r,f)+O\left(r^{\rho(f)-1+\varepsilon}\right)+S(r,f) \end{aligned} \tag{3.32}$$

It is obvious that

$$N_2(r,0;F)=4T(r,f)+O\left(r^{\rho(f)-1+\varepsilon}\right)+S(r,f), \tag{3.33}$$

$$\overline{N}(r,\infty;F)=2T(r,f)+O\left(r^{\rho(f)-1+\varepsilon}\right)+S(r,f), \tag{3.34}$$

$$\begin{aligned} N_L(r,1;F) &\leq N\left(r,\infty;\frac{F}{F'}\right)\leq N\left(r,\infty;\frac{F'}{F}\right)+S(r,f)\\ &\quad \leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+S(r,f)\\ &\quad \leq 4T(r,f)+O\left(r^{\rho(f)-1+\varepsilon}\right)+S(r,f). \end{aligned} \tag{3.35}$$

From (3.8) we have

$$N_2(r,0;G)\leq 2t\mu T(r,f)+S(r,f) \tag{3.36}$$

$$\begin{aligned} N_L(r,1;G) &\leq N\left(r,\infty;\frac{G}{G'}\right)\leq N\left(r,\infty;\frac{G'}{G}\right)+S(r,f)\\ &\quad \leq \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+S(r,f)\\ &\quad \leq (t\mu+1)T(r,f)+S(r,f). \end{aligned} \tag{3.37}$$

Combining (3.31)- (3.37), we deduce

$$(2n-3t\mu-19)T(r,f)\leq O\left(r^{\rho(f)-1+\varepsilon}\right)+S(r,f) \tag{3.38}$$

which contradicts with $n\geq 2t\mu+10$. Therefore $H\equiv 0$. Rest of the result follows from the proof of case I.
Hence this completes the proof of theorem 1.1. □

Corollary 3.1. *Let f be a nonconstant meromorphic function of zero order and n be a positive integer. If one of the following conditions is satisfied:*

$$(I)\ P[f]\ \text{and}\ \Delta_c F(qz)\ \text{share}\ (1,2),(\infty,0)\ \text{and}\ n \geq t\mu+5;$$

$$(II)\ P[f]\ \text{and}\ \Delta_c F(qz)\ \text{share}\ (1,2),(\infty,\infty)\ \text{and}\ n \geq t\mu+4;$$

$$(III)\ P[f]\ \text{and}\ \Delta_c F(qz)\ \text{share}\ (1,0),(\infty,0)\ \text{and}\ n \geq 2t\mu+10;$$

where $\Delta_c F(qz) = f^n(qz+n) - f^n(qz)$ *and* $\mu = \sum_{j=0}^{k}(1+j)$ *then* $P[f] \equiv \Delta_c F(qz)$.

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