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Controllability of the Main Road with an On-Ramp Section in Freeway Traffic Flow

Ali HAMİDOĞLU[1*]

[1]*Mathematics Department, Bahçeşehir University, 34353, Çırağan Caddesi, No: 4-6, Beşiktaş İstanbul/Türkiye*
(ORCID:0000-0003-3584-8848)

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Abstract
This study examines the controllability issue pertaining to the main road with a single on-ramp segment within the context of freeway traffic flow. In this regard, a finite-dimensional nonlinear model is formulated by integrating variable speed limit (VSL) and ramp metering (RM) techniques with the controllability property of the system derived under continuous VSL and RM controllers. This allows us to simultaneously control the density of the main line and reduce the queue density of the merging sector. In the numerical experiment, numerical simulations are conducted on a sample model of a main road to investigate the null controllability of the system and validate the theoretical findings. The numerical findings suggest that it is possible to achieve a consistent queue length for the on-ramp section through the implementation of suitable VSL and RM controls. Therefore, the primary accomplishment of this study is to effectively regulate the traffic flow on the main road segment by managing the density of vehicles within a specified timeframe while also considering the queue density of the on-ramp section.

1. Introduction

Increasing mobility has been a challenging and interesting task in the last decade to provide service and maintenance in urban regions. Hence, with the growing needs of mobility, intelligent transportation systems (ITS) become an important topic in transportation research. The tools, together with applications such as dynamic traffic signaling, RM, VSL, vehicle navigation systems, cooperative driving, and so on, are the major parts of ITS that increase the quality of mobility and the level of safety. On the freeway, it is crucial to maintain the traffic flow without building unnecessary road structures like extra road lanes or ramps. For this reason, various constructions and techniques are provided for building effective control strategies to increase the safety factor and, on the other hand, decrease traffic on the road [1]. As an example, various coordinated RM approaches are investigated in [2]. Model-predictive control approaches have also been emerging for optimal coordination of VSL and RM. The integrated free-way traffic flow management scheme is applied in [3] by means of VSL to design an optimal model of controlling freeway traffic flow with minimum travel time in the network. Several RM algorithms have been used to manage the inputs to freeways from entry ramps in an effort to reduce peak-hour congestion on the roads. In [4], a freeway system made up of an entry/exit ramp and a highway segment is examined as a ramp control problem to reduce the overall system time on the freeway.

Sometimes, VSLs and RM solutions frequently fall short of the anticipated outcomes when dealing with the merging zone of a freeway or managing heavy traffic demand due to poor traffic information transfer and the inability to actively control vehicles. An integrated technique for controlling traffic flow in the expressway merging region is suggested in order to increase traffic efficiency [5].

In a variety of traffic flow models, [6] uses ramp control frameworks via cell transmission. In recent

*Corresponding author: *ali.hamidoglu@eng.bau.edu.tr*

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years, RM and VSL controls have been used for different purposes, such as robust control [7], feedback control [8], optimal control [9], and so on. In the context of the controllability of traffic models, we refer to [10] for studying the controllability problem of traffic network models from the optimal control point of view. The exact boundary controllability problem is considered in [11] for the class of non-local conservation laws that control the traffic flow of the given system. Moreover, a novel methodology is developed in [12] for investigating the problem of controllability on a free-way traffic model by means of knowledge of routing and arcs. A controllability problem is studied in [13] for complex networks under both propagation/spill-back dynamics and drivers' behavior by means of route selection.

In this work, we investigate the controllability problem of a main road with one on-ramp sector in the freeway traffic flow. Roughly speaking, the concept of controllability means investigating whether it is possible to derive the solution of the given control system to a desired final state at the final time by means of a control [14]. More precisely, if any given final target is exactly achieved by a suitable control, then the system becomes exactly controllable. On the other hand, there are two more concepts of controllability: the system becomes null controllable whenever the solution hits zero at the final time [15], [16], and the system becomes approximate controllable whenever the solution nearly achieves the given final input at the terminal time [17]. These notions coincide in finite-dimensional space. This is because the only subspace that is dense in finite-dimensional space is the whole space itself. However, this is not the case in infinite-dimensional spaces. Here, we study the finite-dimensional controllability problem in freeway traffic flow with ramp dynamics. In this regard, we build a finite-dimensional control system via VSL control and RM control by applying the conservation law and Newtonian physics law on the main road as well as taking into account the ramp dynamics of the on-ramp sector. Once we establish the traffic model on the main road, we prove that the system is always controllable by means of continuous VSL and RM controllers.

Hence, the main contributions of the study could be briefly highlighted as follows:

- A road model is proposed, along with parameters and descriptions related to the mainstream and road dynamics.
- Controllability property is obtained for the main road with one on-ramp segment in the freeway traffic flow.
- A special case is regarded as a null controllability problem for controlling the mainstream density at critical levels and reducing the queue length of the on-ramp segment at any given final time by a continuous controller.
- A consistent queue length for the on-ramp segment and a controlled mainstream density are determined through numerical simulations conducted on a sample model of a main road within a specified time frame.

This study is presented as follows. Firstly, several notions and theorems are provided in section 2. In this part, the idea of controllability for finite-dimensional systems is explained, and the Kalman rank condition is given as a necessary and sufficient condition for the controllability property of linear finite-dimensional systems. Later, we state the problem in section 3, where the road model is proposed together with a nonlinear control system designed by means of VSL and RM controls. Here, the property of controllability is derived for the traffic model, and the existence of a continuous controller is proven for the system. In section 4, numerical experiments are conducted on a sample main road model, and the null controllability property of the system is numerically derived by means of VSL and RM controls. Finally, the paper concludes with further remarks and future work in section 5.

2. Material and Method

In this part, we provide several notions and theorems, some of which are novel in the context of matrix theory. Here, we mainly concentrate on the following control model:

$$x'(t) = Ax(t) + Bu(t) + w(t), \qquad (1)$$

with the standard assumption that $w(t)$ is continuous in $[0,T]$ where $T > 0$ stands for a final time and the linear part is provided by means of the real matrices A, B of reasonable dimensions $n \times n, n \times m$ with $n \geq m$.

On the other hand, the control model (1) is the perturbed version of the following finite-dimensional control system:

$$x'(t) = Ax(t) + Bu(t). \qquad (2)$$

Now, we define the notion of controllability for the model (1) in two directions:

Definition 1. (*Exact Controllability***)**

The system (1) is exactly controllable in finite time T if for each $t_0 \in [0,T]$ and arbitrary initial and final state $x_0, x_1 \in \mathbb{R}^n$, one can find a control input $u(\cdot): [t_0,T] \to \mathbb{R}^m$ in such a way that the solution $x(t)$ hits the target at the given time T, i.e., $x(T) = x_1$.

Definition 2. (*Null Controllability***)**

The system (1) is null controllable in finite time T if for each $t_0 \in [0, T]$ and arbitrary initial $x_0 \in \mathbb{R}^n$, one can find a control input $u(\cdot): [t_0, T] \to \mathbb{R}^m$ in such a way that the solution $x(t)$ satisfies $x(T) = 0$.

Theorem 1. (*Duality of Controllability in* $\mathbb{R}^n$**)**
In $\mathbb{R}^n$, the concepts of null controllability and exact controllability are equivalent.

The last theorem comes from the fact that the only affine subspace of $\mathbb{R}^n$ is the whole space itself. Next, we define Kalman rank criteria [18] for the linear control system (2):

Theorem 2. (*Kalman Rank Condition [18]***)**
The continuous time system (2) becomes controllable whenever the following rank condition satisfies

$$rank[B;\ A\,B;\ A^2B; \cdots;\ A^{n-1}B] = n. \qquad (3)$$

Now, we provide the following result [19]:

Theorem 3. (*Controllability of Perturbed System [19]*)
The control system (1) is controllable if Kalman rank condition is satisfied and the function $w(t)$ is continuous and bounded in $(0, T)$. More precisely, if $w(t)$ is bounded and continuous in a given domain, then, the perturbed system (1) is controllable on that domain. In addition, we have the following result:

Corollary 1. (*Existence of Continuous Controller [19]*)
There exists a continuous controller that makes the perturbed control system (1) controllable.

In the following sections, we investigate the problem of controllability on the main road with one on-ramp section to control the traffic flow in the mainstream. Since the concept of exact controllability coincides with the null controllability notion, we build the first-order continuous time system on the traffic density of the main road and on-ramp section in order to control the traffic flow both ways.

3. Controllability of Main Road with On-ramp Segment

In this section, we study the freeway composed of the main road and an on-ramp segment. Firstly, we start by proposing the road model and all related parameters that would be used to build the first-order nonlinear continuous time system. Then, we derive the controllability result from that system by applying the Kalman rank condition. Lastly, we investigate an application of the main result.

3.1. Road Model

In this part, we study a road model with a main road and an on-ramp shown in Figure 1. In this road model, we assume that the flow from the on-ramp sector is continuous, which means that there is no traffic congestion on the on-ramp section. More precisely, the function is assumed to be continuous in time.

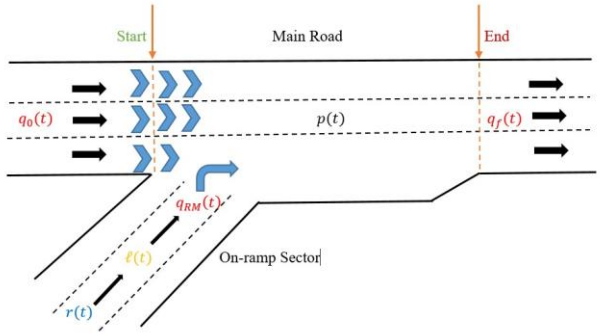

Figure 1. The free-way consisting of the main road with an on-ramp sector.

Define the following parameters for this road model:

- $\rho(t)$ as the density of the main road at the time t.
- $q_0(t)$ and $q_f\,(t)$ as the number of vehicles coming from the initial road segment (Start) into the main road and the main road into the final road segment (End) at a time t respectively.
- $\ell(t)$ as the queue length of the on-ramp segment at the time t.
- $q_{RM}(t)$ as the number of vehicles entering from the on-ramp into the main road at the time t respectively.
- $r(t)$ as the number of vehicles entering the on-ramp at a given time t.
- L and λ as the length and the number of lines of the main road, respectively.

From the conservation law, we have

$$\rho'(t) = \frac{q_0(t) - q_f(t) + q_{RM}(t)}{\lambda\, L}\ . \qquad (5)$$

The linear car-following model is widely used in traffic flow models, which simply depend on Newtonian physics laws. In this regard, the speed and the density are related and represented as the following piecewise function [20]:

$$v(\rho) = \begin{cases} v_f, & \rho < \rho_c \\ C\left(\dfrac{1}{\rho} - \dfrac{1}{\rho_m}\right), & \rho \geq \rho_c \end{cases}$$

where ρ_c and ρ_m represent critical density and maximum density of the traffic flow on the road, respectively, and C is a sensitive constant [20]. Hence, by using $q(\rho) = v\,\rho$, one can get the following flow-density relationship:

$$q(\rho) = \begin{cases} v_f \rho\,, & \rho < \rho_c \\ C(1 - \dfrac{\rho}{\rho_m}), & \rho \geq \rho_c \end{cases}$$

Define, $e(t) = \rho(t) - \rho_c$, then the flow dynamics for the congested state becomes

$$q_f = C\,(1 - \frac{e(t) + \rho_c}{\rho_m}) \quad (6)$$

By using (5) and (6), we finally have the following first-order nonlinear differential system for the proposed road model:

$$e'(t) = \frac{q_0(t) + q_{RM}(t)}{\lambda L} - \frac{C}{\lambda L}\,(1 - \frac{e(t) + \rho_c}{\rho_m}) \quad (7)$$

with the ramp dynamics

$$\ell'(t) = r(t) - q_{RM}(t). \quad (8)$$

To control the solutions of (7) and (8) together, we utilize VSL control as well as RM control in the following sense:

$$u_{VSL}(t) = q_0(t) - C(1 - \frac{\rho_c}{\rho_m}),$$
$$u_{RM}(t) = q_{RM}(t).$$

Now, we concentrate on the following finite-dimensional control system:

$$x'(t) = \boldsymbol{A}x(t) + \boldsymbol{B}u(t) + \boldsymbol{w}(t), \quad (9)$$

where

$$x(t) = \begin{pmatrix} e(t) \\ \ell(t) \end{pmatrix}, u(t) = \begin{pmatrix} u_{VSL}(t) \\ u_{RM}(t) \end{pmatrix}, \boldsymbol{w}(t) = \begin{pmatrix} 0 \\ r(t) \end{pmatrix}$$
$$\boldsymbol{A} = \begin{pmatrix} \dfrac{C}{\lambda L \rho_m} & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \dfrac{1}{\lambda L} & \dfrac{1}{\lambda L} \\ 0 & -1 \end{pmatrix}$$

In the following section, we investigate the problem of controllability on a nonlinear first-order system (9) and obtain the main result of this work.

3.2. Controllability

In this part, we study the controllability problem for the control system (9) as follows:
Given any initial state $x_0 \in \mathbb{R}^2$ and final state $x_f \in \mathbb{R}^2$, is it possible to find a control input $u(t) \in C\big(t_0, t_f\big)^2$ that is the space of continuous functions defined in $\big(t_0, t_f\big)$ such that the solution of the system (9), $x(t)$ satisfies $x\big(t_f\big) = x_f$. For this controllability problem, we prove the following main result of the paper:

Theorem 4. (*Main Result*)
The system (9) is exactly controllable with continuous controllers $u_{VSL}(t)$ and $u_{RM}(t)$.

Proof: Since traffic flow on the on-ramp sector is continuous on time, $\boldsymbol{w}(t)$ becomes continuous and bounded on time t, which provides that the controllability problem of the system (9) becomes the controllability problem of a perturbed version of the following linear control system:

$$x'(t) = \boldsymbol{A}x(t) + \boldsymbol{B}u(t), \quad (10)$$

according to Theorem 3. Hence, it suffices to check the Kalman rank condition for the system (10) according to Teorem 2. Since,

$$A \times B = \begin{pmatrix} \frac{C}{\lambda^2 L^2 \rho_m} & \frac{C}{\lambda^2 L^2 \rho_m} \\ 0 & 0 \end{pmatrix},$$

which makes

$$det[B\ AB] = det\begin{vmatrix} \dfrac{1}{\lambda L} & \dfrac{1}{\lambda L} & \dfrac{C}{\lambda^2 L^2 \rho_m} & \dfrac{C}{\lambda^2 L^2 \rho_m} \\ 0 & -1 & 0 & 0 \end{vmatrix} =$$
$$= det\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 2.$$

Hence, the Kalman rank condition is satisfied for the system (10) which implies that the system (9) is exactly controllable. This concludes the proof. □

3.3. An Application

As an application of Theorem 4, under any given initial data, this result provides $e\big(t_f\big) = \rho\big(t_f\big) - \rho_c = 0$ and $\ell\big(t_f\big) = 0$ under the suitable choice of continuous VSL control and RM control, which is simply the property of null controllability of the system as a result of Teorem 1. As a result, the null controllability issue, which is the special class of exact controllability, provides a way to manage the mainstream density at a critical level and to reduce the queue length of the on-ramp segment by VSL and RM continuous controllers at any given final time.

4. Numerical Results

This section presents numerical simulations conducted on a sample road model in order to provide numerical validation for the primary finding of the study. In this regard, we consider the problem of null controllability on a main road with a length $L = 100$, consisting of $\lambda = 4$ number of lines. Moreover, we assume a sensitivity constant C = 800 and a maximum road density $\rho_m = 2$. Finally, we consider the number of vehicles entering the on-ramp at time t as a nonlinear function, $r(t) = t^2$ **.** Hence, we have the following nonlinear system:

$$\begin{pmatrix} \dot{e}(t) \\ \dot{\ell}(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e(t) \\ \ell(t) \end{pmatrix} + $$
$$+ \begin{pmatrix} \frac{1}{400} & \frac{1}{400} \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u_{VSL}(t) \\ u_{RM}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ t^2 \end{pmatrix}, \quad (11)$$

In the context of this particular system, we address the issue of null controllability pertaining to the road model within a time span of one hour as follows: given $\begin{pmatrix} e(0) \\ \ell(0) \end{pmatrix} = \begin{pmatrix} e^{-1} \\ 0 \end{pmatrix}$, find VSL and RM controls

$\begin{pmatrix} u_{VSL}(t) \\ u_{RM}(t) \end{pmatrix}$ such that the solution of the above system satisfies $\begin{pmatrix} e(1) \\ \ell(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. In this particular problem, the constant e represents Euler's number, while the variable time t is defined within the interval of 0 and 1. Furthermore, we make the initial assumption that there is an absence of traffic congestion by letting $\ell(0) = 0$.

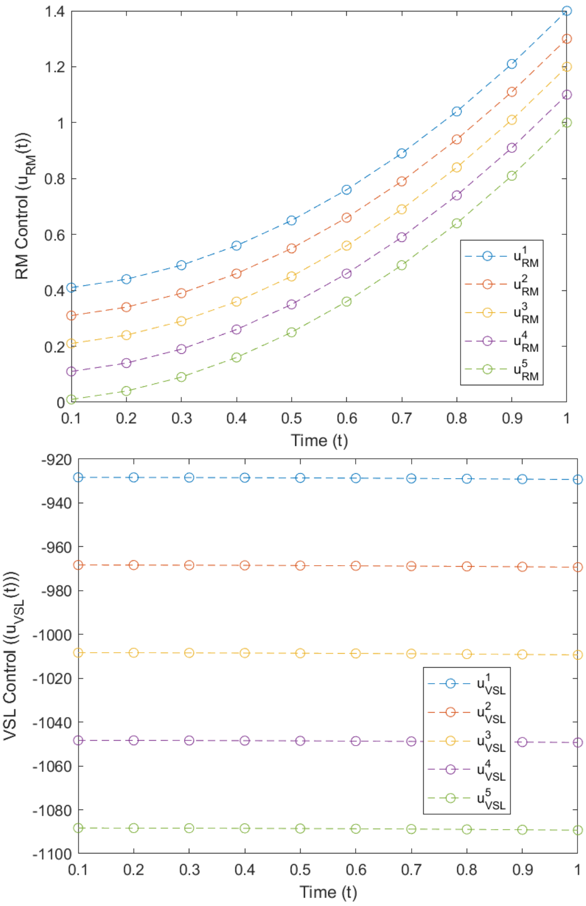

Figure 2. Five distinct VSL and RM control models numerically implemented to achieve the null controllability of the system (11) within a time span of one hour.

According to Theorem 4, it can be deduced that the system (11) fulfills the Kalman rank condition, thereby establishing the theoretical basis for the null controllability of the system (11). On the other hand, in this study, we evaluate and conduct simulations on five distinct VSL and RM control models in order to ascertain their effectiveness in achieving the null controllability outcome, depicted in Figure 2 and Figure 3. These analyses are performed using the MATLAB R2021b version toolbox.

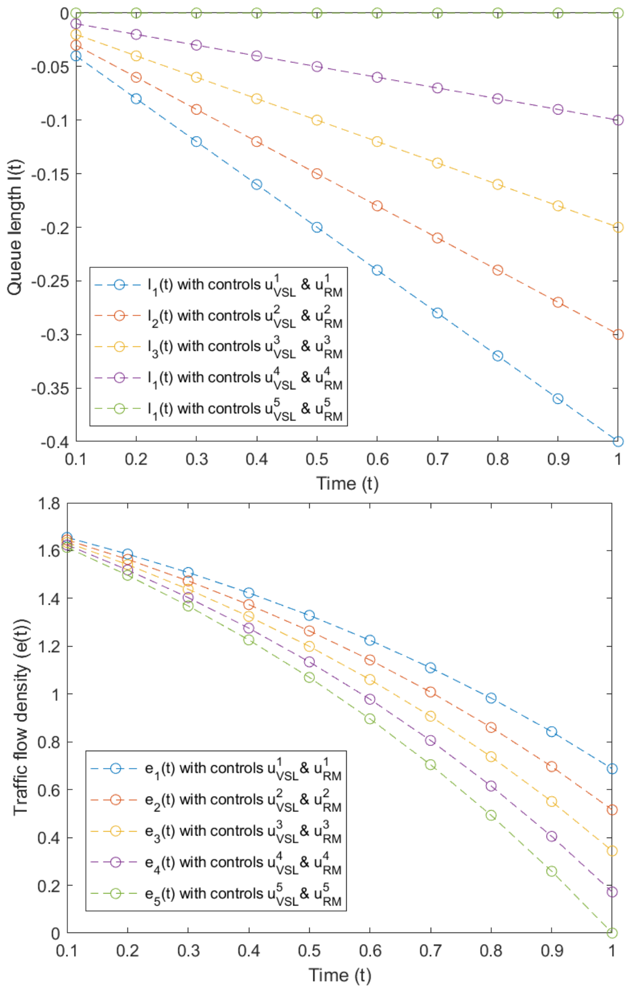

Figure 3. The numerical solutions of the system (11) under five different VSL and RM controls over a duration of one hour.

Figure 2 depicts the computed numerical values of VSL and RM controls over the course of one hour. These parameters are numerically implemented in the system (11) and their corresponding solutions are shown in Figure 3 as traffic flow density $e(t)$ and queue length $\ell(t)$ over the course of one hour.

In these numerical simulations illustrated in Figure 2 and Figure 3, we derive the fact that the green curves establish the null controllability of the system. In other words, numerical results show that the solution of the system (11) under the u_{VSL}^5 and u_{RM}^5 controls provides $e(1) = 0$ and $\ell(1) = 0$ in the final round. This outcome signifies the null controllability of the system (11).

5. Conclusion

This study focuses on the issue of controllability within the traffic model, which comprises a single main road and an on-ramp sector. The controllability is achieved through the implementation of continuous VSL control and RM control strategies. This work demonstrates that by implementing appropriate VSL and RM controls, it is possible to effectively manage the main road density at its critical level while simultaneously reducing the queue length in the on-ramp segment. Theoretical findings are enhanced by conducting numerical simulations on a sample main road model. In this study, the concept of null controllability is derived through the implementation of VSL and RM controls. The numerical results indicate that it is feasible to maintain a constant queue length for the on-ramp segment by implementing appropriate VSL and RM controls. This can be achieved while also ensuring that the density of vehicles on the main road remains at the critical level within a specified time frame. As a future work, It would be an interesting and motivating attempt to focus on the controllability problem of a more general integrated traffic system where the road model consists of N number of main roads with N number of on-ramp segments for $N > 1$.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

References

[1] T.V. Baby, V. Bhattacharyya, P.K. Shahri, A.H. Ghasemi, and B. HomChaudhuri, "A suggestion based fuel efficient control framework for connected and automated vehciles in heterogeneous urban traffic", *Transportation Research Part C: Emerging Technologies,* 134, 103476, 2022.

[2] S.R. Klomp, V.L. Knoop, H. Taale, and S.P. Hoogendoorn, "Ramp metering with microscopic gap detection algorithm design and empirical acceleration verification", *Transportation Research Record,* vol. 2676, no. 1, pp. 91-104, 2021.

[3] S. Li, and D. Cao, "Variable speed limit strategies' analysis with cell transmission model on freeway" *Modern Physics Letters B,* vol. 31, no. 24, 1750219, 2017.

[4] H. Zhang, S. G. Ritchie and W. W. Recker, "Some general results on the optimal ramp control problem", *Transportation Research Part C: Emerging Technologies*, vol. 4, no. 2, pp. 51-69, 1996.

[5] Y. Di, W. Zhang, H. Ding, X. Zheng, and H. Bai, "Integrated Control for Mixed CAV and CV Traffic Flow in Expressway Merge Zones Combined with Variable Speed Limit, Ramp Metering, and Lane Changing", *Journal of Transportation Engineering, Part A: Systems*, vol. 149, no. 2, 04022140, 2023.

[6] G. Gomes, and R. Horowitz, "Optimal freeway ramp metering using the asymmetric cell transmission model", *Transportation Research Part C: Emerging Technologies*, vol. 14, no. 4, pp. 244–262, 2006.

[7] C. Gu, C. Wu, Y. Wu, and B. Wiwatanapataphee, "Distributionally robust ramp metering under traffic demand uncertainty", *Transportmetrica B: Transport Dynamics,* vol. 10, no. 1, pp.652-666, 2022.

[8] C. Gu, T. Zhou, and C. Wu, "Deep Koopman Traffic Modeling for Freeway Ramp Metering", *IEEE Transactions on Intelligent Transportation Systems,* pp. 1-13, 2023.

[9] N. Yuan, M. Ma, S. Liang, W. Wang, and H. Zhang, "Optimal control method of freeway based on tollbooths lane configuration and variable speed limit control", *Physica A: Statistical Mechanics and its Applications*, vol. 603, 127801, 2022.

[10] J. Wang, Y. Zheng, Q. Xu, J. Wang, and K. Li, "Controllability analysis and optimal control of mixed traffic flow with human-driven and autonomous vehicles," *IEEE Transactions on Intelligent Transportation Systems*, vol. 22, no. 12, pp. 7445-7459, 2020.

[11] A. Bayen, J. M. Coron, N. De Nitti, A. Keimer, and L. Pflug, "Boundary controllability and asymptotic stabilization of a nonlocal traffic flow model", *Vietnam Journal of Mathematics*, vol. 49, no. 3, pp. 957-985, 2021.

[12] S. Agarwal and P. Kachroo, "Controllability and observability analysis for intelligent transportation systems", *Transportation in Developing Economies*, vol. 5, no. 1, 2, 2019

[13] M. Rinaldi, "Controllability of transportation networks", *Transportation Research Part B: Methodological*, vol. 118, pp. 381-406, 2018.

[14] A. Hamidoğlu and E. N. Mahmudov, "On construction of sampling patterns for preserving observability/controllability of linear sampled-data systems", *International Journal of Control*, vol. 95, no. 1, pp. 206-217, 2022.

[15] A. Hamidoğlu, "Null controllability of heat equation with switching controls under Robin's boundary condition", *Hacettepe Journal of Mathematics and Statistics*, vol. 45, no. 2, pp. 373-379, 2016.

[16] A. Hamidoglu, "Null controllability of heat equation with switching pointwise controls", *Transactions of NAS of Azerbaijan, Issue Mathematics, Series of Physical-Technical and Mathematical Sciences*, vol. 35, no. 4, pp. 52-60, 2015.

[17] A. Hamidoglu, "Switching lumped controls", *Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan*, vol. 41, no. 2, pp. 83-90, 2015.

[18] E. D. Sontag, "Kalman's controllability rank condition: from linear to nonlinear", *Mathematical system theory: The influence of RE Kalman*, pp. 453-462, 1991.

[19] D. Lukes, "Global controllability of nonlinear systems", *SIAM Journal on Control*, vol. 10, no. 1, pp. 112–126, 1972.

[20] P. I. Richards, "Shock waves on the highway", *Operations research*, vol. 4, no. 1, pp. 42-51, 1956.