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Lightlike Hypersurfaces of an Indefinite Kaehler Manifold with an $(\ell,\, m)$-type Metric Connection

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ABSTRACT

Jin introduced a non-symmetric metric connection, called an (ℓ, m)-*type metric connection* [5,6]. There are two examples of (ℓ, m)-type: a semi-symmetric metric connection when $\ell = 1$ and $m = 0$ and a quater-symmetric connection for $\ell = 0$ and $m = 1$. Our purpose is to investigate lightlike hypersurfaces of an indefinite (complex) Kaehler manifolds with an (ℓ, m)-type metric connection under the tangent characteristic vector field on such hypersurfaces.

Keywords: *(ℓ, m)-type metric connection, lightlike hypersurface, indefinite Kaehler manifold.*

AMS Subject Classification (2020): *Primary: 53C25 ; Secondary: 53C40; 53C50.*

1. Introduction

The notion of a symmetric connection of (ℓ, m)-type on semi-Riemannian manifolds was introduced as follows ([5, 6]):

A *symmetric connection* $\bar\nabla$ *of* $(\ell,\, m)$*-type* on a semi-Riemannian manifold $(\bar M, \bar g)$ is satisfied with the follwoing torsion tensor $\bar T$:

$$\bar T(\bar X, \bar Y) = \ell\{\theta(\bar Y)\bar X - \theta(\bar X)\bar Y\} + m\{\theta(\bar Y)J\bar X - \theta(\bar X)J\bar Y\} \tag{1.1}$$

for smooth functions ℓ and m, a tensor field J of type $(1,1)$ and a 1-form θ associated with a characteristic vector field ζ, which has $\theta(\bar X) = \bar g(\bar X, \zeta)$. Moreover, $\bar\nabla$ is called a *symmetric metric connection of type* $(\ell,\, m)$ (simply, an $(\ell,\, m)$*-type metric connection*) if $\bar g$ is parallel on this connection $\bar\nabla$ (i.e., $\bar\nabla\bar g = 0$).

In case $(\ell, m) = (1,\, 0)$, the $(\ell,\, m)$-type metric connection $\bar\nabla$ becomes a semi-symmetric metric connection, introduced by Hayden [4] and Yano [9]. In case $(\ell, m) = (0,\, 1)$, the connection $\bar\nabla$ becomes a quarter-symmetric metric connection, introduced by Yano-Imai [10]. In the sequel, we shall assume the following:

(1) $\bar X$, $\bar Y$ and $\bar Z$ are the vector fields on $\bar M$.
(2) $(\ell, m) \neq (0, 0)$.
(3) M is a lightlike hypersurface of $\bar M$.
(4) The characteristic vector field ζ is unit spacelike and tangent to M.
(5) $\mathcal{F}(M)$ is the collection of smooth functions on M.
(6) $\Gamma(E)$ is the $\mathcal{F}(M)$ module of smooth sections of any vector bundle E over M.
(7) $(2.1)_i$ is the i-th equation of (2.1).

In this paper, we study the geometry of a lightlike hypersurface of an indefinite Kaehler manifold $\bar M$ with an $(\ell,\, m)$-type metric connection subject such that an indefinite almost complex structure J satisfies (1.1).

Jin studied lightlike hypersurfaces of an indefinite Kaehler manifold with an $(\ell,\, m)$-type metric connection in [6]. However, $(2.1)_3$ and (4.1) in [6] are not correct. First, in [6], this author used the $(\ell,\, m)$-type metric

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connection $\bar{\nabla}$ in $(2.1)_3$, that is, $\bar{\nabla}J = 0$. This is a mistake because the connection in $(2.1)_3$, defined on a Kaehler manifold $\bar{M}$, must be the Levi-Civita connection $\widetilde{\nabla}$ on $\bar{M}$, that is, $\widetilde{\nabla}J = 0$. Next, in [6], Jin used the curvature tensor $\bar{R}$ of the (ℓ, m)-type metric connection $\bar{\nabla}$ as the curvature tensor in (4.1) . This is also a mistake because the curvature tensor in (4.1), defined on an indefinite complex space form $\bar{M}(c)$, must be the curvature tensor $\widetilde{R}$ of the Levi-Civita connection $\widetilde{\nabla}$. In this paper, we rewrite the paper [6] replacing $\bar{\nabla}$ by $\widetilde{\nabla}$ in $(2.1)_3$ and $\bar{R}$ by $\widetilde{R}$ in (4.1) (see new correct equations $(2.1)_3$ and (4.1) in this paper).

Theorem 1.1. *A linear connection $\bar{\nabla}$ on $\bar{M}$ is an (ℓ, m)-type metric connection if and only if it satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \ell\{\theta(\bar{Y})\bar{X} - \bar{g}(\bar{X}, \bar{Y})\zeta\} - m\theta(\bar{X})J\bar{Y}. \tag{1.2}$$

Proof. Let $\bar{\nabla}$ be the linear connection defined by (1.2). By directed calculations from (1.2), we see that $\bar{\nabla}$ satisfies (1.1) and $\bar{\nabla}\bar{g} = 0$. Thus $\bar{\nabla}$ is an (ℓ, m)-type metric connection.
Conversely, if $\bar{\nabla}$ is an (ℓ, m)-type metric connection, then we can write

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \psi(\bar{X}, \bar{Y}). \tag{1.3}$$

Substituting (1.3) into the equation $(\bar{\nabla}_{\bar{X}}\bar{g})(\bar{Y}, \bar{Z}) = 0$ and using the fact that $\widetilde{\nabla}$ is a metric connection, we have

$$\bar{g}(\psi(\bar{X}, \bar{Y}), \bar{Z}) + \bar{g}(\psi(\bar{X}, \bar{Z}), \bar{Y}) = 0. \tag{1.4}$$

Also, from (1.1), (1.3) and the fact that $\widetilde{\nabla}$ is torsion-free, it follows that

$$\psi(\bar{X}, \bar{Y}) - \psi(\bar{Y}, \bar{X}) = \ell\{\theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}\} + m\{\theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}\}. \tag{1.5}$$

From (1.5), we get

$$\begin{aligned}&\bar{g}(\psi(\bar{X}, \bar{Y}), \bar{Z}) - \bar{g}(\psi(\bar{Y}, \bar{X}), \bar{Z})\\ &= \ell\{\theta(\bar{Y})\bar{g}(\bar{X}, \bar{Z}) - \theta(\bar{X})\bar{g}(\bar{Y}, \bar{Z})\} + m\{\theta(\bar{Y})\bar{g}(J\bar{X}, \bar{Z}) - \theta(\bar{X})\bar{g}(J\bar{Y}, \bar{Z})\},\end{aligned}$$

$$\begin{aligned}&\bar{g}(\psi(\bar{X}, \bar{Z}), \bar{Y}) - \bar{g}(\psi(\bar{Z}, \bar{X}), \bar{Y})\\ &= \ell\{\theta(\bar{Z})\bar{g}(\bar{X}, \bar{Y}) - \theta(\bar{X})\bar{g}(\bar{Z}, \bar{Y})\} + m\{\theta(\bar{Z})\bar{g}(J\bar{X}, \bar{Y}) - \theta(\bar{X})\bar{g}(J\bar{Z}, \bar{Y})\}.\end{aligned}$$

Adding these two equations together with (1.4), we have

$$\begin{aligned}&-\bar{g}(\psi(\bar{Y}, \bar{X}), \bar{Z}) - \bar{g}(\psi(\bar{Z}, \bar{X}), \bar{Y})\\ &= \ell\{\theta(\bar{Y})\bar{g}(\bar{X}, \bar{Z}) + \theta(\bar{Z})\bar{g}(\bar{X}, \bar{Y}) - 2\theta(\bar{X})\bar{g}(\bar{Y}, \bar{Z})\}\\ &+ m\{\theta(\bar{Y})\bar{g}(J\bar{X}, \bar{Z}) + \theta(\bar{Z})\bar{g}(J\bar{X}, \bar{Y})\}.\end{aligned}$$

Using (1.4) to the left term of the last equation, we have

$$\begin{aligned}&\bar{g}(\psi(\bar{Y}, \bar{Z}), \bar{X}) - \bar{g}(\psi(\bar{Z}, \bar{Y}), \bar{X})\\ &= \ell\{\theta(\bar{Y})\bar{g}(\bar{X}, \bar{Z}) + \theta(\bar{Z})\bar{g}(\bar{X}, \bar{Y}) - 2\theta(\bar{X})\bar{g}(\bar{Y}, \bar{Z})\}\\ &+ m\{\theta(\bar{Y})\bar{g}(J\bar{X}, \bar{Z}) + \theta(\bar{Z})\bar{g}(J\bar{X}, \bar{Y})\}.\end{aligned}$$

Substituting (1.4) to the last equation, we obtain

$$\bar{g}(\psi(\bar{Y}, \bar{Z}), \bar{X}) = \ell\{\theta(\bar{Z})\bar{g}(\bar{Y}, \bar{X}) - \bar{g}(\bar{Y}, \bar{Z})\bar{g}(\zeta, \bar{X})\} - m\theta(\bar{Y})\bar{g}(J\bar{Z}, \bar{X}).$$

As $\bar{g}$ is non-degenerate, we obtain

$$\psi(\bar{X}, \bar{Y}) = \ell\{\theta(\bar{Y})\bar{X} - \bar{g}(\bar{X}, \bar{Y})\zeta\} - m\theta(\bar{X})J\bar{Y}.$$

Thus $\bar{\nabla}$ satisfies (1.2). This result implies that a linear connection $\bar{\nabla}$ on $\bar{M}$ is an (ℓ, m)-type metric connection if and only if $\bar{\nabla}$ satisfies (1.2). □

2. $(\ell,\, m)$-type metric connections

Let $\bar{M} = (\bar{M},\, \bar{g},\, J)$ be an indefinite Kaehler manifold equipped with a unique Levi-Civita connection $\widetilde{\nabla}$, a semi-Riemannian metric g and an indefinite almost complex structure J such that

$$J^2 = -I, \qquad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}), \qquad (\widetilde{\nabla}_{\bar{X}} J)\bar{Y} = 0. \tag{2.1}$$

Denote by $\bar{\nabla}$ an $(\ell,\, m)$-type metric connection on $\bar{M}$. Using (1.2), we have

$$(\bar{\nabla}_{\bar{X}} J)(\bar{Y}) = \ell\{\theta(J\bar{Y})\bar{X} - \theta(\bar{Y})J\bar{X} - \bar{g}(\bar{X}, J\bar{Y})\zeta + g(\bar{X}, \bar{Y})J\zeta\}. \tag{2.2}$$

For the normal subbundle $TM^{\perp}$([2]) of the tangent bundle TM of rank 1 , a screen distribution $S(TM)$ of $TM^{\perp}$ in TM is non-degenerate on M with the orthgonal direct sum $\oplus_{orth}$:

$$TM = TM^{\perp} \oplus_{orth} S(TM).$$

For a null section η in $\Gamma(TM^{\perp})$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N, called the *null transversal vector field* of M, of a unique vector bundle $tr(TM)$, called the *transversal vector bundle*, in $S(TM)^{\perp}$ such that

$$\bar{g}(\eta, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

Moreover, the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follows:

$$T\bar{M} = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus_{orth} S(TM).$$

Let $P : \Gamma(TM) \longrightarrow \Gamma(S(TM))$ be the natural projection. Then we have thel Gauss and Weingartan formula of M and $S(TM)$ as follows:

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + \mathcal{B}(X, Y)N, & (2.3)\\ \bar{\nabla}_X N &= -A_{_N} X + \tau(X)N, & (2.4)\\ \nabla_X PY &= \nabla^*_X PY + \mathcal{C}(X, PY)\eta, & (2.5)\\ \nabla_X \eta &= -A^*_\eta X - \tau(X)\eta & (2.6) \end{aligned}$$

for $X, Y, Z \in \Gamma(TM)$, the induced connections ∇ on TM, and ∇^* on $S(TM)$, the local second fundamental forms $\mathcal{B}$ and $\mathcal{C}$, the shape operators $A_{_N}$ and A^*_η on TM and $S(TM)$, respectively and a 1-form τ.

Due to [2, Section 6.2], we have $J(TM^{\perp}) \oplus J(tr(TM))$ is a subbundle of $S(TM)$, of rank 2 for subbundles $J(TM^{\perp})$ and $J(tr(TM))$ of $S(TM)$ with rank 1 and $J(TM^{\perp}) \cap J(tr(TM)) = \{0\}$. Therefore, there exist two non-degenerate invariant distributions D_o and D on M in terms of J (that is, $J(D_o) = D_o$ and $J(D) = D$) satisfying

$$S(TM) = J(TM^{\perp}) \oplus J(tr(TM)) \oplus_{orth} D_o,$$
$$D = \{TM^{\perp} \oplus_{orth} J(TM^{\perp})\} \oplus_{orth} D_o.$$

In this case, TM has the decomposition:

$$TM = D \oplus J(tr(TM)). \tag{2.7}$$

Now we consider the null vector fields U and V, corresponding to N and η in terms of J, respectively, and dual 1-forms u and v of U and V, respectively, satisfying

$$U = -JN, \quad V = -J\eta, \quad u(X) = g(X, V), \quad v(X) = g(X, U). \tag{2.8}$$

For the projection $S : TM \longrightarrow D$, arbitrary vector field X in $\Gamma(TM)$ can be written as $X = SX + u(X)U$, and also we have

$$JX = FX + u(X)N, \tag{2.9}$$

where $F = J \circ S$ is a tensor field of type $(1, 1)$ globally defined on M. From (2.9), (2.1) and (2.8), we obtain

$$F^2X = -X + u(X)U. \tag{2.10}$$

Therefore, $(F,\, u,\, U)$ is an indefinite almost contact structure on M as $u(U) = 1$ and $FU = 0$. Here, F is called the *structure tensor field* of M and U the *structure vector field* of M.
The connection ∇ is an (ℓ, m)-type non-metric connection, and satisfies

$$(\nabla_X g)(Y, Z) = \mathcal{B}(X, Y)\mu(Z) + \mathcal{B}(X, Z)\mu(Y), \tag{2.11}$$
$$T(X, Y) = \ell\{\theta(Y)X - \theta(X)Y\} + m\{\theta(Y)FX - \theta(X)FY\}, \tag{2.12}$$
$$\mathcal{B}(X, Y) - \mathcal{B}(Y, X) = m\{\theta(Y)u(X) - \theta(X)u(Y)\}, \tag{2.13}$$

where T is the induced torsion tensor with respect to ∇ on M and μ is a 1-form on TM such that $\mu(X) = \bar{g}(X, N)$. From the fact that $\mathcal{B}(X, Y) = \bar{g}(\bar{\nabla}_X Y, \eta)$, we obtain

$$\mathcal{B}(X, \eta) = 0, \qquad\qquad \mathcal{B}(\eta, X) = 0, \tag{2.14}$$

and also we have

$$g(A^*_\eta X, Y) = B(X, Y), \qquad \bar{g}(A^*_\eta X, N) = 0, \tag{2.15}$$
$$g(A_{_N} X, PY) = \mathcal{C}(X, PY), \quad \bar{g}(A_{_N} X, N) = 0. \tag{2.16}$$

From (2.14)$_2$, (2.15) and the non-degeneracy of $S(TM)$, we have

$$A^*_\eta \eta = 0. \tag{2.17}$$

We set $b = \theta(N)$. Applying $\bar{\nabla}_X$ to (2.8) and (2.9) by turns, we have

$$\mathcal{B}(X, U) = \mathcal{C}(X, V) + \ell\{bu(X) - \theta(V)\mu(X)\}, \tag{2.18}$$
$$\begin{aligned}\nabla_X U = {} & F(A_{_N} X) + \tau(X)U \\ & + \ell\{\theta(U)X + bFX - v(X)\zeta - \mu(X)F\zeta\},\end{aligned} \tag{2.19}$$
$$\nabla_X V = F(A^*_\eta X) - \tau(X)V + \ell\{\theta(V)X - u(X)\zeta\}, \tag{2.20}$$
$$\begin{aligned}(\nabla_X F)(Y) = {} & u(Y)A_{_N} X - \mathcal{B}(X, Y)U \\ & + \ell\{\theta(JY)X - \theta(Y)FX - \bar{g}(X, JY)\zeta + g(X, Y)F\zeta\},\end{aligned} \tag{2.21}$$
$$\begin{aligned}(\nabla_X u)(Y) = {} & -u(Y)\tau(X) - \mathcal{B}(X, FY) \\ & + \ell\{\theta(V)g(X, Y) - \theta(Y)u(X)\},\end{aligned} \tag{2.22}$$
$$\begin{aligned}(\nabla_X v)(Y) = {} & v(Y)\tau(X) - g(A_{_N} X, FY) - \ell\theta(U)g(X, Y)\} \\ & - \ell\{bg(FX, Y) - \theta(Y)v(X) - \mu(X)g(F\zeta, Y)\}.\end{aligned} \tag{2.23}$$

Example 2.1. Let $\bar{M}$ be a semi-Euclidean manifold $\mathbf{R}^4_2$, covered by coordinate neighborhoods with coordinates $\{x_1,\, y_1,\, x_2,\, y_2\}$. There exist a non-degenerate metric $\bar{g}$, an endomorphism J and a natural connection $\widetilde{\nabla}$ of the forms

$$\bar{g}((x_1,\, y_1,\, x_2,\, y_2),\, (u_1,\, v_1,\, u_2,\, v_2)) = -x_1u_1 - y_1v_1 + x_2u_2 + y_2v_2,$$

$$J(x_1,\, y_1,\, x_2,\, y_2) \;=\; (-y_1,\, x_1,\, -y_2,\, x_2).$$

From the second equation of the last relations, we see that

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \qquad\qquad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i},$$

where i run over $1,\, 2$. Then $(\bar{M},\, \bar{g},\, J)$ is an almost complex manifold. We let

$$\bar{\nabla}_{\bar{X}}\bar{Y} \;=\; \widetilde{\nabla}_{\bar{X}}\bar{Y} + \ell\{\theta(\bar{Y})\bar{X} - \bar{g}(\bar{X}, \bar{Y})\zeta\} - m\theta(\bar{X})J\bar{Y},$$

where ℓ and m are smooth functions and θ is a 1-form associated with a smooth vector field ζ. Then $\bar{\nabla}$ is an (ℓ, m)-type metric connection on $(\bar{M},\, \bar{g},\, J)$.

Consider a hypersurface M of $\bar{M} = \mathbf{R}^4_2$ given by

$$x_1 = y_1 + \sqrt{2}\,\sqrt{x_2^2 + y_2^2}.$$

For simplicity, we set $\mathbf{f} = \sqrt{x_2^2 + y_2^2}$, $\frac{\partial}{\partial x_i} = \partial_{x_i}$ and $\frac{\partial}{\partial y_i} = \partial_{y_i}$ for $i = 1,\, 2$. Let

$$\Lambda(x_1,\, y_1,\, x_2,\, y_2) \;=\; x_1 - y_1 - \sqrt{2}\sqrt{x_2^2 + y_2^2},$$

then $M = \Lambda^{-1}(0)$, *i.e.*, M is a level surface of the function Λ. As ∂_{x_1} and ∂_{y_1} are timelike and ∂_{x_2} and ∂_{y_2} are spacelike, the gradient vector field

$$\begin{aligned} \nabla\Lambda &= -\frac{\partial\Lambda}{\partial x_1}\partial_{x_1} - \frac{\partial\Lambda}{\partial y_1}\partial_{y_1} + \frac{\partial\Lambda}{\partial x_2}\partial_{x_2} + \frac{\partial\Lambda}{\partial y_2}\partial_{y_2} \\ &= -\partial_{x_1} + \partial_{y_1} - \frac{\sqrt{2}}{\mathbf{f}}(x_2\partial_{x_2} + y_2\partial_{y_2}) \end{aligned}$$

is orthogonal to all vectors tangent to the level surface M, it is easy to check that M is a lightlike hypersurface whose normal bundle $TM^{\perp}$ is spanned by

$$\xi = \mathbf{f}\,(\partial_{x_1} - \partial_{y_1}) + \sqrt{2}\,(x_2\,\partial_{x_2} + y_2\,\partial_{y_2}).$$

Then the transversal vector bundle is given by

$$tr(TM) = Span\left\{N = \frac{1}{4\mathbf{f}^2}\left\{\mathbf{f}(-\partial_{x_1} + \partial_{y_1}) + \sqrt{2}\,(x_2\,\partial_{x_2} + y_2\,\partial_{y_2})\right\}\right\}.$$

Since $u_1 = y_1,\ u_2 = x_2,\ u_3 = y_2,\ x_1 = u_1 + \sqrt{2}\mathbf{f}$ and $\partial_{u_i} = \sum \frac{\partial x_A}{\partial u_i}\,\partial_A$, the tangent bundle $T\mathbf{R}^4_2$ is spanned by

$$\left\{\partial_{u_1} = \partial_{x_1} + \partial_{y_1},\quad \partial_{u_2} = \frac{\sqrt{2}x_2}{\mathbf{f}}\partial_{x_1} + \partial_{x_2},\quad \partial_{u_3} = \frac{\sqrt{2}y_2}{\mathbf{f}}\partial_{x_1} + \partial_{y_2}\right\}.$$

It follows that the corresponding screen distribution $S(TM)$ is spanned by

$$\{W_1 = \partial_{x_1} + \partial_{y_1},\ W_2 = -y_2\,\partial_{x_2} + x_2\,\partial_{y_2}\}.$$

By direct calculations we obtain

$$\begin{aligned} \widetilde{\nabla}_X W_1 &= \widetilde{\nabla}_{W_1} X = 0, \\ \widetilde{\nabla}_{W_2} W_2 &= -x_2\,\partial_{x_2} - y_2\,\partial_{y_2}, \\ \widetilde{\nabla}_\xi \xi = \sqrt{2}\,\xi, \qquad & \widetilde{\nabla}_{W_2}\xi = \widetilde{\nabla}_\xi W_2 = \sqrt{2}\,W_2, \end{aligned}$$

for any $X \in \Gamma(TM)$. Now we set $\zeta = \omega + \lambda N = a_1\partial_{x_1} + b_1\partial_{y_1} + a_2\partial_{x_2} + b_2\partial_{y_2}$. By using (1.2), we obtain

$$\begin{aligned} \bar{\nabla}_X W_1 &= \ell\{\theta(W_1)X - g(X, W_1)\zeta\} - m\theta(X)JW_1 \\ &= \ell\{-(a_1 + b_1)X + (x_1 + y_1)\zeta\} + m(a_1x_1 + y_1b_1)JW_1. \end{aligned}$$

Using (2.3) and the fact that $JW_1 = \frac{1}{2}\{\xi - 4\mathbf{f}^2 N\}$, we obtain

$$\nabla_X W_1 = \ell\{-(a_1 + b_1)X + (x_1 + y_1)\omega\} + \frac{1}{2}m(a_1x_1 + y_1b_1)\xi,$$
$$B(X, W_1) = \lambda\ell(x_1 + y_1) - 2m(a_1x_1 + y_1b_1)\mathbf{f}^2.$$

Thus $B(W_1, W_1) = 2\lambda\ell - 2m(a_1 + b_1)\mathbf{f}^2$ and $B(W_2, W_1) = 0$.
By the same method, we see that

$$\begin{aligned} \bar{\nabla}_{W_1} X &= \ell\{\theta(X)W_1 + g(X, W_1)\zeta\} + m\theta(W_1)JX, \\ &= \ell\{-(a_1x_1 + b_1y_1)W_1 + (x_1 + y_1)\zeta\} - m(a_1 + b_1)JX. \end{aligned}$$

Using (2.3) and (2.9), we have

$$\nabla_{W_1} X = \ell\{-(a_1x_1 + b_1y_1)W_1 + (x_1 + y_1)\omega\} - m(a_1 + b_1)FX,$$
$$B(W_1, X) = \lambda\ell(x_1 + y_1) - m(a_1 + b_1)u(X).$$

Thus $B(W_1, W_1) = 2\lambda\ell - m(a_1 + b_1)u(W_1)$ and $B(W_1, W_2) = -m(a_1 + b_1)u(W_2)$. From the last equations, we obtain

$$B(W_1, W_2) - B(W_2, W_1) = -m(a_1 + b_1)u(W_2).$$

By the similar produce, we obtain all forms of $\nabla_X Y$ and $B(X, Y)$.

3. Some results

Theorem 3.1. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with an (ℓ, m)-type metric connection $\bar{\nabla}$ subject to $\zeta \in \Gamma(TM)$ and integrable D. If F is parallel in terms of ∇ on M, then*

(1) $\ell = 0$ and $\bar{\nabla}$ is a quarter-symmetric metric connection,
(2) D and $J(tr(TM))$ are parallel on M, and
(3) $M = \mathcal{C}_U \times M^\sharp$ is locally a product manifold, where $\mathcal{C}_U$ is a null curve tangent to $J(tr(TM))$ and $M^\sharp$ is a leaf of the distribution D.

Proof. (1) Replacing Y by η to (2.21) and using $(2.14)_1$, we have

$$\ell\{\theta(V)X - u(X)\zeta\} = 0.$$

Taking $X = \eta$, we have $\ell\theta(V)\eta = 0$. Thus $\ell\theta(V) = 0$. Consequently, we get $\ell u(X) = 0$. Setting $X = U$, we have $\ell = 0$. Therefore, $\bar{\nabla}$ is a quarter-symmetric metric connection.

(2) Taking the product with V to (2.21): $B(X,Y)U = u(Y)A_N X$, we obtain

$$\mathcal{B}(X,Y) = u(Y)u(A_N X).$$

Putting $Y = V$ and $Y = FZ$, we obtain

$$\mathcal{B}(X,V) = 0, \qquad \mathcal{B}(X,FZ) = 0.$$

In general, by using (2.7), (2.9), (2.11), (2.15) and (2.20), we derive

$$g(\nabla_X\eta, V) = -\mathcal{B}(X,V) = 0, \qquad g(\nabla_X V, V) = 0,$$
$$g(\nabla_X Z_o, V) = \mathcal{B}(X, FZ_o) = 0, \qquad \forall\, Z_o \in \Gamma(D_o).$$

It follows that D is a parallel distribution on M, that is,

$$\nabla_X Y \in \Gamma(D), \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(D).$$

Also, taking $Y = U$ to (2.21): $\mathcal{B}(X,Y)U = u(Y)A_N X$, we get

$$A_N X = \mathcal{B}(X,U)U. \tag{3.1}$$

From (3.1), we obtain $F(A_N X) = 0$ and hance, from (2.19), we get

$$\nabla_X U = \tau(X)U. \tag{3.2}$$

Moreover,

$$\nabla_X U \in \Gamma(J(tr(TM))), \quad \forall\, X \in \Gamma(TM),$$

which means $J(tr(TM))$ is parallel on M.

(3) From (2) and (2.7), by the decomposition theorem [3], $M = \mathcal{C}_U \times M^\sharp$ is locally a product manifold, where $\mathcal{C}_U$ is a null curve tangent to $J(tr(TM))$ and $M^\sharp$ is a leaf of D. □

Definition 3.1. The structure vector field U of M is said to be *principal* in terms of A^*_η if there exists a smooth function α such that

$$A^*_\eta U = \alpha\, U.$$

A lightlike hypersurface M of an indefinite almost complex manifold $\bar{M}$ is called a *Hopf lightlike hypersurface* if it admits a principal structure vector field U.

Example 3.1. We consider a complex metric as the polynomial $Q(z) = -\sum_{j=1}^{p} z_j^2 + \sum_{j=p+1}^{n+1} z_j^2 = g_{\mathbb{C}}(z,\bar{z})$. We define $\mathbb{S}^1$-invariant hypersurface

$$\widetilde{M}_1 = \{z = (z_1, \ldots, z_{n+1}) \in \mathbb{S}^{2n+1}_{2p} \mid Q(z)\bar{Q}(z) = 1,\ \mathrm{rank}_{\mathbb{R}}\{z, iz, \bar{z}, i\bar{z}\} = 4\}.$$

We define the action and its corresponding quotient

$$\mathbb{S}^1 \times \mathbb{S}^{2n+1}_{2p} \to \mathbb{S}^{2n+1}_{2p},\ (a,(z_1 \ldots, z_{2n+1})) \mapsto (az_1, \ldots, az_{n+1}),$$
$$\pi : \mathbb{S}^{2n+1}_{2p} \to \mathbb{C}P^n_p = S^{2n+1}_{2p}/\sim.$$

From a semi-Riemannian submersion π, $M_1 = \pi\left(\widetilde{M}_1\right)$ is Hopf (see [1], [8]).

Theorem 3.2. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with an $(\ell,\, m)$-type metric connection $\bar{\nabla}$ subject to $\zeta \in \Gamma(TM)$. If V is parallel in terms of ∇ on M, then*

(1) $\ell = 0$ and $\bar{\nabla}$ is a quarter-symmetric metric connection,
(2) the 1-form τ satisfies $\tau = 0$,
(3) M is Hopf lightlike hypersurface of $\bar{M}$ such that $\alpha = m\theta(V)$,
(4) the functions $\theta(U)$ and $\theta(V)$ are satisfied $2\theta(U)\theta(V) = 1$.

Proof. (1) Assume that V is parallel in terms of ∇ on M. Applying the scalar product with N to (2.20), we get

$$\mathcal{B}(X, U) = \ell\{bu(X) - \theta(V)\mu(X)\}.$$

Taking $X = \eta$ to this equation and using $(2.14)_2$, we get $\ell\theta(V) = 0$. Thus

$$\mathcal{B}(X, U) = \ell bu(X).$$

Taking $X = \zeta$, $X = U$, $X = V$ and $X = F\zeta$ to this by turns, we obtain

$$\mathcal{B}(\zeta, U) = 0, \quad \mathcal{B}(U, U) = \ell b, \quad \mathcal{B}(V, U) = 0, \quad \mathcal{B}(F\zeta, U) = 0. \tag{3.3}$$

Applying the scalar product with U to (2.20) and using $\ell\theta(V) = 0$, we obtain

$$\tau(X) = -\ell\theta(U)u(X). \tag{3.4}$$

Taking $X = U$ and $Y = V$ to (2.13) and using $(3.3)_3$, we obtain

$$\mathcal{B}(U, V) = m\theta(V). \tag{3.5}$$

Taking the scalar product with ζ to $J\zeta = F\zeta + \theta(V)N$ and using the facts that $\bar{g}(J\zeta, \zeta) = 0$ and $\theta(N) = b$, we obtain $\theta(F\zeta) = -b\theta(V)$. Taking $X = U$ and $Y = F\zeta$ to (2.13) and using $(3.3)_4$ and $\theta(F\zeta) = -b\theta(V)$, we obtain

$$\mathcal{B}(U, F\zeta) = -mb\theta(V). \tag{3.6}$$

Taking the scalar product with ζ to (2.20) and using (3.4), we obtain

$$\mathcal{B}(X, F\zeta) + b\mathcal{B}(X, V) + \ell u(X) = 0.$$

Replacing X by U to this and using (3.5) and (3.6), we have $\ell = 0$.

(2) As $\ell = 0$, from (3.4), we see that $\tau = 0$.

(3) As $\tau = \ell = 0$, (2.20) reduces $F(A^*_\eta X) = 0$. Thus $J(A^*_\eta X) = \mathcal{B}(X, V)N$. Applying J to this equation and using $(2.1)_1$, we obtain

$$A^*_\eta X = \mathcal{B}(X, V)U. \tag{3.7}$$

Taking $X = U$ to this equation and using (3.5), we obtain $A^*_\eta U = m\theta(V)U$. Thus M is Hopf lightlike hypersurface of $\bar{M}$ such that $\alpha = m\theta(V)$.

(4) Taking the scalar product with ζ to $A^*_\eta U = m\theta(V)U$, we have

$$\mathcal{B}(U, \zeta) = m\theta(U)\theta(V).$$

On the other hand, taking $X = U$ and $Y = \zeta$ to (2.13) and using $(3.3)_1$, we get

$$\mathcal{B}(U, \zeta) = m\{1 - \theta(U)\theta(V)\}.$$

From the last two equations, we get $m\{1 - 2\theta(U)\theta(V)\} = 0$. As $\ell = 0$, we see that $m \neq 0$ as $(\ell, m) \neq (0,\, 0)$. Therefore, we obtain $2\theta(U)\theta(V) = 1$. □

Definition 3.2. The structure tensor field F of M is said to be *recurrent* [7] if there exists a 1-form ϖ on M such that

$$(\nabla_X F)Y = \varpi(X)FY.$$

Theorem 3.3. *If the structure tensor field F of a lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$ with an $(\ell,\, m)$-type metric connection subject to $\zeta \in \Gamma(TM)$ is recurrent, then F is parallel in terms of ∇ on M.*

Proof. If M is recurrent, then, from (3.6), we obtain

$$\begin{aligned}\varpi(X)FY = \; & u(Y)A_N X - \mathcal{B}(X,Y)U \\ & + \ell\{\theta(JY)X - \theta(Y)FX - \bar{g}(X,JY)\zeta + g(X,Y)F\zeta\}.\end{aligned} \tag{3.8}$$

Taking $Y = \eta$ and $Y = V$ at (3.8) by turns and using $(2.14)_1$, we have

$$\varpi(X)V = \ell\{\theta(V)X - u(X)\zeta\}. \tag{3.9}$$

$$\varpi(X)\eta = -\mathcal{B}(X,V)U - \ell\{\theta(V)FX - u(X)F\zeta\}.$$

Applying F to the second equation and using (2.10), we have

$$-\varpi(X)V = \ell\{\theta(V)X - u(X)\zeta\}.$$

Comparing this equation with (3.9), we obtain $\varpi(X)V = 0$, and hence $\varpi = 0$. Therefore, $\nabla_X F = 0$ and F is parallel in terms of ∇. □

Corollary 3.1. *If the structure tensor field F of a lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$ with an (ℓ, m)-type metric connection subject to $\zeta \in \Gamma(TM)$ is recurrent, then we have Theorem 3.1 is satisfied.*

Definition 3.3. The structure tensor field F of M is said to be *Lie recurrent* [7] if there exists a 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

where $(\mathcal{L}_X F)Y = [X, FY] - F[X, Y]$ is the Lie derivative on M with respect to X, In case $\vartheta = 0$, we say that F is *Lie parallel*.

Theorem 3.4. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with an (ℓ, m)-type metric connection such that $\zeta \in \Gamma(TM)$. If F is Lie recurrent, then the following statements are satisfied:*

(1) the structure tensor field F is Lie parallel,
(2) the 1-form τ vanishes, i.e., $\tau = 0$,
(3) $A^*_\eta U = -m\theta(U)V, \quad A^*_\eta V = -m\theta(V)V$.

Proof. (1) Using (2.10), (2.12) and (2.21), we obtain

$$\begin{aligned}\vartheta(X)FY = \; & u(Y)A_N X - \mathcal{B}(X,Y)U - \nabla_{FY}X + F\nabla_Y X \\ & + \ell\{bu(Y)X + g(X,Y)F\zeta - \bar{g}(X,JY)\zeta\} \\ & - m\{\theta(Y)X + \theta(FY)FX - \theta(Y)u(X)U\}.\end{aligned} \tag{3.10}$$

Taking $Y = \eta$ and $Y = V$ to (3.10) by turns and using $(2.14)_1$, we have

$$-\vartheta(X)V = \nabla_V X + F\nabla_\eta X + \ell u(X)\zeta + m\theta(V)FX, \tag{3.11}$$

$$\begin{aligned}\vartheta(X)\eta = \; & -\mathcal{B}(X,V)U - \nabla_\eta X + F\nabla_V X + \ell u(X)F\zeta \\ & - m\theta(V)\{X - u(X)U\}.\end{aligned} \tag{3.12}$$

Taking the scalar product with U to (3.11) and N to (3.12) by turns and comparing two resulting equations, we get $\vartheta = 0$. Thus F is Lie parallel.

(2) Taking the scalar product with V to (3.11) with $X = U$, we get

$$\tau(V) = 0. \tag{3.13}$$

Taking $X = \eta$ to (3.11) and using (2.6), (2.17) and (3.13), we have

$$A^*_\eta V = \{\tau(\eta) - m\theta(V)\}V; \qquad \mathcal{B}(V,U) = \tau(\eta) - m\theta(V).$$

Taking the scalar product with V to (3.12) with $X = U$, we obtain

$$\mathcal{B}(U,V) = -\tau(\eta).$$

Taking $X = V$ and $Y = U$ to (2.13) and using the last two equations, we have

$$\tau(\eta) = 0\,; \qquad \mathcal{B}(U, V) = 0, \qquad A^*_\eta V = -m\theta(V)V. \tag{3.14}$$

Taking $X = U$ to (3.10) and using (2.10), (2.13), (2.18) and (2.19), we get

$$\begin{aligned} &u(Y)A_N U - F(A_N FY) - \tau(FY)U - A_N Y \\ &+ \ell\{v(Y)F\zeta + \mu(Y)\zeta\} - m\{\theta(Y) - \theta(U)u(Y)\}U = 0. \end{aligned} \tag{3.15}$$

Taking $Y = V$ to (3.15) and using $(3.14)_1$, we have

$$A_N V = -F(A_N\eta) + \ell F\zeta - m\theta(V)U. \tag{3.16}$$

Taking the scalar product with U to (3.16) and using $(2.1)_2$ and (2.9), we have

$$\mathcal{C}(V, U) = -\ell b. \tag{3.17}$$

Replacing Y by U to (3.10) and using the fact that $FU = 0$, we have

$$\begin{aligned} A_N X \;=\; &\mathcal{B}(X, U)U - F\nabla_U X - \ell\{bX + v(X)F\zeta - \mu(X)\zeta\} \\ &+ m\theta(U)\{X - u(X)U\}. \end{aligned} \tag{3.18}$$

Taking $X = V$ to this equation and using (2.13), (2.20) and $(3.14)_3$, we get

$$A_N V = A^*_\eta U + \tau(U)\eta - \ell bV + m\theta(U)V - m\theta(V)U.$$

Taking the scalar product with N and U by turns and using (3.17), we have

$$\tau(U) = 0, \qquad\qquad \mathcal{B}(U, U) = -m\theta(U). \tag{3.19}$$

$$A_N V = A^*_\eta U - \ell bV + m\theta(U)V - m\theta(V)U. \tag{3.20}$$

From (2.18) and $(3.19)_2$, we obtain

$$\mathcal{C}(U, V) = -\ell b - m\theta(U). \tag{3.21}$$

Taking the product with V to (3.15) and using (2.18) and (3.21), we have

$$\mathcal{B}(Y, U) = -\tau(FY) - m\theta(Y). \tag{3.22}$$

Taking $X = V$ to (3.10) and using (2.10), (2.13), (2.20) and $(3.14)_1$, we get

$$\begin{aligned} &u(Y)A_N V - F(A^*_\eta FY) - A^*_\eta Y + \tau(FY)V - \tau(Y)\eta \\ &+ \ell bu(Y)V - m\{\theta(Y)V + \theta(FY)\eta - \theta(V)u(Y)U\} = 0. \end{aligned} \tag{3.23}$$

Taking the scalar product with U and using (2.15) and (3.17), we have

$$\mathcal{B}(Y, U) = \tau(FY) - m\theta(Y).$$

Comparing this equation with (3.22), we see that $\tau(FY) = 0$. Replacing Y by FX to this result and using (2.10) and $(3.19)_1$, we have $\tau = 0$.

(3) From $(3.14)_3$, we show that $A^*_\eta V = -m\theta(V)V$. Replacing Y by U to (2.13) and using (3.22) with $\tau = 0$, we have $B(U, X) = -m\theta(U)u(X)$. From this result and (2.15), we see that $A^*_\eta U = -m\theta(U)V$. From this result and (3.20), we have $A_N V = -\ell bV - m\theta(V)U$. Thus we have our theorem. □

4. Indefinite complex space forms

A connected indefinite Kaehler manifold $\bar{M}(c)$ of constant holomorphic sectional curvature c is called an *indefinite complex space form* if its curvature tensor $\widetilde{R}$ satisfies

$$\begin{aligned} \widetilde{R}(X, Y)Z \;=\; &\frac{c}{4}\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(JY, Z)JX \\ &- \bar{g}(JX, Z)JY + 2\bar{g}(X, JY)JZ\}. \end{aligned} \tag{4.1}$$

For the curvature tensor $\bar{R}$ of the $(\ell,\, m)$-type metric connection $\bar{\nabla}$ on $\bar{M}$, we have the following relation:

$$\begin{aligned}\bar{R}(\bar{X},\bar{Y})\bar{Z} \;=\; & \widetilde{R}(\bar{X},\bar{Y})\bar{Z} \\ & + (X\ell)\{\theta(Z)Y - g(Y,Z)\zeta\} - (Xm)\theta(Y)JZ \\ & - (Y\ell)\{\theta(Z)X - g(X,Z)\zeta\} + (Ym)\theta(X)JZ \\ & + \ell\{(\bar{\nabla}_X\theta)(Z)Y - (\bar{\nabla}_Y\theta)(Z)X \\ & \qquad + g(X,Z)\bar{\nabla}_Y\zeta - g(Y,Z)\bar{\nabla}_X\zeta \\ & \qquad + \ell[g(Y,Z)X - g(X,Z)Y]\} \\ & - m\{(\bar{\nabla}_X\theta)(Y) - (\bar{\nabla}_Y\theta)(X) \\ & \qquad + m[\theta(Y)\theta(JX) - \theta(X)\theta(JY)]\}JZ \\ & + \ell m\{[\theta(Y)JX - \theta(X)JY]\theta(Z) \\ & \qquad - [\theta(Y)g(JX,Z) - \theta(X)g(JY,Z)]\zeta\}.\end{aligned} \tag{4.2}$$

For the curvature tensors R and R^* of the connection ∇ and ∇^* on M and $S(TM)$, respectively, we have the Gauss equations for M and $S(TM)$ such that

$$\begin{aligned}\bar{R}(X,Y)Z = {} & R(X,Y)Z + \mathcal{B}(X,Z)A_{_N}Y - \mathcal{B}(Y,Z)A_{_N}X \\ & + \{(\nabla_X\mathcal{B})(Y,Z) - (\nabla_Y\mathcal{B})(X,Z) \\ & \quad + \tau(X)\mathcal{B}(Y,Z) - \tau(Y)\mathcal{B}(X,Z) \\ & \quad - \ell[\theta(X)\mathcal{B}(Y,Z) - \theta(Y)\mathcal{B}(X,Z)] \\ & \quad - m[\theta(X)\mathcal{B}(FY,Z) - \theta(Y)\mathcal{B}(FX,Z)]\}N,\end{aligned} \tag{4.3}$$

$$\begin{aligned}R(X,Y)PZ = {} & R^*(X,Y)PZ + \mathcal{C}(X,PZ)A^*_\eta Y - \mathcal{C}(Y,PZ)A^*_\eta X \\ & + \{(\nabla_X\mathcal{C})(Y,PZ) - (\nabla_Y\mathcal{C})(X,PZ) \\ & \quad - \tau(X)\mathcal{C}(Y,PZ) + \tau(Y)\mathcal{C}(X,PZ) \\ & \quad - \ell[\theta(X)\mathcal{C}(Y,PZ) - \theta(Y)\mathcal{C}(X,PZ)] \\ & \quad - m[\theta(X)\mathcal{C}(FY,PZ) - \theta(Y)\mathcal{C}(FX,PZ)]\}\eta.\end{aligned} \tag{4.4}$$

Differentiating $\bar{g}(\zeta,\eta)=0$ with respect to $\bar{\nabla}_X$ and using (2.6) and (2.15), we have

$$\bar{g}(\bar{\nabla}_X\zeta,\eta) = \mathcal{B}(X,\zeta). \tag{4.5}$$

Taking the scalar product with η and N to (4.2) by turns and using $(2.16)_2$, (4.1), (4.3), (4.4) and (4.5), we get

$$\begin{aligned}& (\nabla_X\mathcal{B})(Y,Z) - (\nabla_Y\mathcal{B})(X,Z) \\ & + \{\tau(X) - \ell\theta(X)\}\mathcal{B}(Y,Z) - \{\tau(Y) - \ell\theta(Y)\}\mathcal{B}(X,Z) \\ & - m\{\theta(X)\mathcal{B}(FY,Z) - \theta(Y)\mathcal{B}(FX,Z)\} \\ & + \{(Xm)\theta(Y) - (Ym)\theta(X)\}u(Z) \\ & - \ell\{g(X,Z)\mathcal{B}(Y,\zeta) - g(Y,Z)\mathcal{B}(X,\zeta)\} \\ & + m\{(\bar{\nabla}_X\theta)(Y) - (\bar{\nabla}_Y\theta)(X) \\ & \qquad + m[\theta(Y)\theta(JX) - \theta(X)\theta(JY)]\}u(Z) \\ & - \ell m\{\theta(Y)u(X) - \theta(X)u(Y)\}\theta(Z) \\ & = \frac{c}{4}\{u(X)\bar{g}(JY,Z) - u(Y)\bar{g}(JX,Z) + 2u(Z)\bar{g}(X,JY)\},\end{aligned} \tag{4.6}$$

$$\begin{aligned}
&(\nabla_X\mathcal{C})(Y,PZ)-(\nabla_Y\mathcal{C})(X,PZ)\\
&-\{\tau(X)+\ell\theta(X)\}\mathcal{C}(Y,PZ)+\{\tau(Y)+\ell\theta(Y)\}\mathcal{C}(X,PZ)\\
&-m\{\theta(X)\mathcal{C}(FY,PZ)-\theta(Y)\mathcal{C}(FX,PZ)\}\\
&-(X\ell)\{\theta(PZ)\mu(Y)-bg(Y,PZ)\}\\
&+(Y\ell)\{\theta(PZ)\mu(X)-bg(X,PZ)\}\\
&+\{(Xm)\theta(Y)-(Ym)\theta(X)\}v(PZ)\\
&-\ell\{(\bar\nabla_X\theta)(PZ)\eta(Y)-(\bar\nabla_Y\theta)(PZ)\mu(X)\}\\
&-\ell\{g(X,PZ)\bar g(\bar\nabla_Y\zeta,N)-g(Y,PZ)\bar g(\bar\nabla_X\zeta,N)\}\\
&-\ell^2\{g(Y,PZ)\mu(X)-g(X,PZ)\mu(Y)\}\\
&+m\{(\bar\nabla_X\theta)(Y)-(\bar\nabla_Y\theta)(X)\\
&\qquad+m[\theta(Y)\theta(JX)-\theta(X)\theta(JY)]\}v(PZ)\\
&-\ell m\{\theta(Y)v(X)-\theta(X)v(Y)\}\theta(PZ)\\
&+\ell mb\{\theta(Y)\bar g(JX,PZ)-\theta(X)\bar g(JY,PZ)\}\\
&=\frac{c}{4}\{\mu(X)g(Y,PZ)-\mu(Y)g(X,PZ)+v(X)g(FY,PZ)\\
&\qquad-v(Y)g(FX,PZ)+2v(PZ)\bar g(X,JY)\}.
\end{aligned}\tag{4.7}$$

Theorem 4.1. *Let M be a lightlike hypersurface of an indefinite complex space form $\bar M(c)$ with an (ℓ,m)-type metric connection subject such that ζ is tangent to M. If one of the following four statements is satisfied, then $c=0$.*

(1) F is parallel with respect to the connection ∇,
(2) F is recurrent,
(3) F is Lie recurrent,
(4) U is parallel with respect to ∇ and $\ell=0$.

Moreover, in case (4), the 1-form τ satisfies $\tau=0$.

Proof. (1) As F is parallel with respect to ∇, we show that $\ell=0$ by Theorem 3.1. Taking the scalar product with U to (3.1) and using (2.16), we have

$$\mathcal{C}(X,U)=0.$$

Differentiating $\mathcal{C}(Y,U)=0$ with respect to ∇_X and using (3.2), we obtain

$$(\nabla_X\mathcal{C})(Y,U)=0.$$

Taking $PZ=U$ to (4.7) and using the last two equations and $\ell=0$, we get

$$\frac{c}{2}\{\mu(X)v(Y)-\mu(Y)v(X)\}=0.$$

Taking $X=\eta$ and $Y=V$ to this equation, we obtain $c=0$.

(2) By Theorem 3.3 and (1) of this theorem, we obtain $c=0$.

(3) As $\tau=0$ by (2) of Theorem 3.2, the equation (3.22) reduce to

$$\mathcal{B}(Y,U)=-m\theta(Y).\tag{4.8}$$

Differentiating (4.8) with respect to ∇_X and using (2.19) and the fact that $\tau=0$, we obtain

$$\begin{aligned}
(\nabla_X\mathcal{B})(Y,U)=&-(Xm)\theta(Y)-m\{(\bar\nabla_X\theta)(Y)+b\mathcal{B}(X,Y)\}\\
&-g(A^*_\eta Y,F(A_{\scriptscriptstyle N}X))-\ell\theta(U)\mathcal{B}(Y,X)-\ell bg(A^*_\eta Y,FX)\\
&+\ell v(X)\mathcal{B}(Y,\zeta)+\ell\mu(X)g(A^*_\eta Y,F\zeta).
\end{aligned}$$

Taking $Z=U$ to (4.6) and using (4.8) and the last equation, we obtain

$$\begin{aligned}
&g(A^*_\eta X,F(A_{\scriptscriptstyle N}Y))-g(A^*_\eta Y,F(A_{\scriptscriptstyle N}X))\\
&+\ell b\{g(A^*_\eta X,FY)-g(A^*_\eta Y,FX)\}\\
&+\ell\{\mu(X)g(A^*_\eta Y,F\zeta)-\mu(Y)g(A^*_\eta X,F\zeta)\}\\
&=\frac{c}{4}\{u(Y)\mu(X)-u(X)\mu(Y)+2\bar g(X,JY)\}.
\end{aligned}$$

Taking $Y = U$ and $X = \eta$ to this equation and using (2.17) and the facts that $A^*_\eta U = -m\theta(U)V$ and $g(V, FX) = 0$, we get $c = 0$.

(4) Assume that U is parallel with respect to ∇ and $\ell = 0$. Taking the scalar product with V and N to (2.19) by turns such that $\nabla_X U = 0$, we get

$$\tau(X) = 0, \qquad \mathcal{C}(X, U) = 0.$$

Differentiating $\mathcal{C}(Y, U) = 0$ with respect to ∇_X and using $\nabla_X U = 0$, we obtain

$$(\nabla_X \mathcal{C})(Y, U) = 0.$$

Taking $PZ = U$ to (4.7), we obtain

$$c\{\mu(X)v(Y) - \mu(Y)v(X)\} = 0.$$

Taking $X = \eta$ and $Y = V$ to this equation, we have $c = 0$. □

Theorem 4.2. *Let M be a lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with an (ℓ, m)-type metric connection such that $\zeta \in \Gamma(TM)$. If V is parallel in terms of ∇, then the following equation holds*

$$(\eta m)\theta(U) + m(\bar{\nabla}_\eta \theta)(U) - m^2 = \frac{3}{4}c.$$

Moreover, if ζ is an asymtotic direction, i.e., $\mathcal{B}(\zeta, \zeta) = 0$, then

$$2(\eta m)\theta(U) = m^2 + \frac{3}{4}c.$$

Proof. As V is parallel with respect to ∇, we show that $\ell = \tau = 0$ by Theorem 3.2. Taking the scalar product with U to (3.7) and using (2.18), we have

$$\mathcal{C}(X, V) = 0.$$

Differentiating $C(Y, V) = 0$ with respect to ∇_X and using $\nabla_X V = 0$, we obtain

$$(\nabla_X \mathcal{C})(Y, V) = 0.$$

Taking $PZ = V$ to (4.7), we get

$$\begin{aligned}&(Xm)\theta(Y) - (Ym)\theta(X)\\&+ m\{(\bar{\nabla}_X\theta)(Y) - (\bar{\nabla}_Y\theta)(X) + m[\theta(Y)\theta(JX) - \theta(X)\theta(JY)]\}\\&= \frac{c}{4}\{\mu(X)u(Y) - \mu(Y)u(X) + 2\bar{g}(X, JY)\}.\end{aligned} \tag{4.9}$$

Differentiating $\theta(\eta) = 0$ in terms of ∇_X and using (2.6) and (2.15), we have

$$(\bar{\nabla}_X\theta)(\eta) = \mathcal{B}(X, \zeta). \tag{4.10}$$

Taking $X = U$ to (4.10) and using $A^*_\eta U = m\theta(V)U$, we have

$$(\bar{\nabla}_U\theta)(\eta) = g(A^*_\eta U, \zeta) = m\theta(U)\theta(V).$$

Taking $X = \eta$ and $Y = U$ to (4.9) and using the above equation and $2\theta(U)\theta(V) = 1$, we have

$$(\eta m)\theta(U) + m(\bar{\nabla}_\eta\theta)(U) = m^2 + \frac{3}{4}c.$$

Applying $\bar{\nabla}_X$ to $\theta(\zeta) = 1$, we have $(\bar{\nabla}_X\theta)(\zeta) = 0$. Taking $X = \eta$ and $Y = \zeta$ to (4.9) and using (4.10) and $(\bar{\nabla}_\eta\theta)(\zeta) = 0$, we obtain

$$\eta m = m\mathcal{B}(\zeta, \zeta) + \{m^2 + \frac{3}{4}c\}\theta(V).$$

Assume that $\mathcal{B}(\zeta, \zeta) = 0$. Taking the product with $\theta(U)$ to the above equation and using $2\theta(U)\theta(V) = 1$, we have $2(\eta m)\theta(U) = m^2 + \frac{3}{4}c$. □

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Author's contributions

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