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A Study on the Solutions of $(3 + 1)$ Conformal Time Derivative Generalized q-deformed Sinh-Gordon Equation

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Abstract

This article is about examining the solutions of the $(3 + 1)$ conformal time derivative generalized q-deformed Sinh-Gordon equation. The integration method used to reach the solutions of the equation is the generalized exponential rational function method. In this article, the process of examining the solutions goes step by step, first the basic steps of the proposed method are given, then the reduction of the equation is examined, and then the solutions are obtained by applying the method. The obtained wave solutions include hyperbolic soliton solutions. In addition, dark and bright solitons have been obtained. To perceive the physical phenomena, 2D and 3D graphical patterns of some of solutions obtained in this study are plotted by using Maple programming. The worked-out solutions ascertained that the suggested method is effectual, simple and direct.

Keywords: Conformal time derivative; The generalized exponential rational function method; The generalized q- deformed Sinh–Gordon equation.

1. Introduction

Fractional differential equations, which have attracted a lot of attention since they were discovered, have found many applications in fields such as physics, engineering, optics, biology, technology and so on. Theoretical models expressed in fractional analysis are more compatible with experimental data than models expressed in integer orders. When describing physical mechanical problems, the model expressed by fractional analysis has been revealed to have a clearer physical meaning and a simpler expression [1]. Thanks to these advantages, the models obtained using fractional order differentials attract a lot of attention, and researches and studies on this subject are increasing. Therefore, the solutions of these equations and the behavioral interpretations about them gain importance. Therefore, the solutions and behavioral interpretations of these equations gain importance. A soliton solution is a large amplitude, permanent pulse whose shape and velocity do not change due to collisions with other soliton waves, and is the exact solution of a nonlinear equation. The solitary wave was discovered experimentally by John Scott Russell in 1834. Optical solitons are a type of solitary wave that have the ability to propagate waves long distance without scattering, i.e. retain their shape over a long distance, and optical soliton models have found use in solitary wave-based communication links, amplifiers, optical pulse compressors, fiber optics, and some other mechanisms.Since soliton theory has a wide application area, direct and indirect methods that provide exact solutions of nonlinear differential equations have been brought to the literature by scientists. Some of these methods are the Jacobi elliptic function method [2], the (G'/G) method [3], the Sardar subequation method [4], the exponential rational function method [5,6], the Bernoulli sub-ODE method [7], the Hirota bilinear method [8], the new extended direct algebraic method [9], the Cole-Hopf transformation method [10], the local fractional generalized-exp function method [11], Kudryashov and exponential methods [12, 13], the variational direct method [14] and so on.

The classical sinh-Gordon equation given as

$$u_{tt} - u_{xx} = \sinh u \tag{1.1}$$

is well-known equation and appears in integrable quantum field theory, kink dynamics, fluid dynamics, the propagation of fluxing in Josephson junctions (a junction between two superconductors), the motion of rigid pendulum attached to a stretched wire, and dislocations in crystals and in many other scientific

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applications [15-18]. The generalized q-deformed Sinh-Gordon equation [19] described as

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \left[\sinh_q(L^\theta)\right]^l - \omega. \quad \textbf{(1.2)}$$

Definitions and basic properties of q-calculus are reviewed by Victor and Pokman [20]. The q-deformed function which is introduced by Arai [21]. When this function is included in the dynamic system, the symmetry of the system and the solution is broken. Symmetry breaking [22] is a fundamental phenomenon in particle physics. In its most basic form, spontaneous symmetry breaking happens when a dynamical system's symmetry is not visible in its ground state or equilibrium state. Many classical and quantum systems have this property. Alrebdi et al. investigated the (2+1)-dimensional q-deformed Sinh-Gordon model given as [23]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial t^2} = \left[\sinh_q(L^\theta)\right]^l - \omega \quad \textbf{(1.3)}$$

And the (G'/G,1/G) expansion and sine-Gordon expansion methods are applied. The Sinh-Gordon equation with conformal time derivative generalized q-deformation has the following form:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^{2\beta} u}{\partial t^{2\beta}} = \left[\sinh_q(L^\theta)\right]^l - \omega \quad \textbf{(1.4)}$$

where $\frac{\partial^\beta u}{\partial t^\beta}$ is the conformable derivative operator. The definition of conformable fractional derivative of order $\beta \in (0,1)$ [24] defined as

$$\frac{\partial^\beta f(t)}{\partial t^\beta} = \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\beta}) - f(t)}{\epsilon}, f:(0,\infty) \to \mathbb{R} \quad \textbf{(1.5)}$$

Substituting $\beta = 1$ in Eq. (1.4), Eq. (1.3) is obtained. In this study, we consider the three-dimensional conformal time derivative generalized q- deformed Sinh-Gordon equation [25] given as

$$\frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2} + \frac{\partial^2 L}{\partial z^2} - \frac{\partial^{2\beta} L}{\partial t^{2\beta}} = \left[\sinh_q(L^\theta)\right]^l - \omega \quad \textbf{(1.6)}$$

The proposed equation has expanded modeling possibilities for complex processes with broken symmetry. To obtain optical soliton solutions of (1.6) we utilized the generalized exponential rational function method which was introduced by Ghanbari and Inc [26] in 2018. This method, which was reduced to the "exponential rational function" method in a special case, has been used many times since then. The following studies can be considered for the efficiency and effectiveness of the method [27-33].

This paper is organized as follows: the second section is devoted to methodology. In the third section, the mathematical model is investigated. The fourth section contains the solutions. We provide several figures for solutions in the fifth section. We conclude in the last section.

2. Materials and Methods

2.1 The Generalized Exponential Rational Function Method (GERFM)

This section is devoted to explain the basic steps of GERFM. For this, consider the following partial differential equation (PDE)

$$P(u, u_t, u_x, u_{tt}, u_{xt}, \dots) = 0, \quad \textbf{(2.1)}$$

where P is a polynomial in dependent function u and its partial derivatives with respect to x and t. With the help of the traveling wave transformation $u = u(\xi), \xi = k(x - ct)$, where c is a constant, Eq. (2.1) is transformed to an ordinary differential equation (ODE)

$$Q(v, v', v'', \dots) = 0, \quad \textbf{(2.2)}$$

where $(.)' = \frac{d}{d\xi}(.)$.

Step 1. Exact solutions of the Eq. (2.2) can be constructed as [26, 34]:

$$v(\xi) = A_0 + \sum_{k=1}^{N} A_k \phi(\xi)^k + \sum_{k=1}^{N} B_k \phi(\xi)^{-k}, \quad \textbf{(2.3)}$$

where

$$\phi(\xi) = \frac{p_1 e^{q_1 \xi} + p_2 e^{q_2 \xi}}{p_3 e^{q_3 \xi} + p_4 e^{q_4 \xi}}. \quad \textbf{(2.4)}$$

Here $p_1, \dots, p_4$, $q_1, \dots, q_4$, A_0, A_k and B_k $(k = 1, \dots, N)$ are constants.

Step 2. The positive integer N is determined using the homogeneous balance principle.

Step 3. An algebraic equation $T(\xi, e^{q_1\xi}, e^{q_2\xi}, e^{q_3\xi}, e^{q_4\xi}) = 0$ is obtained inserting Eq. (2.3) into Eq. (2.2) and arranging all terms.

Step 4. Equating coefficients of powers of T to zero, a system with respect to A_0, A_k and B_k and $p_1, \dots, p_4$, $q_1, \dots, q_4$ is obtained.

Step 5. Using the obtained values with solving the set of equations by use of a computer program, the soliton solutions of Eq. (2.1) is found.

2.2 The Mathematical Model

We must reduce Eq. (1.6) to an ODE to examine the soliton solutions of this equation with the proposed method. For this, the following transformation should be used.

$$\begin{cases} L(x, y, z, t) = V(\xi), \\ \xi = \sigma x + v y + R z - \frac{\kappa}{\beta} t^\beta. \end{cases} \quad \textbf{(2.5)}$$

Here, σ, v, R constants and κ shows the speed of traveling wave. Using (2.5), Eq. (1.6) can be converted into:

$$(-\kappa^2+v^2+\sigma^2+R^2)V''+\omega-\left[\sinh_q(V^\theta)\right]^l=0, \quad \textbf{(2.6)}$$

where $(.)'=\frac{d}{d\xi}$. There are two cases according to choice of l, θ, ω.

Case one: $l=\theta=1, \omega=0$. Therefore, Eq. (2.6) can be written as:

$$(-\kappa^2+v^2+\sigma^2+R^2)V''-\sinh_q(V)=0. \quad \textbf{(2.7)}$$

If both sides of the above equation are multiplied by V' and integrated with respect to ξ once, the following equation is obtained.

$$(-\kappa^2+v^2+\sigma^2+R^2)(V'^2-\cosh_q(V))-2M_1=0, \quad \textbf{(2.8)}$$

where M_1 is the integration constant. Now, if the transformation $V=\ln(u)$, $u=u(\xi)$ is used, we get

$$(\kappa^2-v^2-\sigma^2-R^2)u'^2+2M_1u^2+Qu+u^3=0. \quad \textbf{(2.9)}$$

Case two: $l=2, \theta=1, \omega=-\frac{Q}{2}$. Eq. (2.6) can be rewritten as:

$$(-\kappa^2+v^2+\sigma^2+R^2)V''-\left(\sinh_q(V)\right)^2+\frac{Q}{2}=0. \quad \textbf{(2.10)}$$

After simplifying Eq. (2.10) and using the transformation $V=\frac{1}{2}\ln(u)$, we get

$$2(-\kappa^2+v^2+\sigma^2+R^2)u'^2-2(-\kappa^2+v^2+\sigma^2+R^2)uu''+Q^2u+u^3=0. \quad \textbf{(2.11)}$$

2.3 Application of GERFM to (2.9)

According to the homogeneous balance principle, it is clear to have $2N+2=3N, N=2$. Therefore, the solution can be written as follows:

$$u(\xi)=A_0+A_1\phi(\xi)+\frac{B_1}{\phi(\xi)}+A_2\phi(\xi)^2+\frac{B_2}{\phi(\xi)^2}. \quad \textbf{(2.12)}$$

Inserting (2.12) to (2.9), we obtain a system of algebraic equations made up of tedious and rather long equations. Solving this system with computer program, we get the following results:

Group 1: $p_1=-3, p_2=-2, p_3=1, p_4=1$ and $q_1=1, q_2=0, q_3=1, q_4=0$ provides:

$$\phi(\xi)=-\frac{5+\sinh(\xi)+5\cosh(\xi)}{2(\cosh(\xi)+1)} \quad \textbf{(2.13)}$$

Set 1.1:

$$\left\{\begin{array}{c} A_0=\frac{25}{4}A_2, A_1=5A_2, B_1=B_2=0, \\ M_1=-\frac{A_2}{4}, Q=\frac{A_2{}^2}{16}, \\ R=\frac{\sqrt{A_2-4\sigma^2+4\kappa^2-4v^2}}{2}. \end{array}\right\} \quad \textbf{(2.14)}$$

Placing values in Eqs. (2.12) and (2.13), yields the following solution

$$u_{1.1}(\xi)=\frac{(\cosh(\xi)-1)A_2}{4(\cosh(\xi)+1)}. \quad \textbf{(2.15)}$$

Then we get the following dark soliton solution of Eq. (1.6):

$$L_{1_{1.1}}(x,y,z,t)=\ln\left(\frac{\left(\cosh\left(\sigma x+vy+Rz-\frac{\kappa t^\beta}{\beta}\right)-1\right)A_2}{4\left(\cosh\left(\sigma x+vy+Rz-\frac{\kappa t^\beta}{\beta}\right)+1\right)}\right). \quad \textbf{(2.16)}$$

Set 1.2:

$$\left\{\begin{array}{c} A_0=\frac{25B_2}{144}, A_1=A_2=0, B_1=\frac{5}{6}B_2, \\ M_1=-\frac{1}{144}B_2, Q=\frac{1}{20736}B_2{}^2, \\ R=\frac{\sqrt{-144v^2+B_2-144\sigma^2+144\kappa^2}}{12}. \end{array}\right\} \quad \textbf{(2.17)}$$

Inserting these values into Eq. (2.12), we have:

$$u_{1.2}(\xi)=\frac{1}{144}\frac{B_2\left(5\sinh(\xi)-12+13\cosh(\xi)\right)}{5\sinh(\xi)+12+13\cosh(\xi)}. \quad \textbf{(2.18)}$$

Consequently, we get

$$L_{1_{1.2}}(x,y,z,t)=\ln\left(\frac{1}{144}B_2\frac{5\sinh\left(\sigma x+vy+Rz-\frac{\kappa t^\beta}{\beta}\right)-12+13\cosh\left(\sigma x+vy+Rz-\frac{\kappa t^\beta}{\beta}\right)}{5\sinh\left(\sigma x+vy+Rz-\frac{\kappa t^\beta}{\beta}\right)+12+13\cosh\left(\sigma x+vy+Rz-\frac{\kappa t^\beta}{\beta}\right)}\right). \quad \textbf{(2.19)}$$

Group 2: $p_1=2, p_2=0, p_3=1, p_4=1$ and $q_1=-1, q_2=0, q_3=1, q_4=-1$ provides:

$$\phi=\frac{\cosh(\xi)-\sinh(\xi)}{\cosh(\xi)}. \quad \textbf{(2.20)}$$

Set 2.1:

$$\left\{\begin{array}{c} A_0=\sqrt{Q}, A_1=-2\sqrt{Q}, A_2=\sqrt{Q}, \\ B_1=B_2=0, \\ M_1=-\sqrt{Q}, R=\frac{\sqrt{\sqrt{Q}+4\kappa^2-4\sigma^2-4v^2}}{2} \end{array}\right\} \quad \textbf{(2.21)}$$

The $u(\xi)$ in (2.12) can be written as following

$$u_{2.1}(\xi) = \frac{\sqrt{Q}(\sinh(\xi))^2}{(\cosh(\xi))^2}. \quad \textbf{(2.22)}$$

Then we get the following solution of Eq. (1.6):

$$L_{1_{2.1}}(x,y,z,t) = \ln\left(\sqrt{Q}\frac{\left(\sinh\left(\sigma x+vy+Rz-\frac{\kappa t^\beta}{\beta}\right)\right)^2}{\left(\cosh\left(\sigma x+vy+Rz-\frac{\kappa t^\beta}{\beta}\right)\right)^2}\right). \quad \textbf{(2.23)}$$

Group 3: $p_1 = -3, p_2 = -1, p_3 = 1, p_4 = 1$ and $q_1 = 1, q_2 = -1, q_3 = 1, q_4 = -1$ provides:

$$\phi(\xi) = -\frac{2\cosh(\xi)+\sinh(\xi)}{\cosh(\xi)}. \quad \textbf{(2.24)}$$

Set 3.1:

$$\left\{\begin{matrix} A_0 = \frac{4}{9}B_2, A_1 = 0, A_2 = 0, B_1 = \frac{4}{3}B_2, \\ M_1 = -\frac{1}{9}B_2, Q = \frac{1}{81}B_2^2, \\ R = \frac{\sqrt{-36\sigma^2-36v^2+B_2+36\kappa^2}}{6}. \end{matrix}\right\} \quad \textbf{(2.25)}$$

Inserting these values in Eqs. (2.12) and (2.24), we get

$$u_{3.1}(\xi) = \frac{B_2\left(4\cosh(\xi)\sinh(\xi)+5(\cosh(\xi))^2-4\right)}{9\left(4\cosh(\xi)\sinh(\xi)+5(\cosh(\xi))^2-1\right)} \quad \textbf{(2.26)}$$

Then we get the following solution of Eq. (1.6):

$$L_{1_{3.1}}(x,y,z,t) = \ln\left(\frac{B_2\left(4\cosh(\xi)\sinh(\xi)+5(\cosh(\xi))^2-4\right)}{9\left(4\cosh(\xi)\sinh(\xi)+5(\cosh(\xi))^2-1\right)}\right), \quad \textbf{(2.27)}$$

where $\xi = \sigma x + vy + Rz - \frac{\kappa t^\beta}{\beta}$.

Group 4: $p_1 = 1, p_2 = 1, p_3 = -1, p_4 = 1$ and $q_1 = 1, q_2 = -1, q_3 = 1, q_4 = -1$ provides:

$$\phi(\xi) = -\frac{\cosh(\xi)}{\sinh(\xi)}. \quad \textbf{(2.28)}$$

Set 4.1:

$$\left\{\begin{matrix} A_0 = 8(-\kappa^2 + v^2 + \sigma^2 + R^2), \\ A_1 = 0, B_1 = 0, \\ A_2 = 4(v^2 + \sigma^2 + R^2 - \kappa^2), \\ M_1 = -16(v^2 + \sigma^2 + R^2 - \kappa^2), \\ B_2 = 4(v^2 + \sigma^2 + R^2 - \kappa^2),, \\ Q = 512(-R^2\kappa^2 + R^2\sigma^2 - v^2\kappa^2 \\ +v^2\sigma^2 - \kappa^2\sigma^2 + R^2v^2) \\ +256(\kappa^4 + \sigma^4 + R^4 + v^4). \end{matrix}\right\} \quad \textbf{(2.29)}$$

Placing values in (2.29) into Eq. (2.12) and using Eq. (2.28), we achieve

$$u_{4.1}(\xi) = 4\frac{\left(4(\cosh(\xi))^4-4(\cosh(\xi))^2+1\right)(-\kappa^2+v^2+\sigma^2+R^2)}{\sinh(\xi)^2\cosh(\xi)^2}. \quad \textbf{(2.30)}$$

Then we get the following solution of Eq. (1.6):

$$L_{1_{4.1}}(x,y,z,t) = \ln\left(\frac{4(-\kappa^2+v^2+\sigma^2+R^2)\left(2\cosh\left(\sigma x+vy+Rz-\frac{\kappa t^\beta}{\beta}\right)^2-1\right)^2}{\left(\sinh\left(\sigma x+vy+Rz-\frac{\kappa t^\beta}{\beta}\right)\right)^2\left(\cosh\left(\sigma x+vy+Rz-\frac{\kappa t^\beta}{\beta}\right)\right)^2}\right), \quad \textbf{(2.31)}$$

where $\xi = \sigma x + vy + Rz - \frac{\kappa t^\beta}{\beta}$.

Group 5: $p_1 = -1, p_2 = 3, p_3 = 1, p_4 = -1$ and $q_1 = 1, q_2 = -1, q_3 = 1, q_4 = -1$ provides:

$$\phi(\xi) = \frac{\cosh(\xi)-2\sinh(\xi)}{\sinh(\xi)}. \quad \textbf{(2.32)}$$

Set 5.1:

$$\left\{\begin{matrix} A_0 = 4A_2, A_1 = 4A_2, B_1 = B_2 = 0, \\ M_1 = -A_2, Q = A_2^2, \\ R = \frac{\sqrt{A_2-4\sigma^2+4\kappa^2-4v^2}}{2}. \end{matrix}\right\} \quad \textbf{(2.33)}$$

Inserting values in Eqs. (2.12) and (2.32), we have

$$u_{5.1}(\xi) = \frac{A_2\cosh(\xi)^2}{\sinh(\xi)^2}. \quad \textbf{(2.34)}$$

Then we get the following bright soliton solution of Eq. (1.6):

$$L_{1_{5.1}}(x,y,z,t) = \ln\left(\frac{A_2\left(\cosh\left(\sigma x+vy+Rz-\frac{\kappa t^\beta}{\beta}\right)\right)^2}{\left(\sinh\left(\sigma x+vy+Rz-\frac{\kappa t^\beta}{\beta}\right)\right)^2}\right). \quad \textbf{(2.35)}$$

Set 5.2:

$$\left\{\begin{matrix} A_0 = \frac{4}{9}B_2, A_1 = 0, A_2 = 0, B_1 = \frac{4B_2}{3}, \\ M_1 = -\frac{1}{9}B_2, Q = \frac{1}{81}B_2^2, \\ R = \frac{\sqrt{-36v^2+36\kappa^2+B_2-36\sigma^2}}{6}. \end{matrix}\right\} \quad \textbf{(2.36)}$$

Placing values in Eq. (2.12) and Eq. (2.32), we obtain

$$u_{5.2}(\xi) = \frac{B_2\left(-4\cosh(\xi)\sinh(\xi)+5(\cosh(\xi))^2-1\right)}{9\left(-4\cosh(\xi)\sinh(\xi)+5(\cosh(\xi))^2-4\right)} \quad \textbf{(2.37)}$$

Then we get the following solution of Eq. (1.6):

$$L_{1_{5.2}}(x,y,z,t) = \ln\left(\frac{B_2\left(-4\cosh(\xi)\sinh(\xi)+5\left(\cosh(\xi)\right)^2-1\right)}{9\left(-4\cosh(\xi)\sinh(\xi)+5\left(\cosh(\xi)\right)^2-4\right)}\right), \quad \textbf{(2.38)}$$

where $\xi = \sigma x + vy + Rz - \frac{\kappa t^{\beta}}{\beta}$.

Group 6: $p_1 = 1, p_2 = 2, p_3 = 1, p_4 = 1$ and $q_1 = -1, q_2 = 1, q_3 = -1, q_4 = 1$ provides:

$$\phi(\xi) = \frac{3\cosh(\xi)+\sinh(\xi)}{2\cosh(\xi)}. \quad \textbf{(2.39)}$$

Set 6.1:

$$\left\{\begin{array}{c} A_0 = \frac{9B_2}{16}, A_1 = A_2 = 0, B_1 = -\frac{3B_2}{2}, \\ M_1 = -\frac{1}{16}B_2, Q = \frac{1}{256}B_2{}^2, \\ R = \frac{\sqrt{-64\sigma^2+B_2+64\kappa^2-64v^2}}{8}. \end{array}\right\} \quad \textbf{(2.40)}$$

Putting these results in Eqs. (2.12) and (2.39) leads to

$$u_{6.1}(\xi) = \frac{B_2\left(6\cosh(\xi)\sinh(\xi)+10\left(\cosh(\xi)\right)^2-9\right)}{16\left(6\cosh(\xi)\sinh(\xi)+10\left(\cosh(\xi)\right)^2-1\right)} \quad \textbf{(2.41)}$$

Then we get the following solution of Eq. (1.6):

$$L_{1_{6.1}}(x,y,z,t) = \ln\left(\frac{B_2\left(6\cosh(\xi)\sinh(\xi)+10\left(\cosh(\xi)\right)^2-9\right)}{16\left(6\cosh(\xi)\sinh(\xi)+10\left(\cosh(\xi)\right)^2-1\right)}\right), \quad \textbf{(2.42)}$$

where $\xi = \sigma x + vy + Rz - \frac{\kappa t^{\beta}}{\beta}$.

Group 7: $p_1 = -1, p_2 = 0, p_3 = 1, p_4 = 1$ and $q_1 = 0, q_2 = 0, q_3 = 0, q_4 = 1$ provides:

$$\phi(\xi) = -\left(1+\cosh(\xi)+\sinh(\xi)\right)^{-1} \quad \textbf{(2.43)}$$

Set 7.1:

$$\left\{\begin{array}{c} A_0 = -\kappa^2 + v^2 + \sigma^2 + R^2, \\ , A_1 = 4R^2 - 4\kappa^2 + 4v^2 + 4\sigma^2, \\ A_2 = 4R^2 - 4\kappa^2 + 4v^2 + 4\sigma^2, \\ B_1 = 0, B_2 = 0, \\ M_1 = \kappa^2 - v^2 - \sigma^2 - R^2, \\ Q = R^4 + 2R^2v^2 + 2R^2\sigma^2 \\ -2\kappa^2v^2 - 2R^2\kappa^2 + 2v^2\sigma^2 + \kappa^4 + \\ +v^4 - 2\kappa^2\sigma^2 + \sigma^4. \end{array}\right\} \quad \textbf{(2.44)}$$

Substituting these results into Eq. (2.12) and (2.43), we get

$$u_{7.1}(\xi) = \frac{(-1+\cosh(\xi))(-\kappa^2+v^2+\sigma^2+R^2)}{1+\cosh(\xi)}. \quad \textbf{(2.45)}$$

Then we get the following solution of Eq. (1.6):

$$L_{1_{7.1}}(x,y,z,t) = \ln\left(\frac{(-\kappa^2+v^2+\sigma^2+R^2)\left(-1+\cosh\left(\sigma x+vy+Rz-\frac{\kappa t^{\beta}}{\beta}\right)\right)}{1+\cosh\left(\sigma x+vy+Rz-\frac{\kappa t^{\beta}}{\beta}\right)}\right). \quad \textbf{(2.46)}$$

Group 8: $p_1 = -2, p_2 = -1, p_3 = 1, p_4 = 1$ and $q_1 = 0, q_2 = 1, q_3 = 0, q_4 = 1$ provides:

$$\phi(\xi) = -\frac{3-\sinh(\xi)+3\cosh(\xi)}{2(1+\cosh(\xi))}. \quad \textbf{(2.47)}$$

Set 8.1:

$$\left\{\begin{array}{c} A_0 = \frac{9B_2}{16}, A_1 = 0, A_2 = 0, B_1 = \frac{3B_2}{2}, \\ M_1 = -\frac{1}{16}B_2, Q = \frac{1}{256}B_2{}^2, \\ R = \frac{\sqrt{16\kappa^2+B_2-16v^2-16\sigma^2}}{4}. \end{array}\right\} \quad \textbf{(2.48)}$$

Substituting these results into Eq. (2.12) and (2.47), we obtain

$$u_{8.1}(\xi) = \frac{B_2\left(-3\sinh(\xi)-4+5\cosh(\xi)\right)}{16(-3\sinh(\xi)+5\cosh(\xi)+4)}. \quad \textbf{(2.49)}$$

Then we get the following solution of Eq. (1.6):

$$L_{1_{8.1}}(x,y,z,t) = \ln\left(\frac{B_2\left(-3\sinh(\xi)-4+5\cosh(\xi)\right)}{16(-3\sinh(\xi)+5\cosh(\xi)+4)}\right). \quad \textbf{(2.50)}$$

where $\xi = \sigma x + vy + Rz - \frac{\kappa t^{\beta}}{\beta}$.

2.4 Application of GERFM to (2.11)

According to the homogeneous balance principle, it is clear to have $2N+2 = 3N, N = 2$ for (2.11). Therefore, the solution can be written as follows:

$$w(\xi) = A_0 + A_1\phi(\xi) + \frac{B_1}{\phi(\xi)} + A_2\phi(\xi)^2 + \frac{B_2}{\phi(\xi)^2}. \quad \textbf{(2.51)}$$

Inserting (2.51) to (2.11), we obtain a system of algebraic equations made up of tedious and rather long equations. Solving this system with computer program, we get the following results:

Using the values in Group 1 and (2.13) for Eq. (2.11) , we have

Set 1.1:

$$\left\{\begin{array}{c} A_0 = \frac{25A_2}{4}, A_1 = 5A_2, B_1 = 0, \\ B_2 = 0, Q = \frac{1}{4}i\ A_2, \\ R = 1/2\sqrt{4\kappa^2 - 4v^2 - 4\sigma^2 + A_2}. \end{array}\right\} \quad \textbf{(2.52)}$$

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Placing values in Eqs. (2.51) and (2.13), yields the following solution

$$w_{1.1}(\xi) = \frac{1}{4}\frac{(\cosh(\xi)-1)A_2}{\cosh(\xi)+1}. \quad \textbf{(2.53)}$$

Then we get the following solution of Eq. (1.6):

$$L_{2_{1.1}}(x,y,z,t) = \frac{1}{2}\ln\left(\frac{\left(\cosh\left(\sigma x+\nu y+Rz-\frac{\kappa t^{\beta}}{\beta}\right)-1\right)}{\left(\cosh\left(\sigma x+\nu y+Rz-\frac{\kappa t^{\beta}}{\beta}\right)+1\right)}\frac{A_2}{4}\right). \quad \textbf{(2.54)}$$

Set 1.2:

$$\left\{\begin{array}{c} A_0 = \frac{25}{144}\ B_2,\ A_1 = 0,\ A_2 = 0, \\ B_1 = \frac{5}{6}\ B_2,\ B_2 = \ B_2, Q = \frac{1}{144}\ i\ B_2, \\ R = \frac{\sqrt{B_2 - 144\,\nu^2 - 144\,\sigma^2 + 144\,\kappa^2}}{12}. \end{array}\right\} \quad \textbf{(2.55)}$$

Placing values in Eqs. (2.51) and (2.13), yields the following solution

$$w_{1.2}(\xi) = \frac{1}{144}\frac{B_2\,(5\sinh(\xi)-12+13\cosh(\xi))}{5\sinh(\xi)+12+13\cosh(\xi)}. \quad \textbf{(2.56)}$$

Then we get the following solution of Eq. (1.6):

$$L_{2_{1.2}}(x,y,z,t) = \frac{1}{2}\ln\left(\frac{1}{144}\ B_2\ \frac{(5\sinh(\xi)-12+13\cosh(\xi))}{(5\sinh(\xi)+12+13\cosh(\xi))}\right) \quad \textbf{(2.57)}$$

where $\xi = \sigma\, x + \nu y + Rz - \frac{\kappa\, t^{\beta}}{\beta}$.

Using the values in Group 2 and (2.20) for Eq. (2.11) , we have

Set 2.1:

$$\left\{\begin{array}{c} A_0 = iQ, A_1 = -2\,iQ, A_2 = iQ, \\ B_1 = 0, B_2 = 0, \\ R = \frac{1}{2}\sqrt{iQ - 4\,\sigma^2 - 4\,\nu^2 + 4\,\kappa^2} \end{array}\right\} \quad \textbf{(2.58)}$$

Placing values in Eqs. (2.51) and (2.20), yields the following solution

$$w_{2.1}(\xi) = \frac{iQ(\sinh(\xi))^2}{(\cosh(\xi))^2}. \quad \textbf{(2.59)}$$

Then we get the following solution of Eq. (1.6):

$$L_{2_{2.1}}(x,y,z,t) = \frac{1}{2}\ln\left(\frac{iQ\left(\sinh\left(\sigma x+\nu y+Rz-\frac{\kappa t^{\beta}}{\beta}\right)\right)^2}{\left(\cosh\left(\sigma x+\nu y+Rz-\frac{\kappa t^{\beta}}{\beta}\right)\right)^2}\right). \quad \textbf{(2.60)}$$

Using the values in Group 3 and (2.24) for Eq. (2.11) , we have

Set 3.1:

$$\left\{\begin{array}{c} A_0 = A_1,, A_2 = \frac{1}{4}A_1, B_1 = 0, \\ B_2 = 0, Q = \frac{1}{4}i\ A_1, \\ R = \frac{\sqrt{16\,\kappa^2 + A_1 - 16\,\nu^2 - 16\,\sigma^2}}{4}. \end{array}\right\} \quad \textbf{(2.61)}$$

Placing values in Eqs. (2.51) and (2.24), yields the following solution

$$w_{3.1}(\xi) = \frac{A_1\,(\sinh(\xi))^2}{4(\cosh(\xi))^2}. \quad \textbf{(2.62)}$$

Then we get the following solution of Eq. (1.6):

$$L_{2_{3.1}}(x,y,z,t) = \frac{1}{2}\ln\left(\frac{1}{4}\ A_1\ \frac{\left(\sinh\left(\sigma x+\nu y+Rz-\ \frac{\kappa t^{\beta}}{\beta}\right)\right)^2}{\left(\cosh\left(\sigma x+\nu y+Rz-\frac{\kappa t^{\beta}}{\beta}\right)\right)^2}\right). \quad \textbf{(2.63)}$$

Using the values in Group 5 and (2.32) for Eq. (2.11) , we have

Set 5.1:

$$\left\{\begin{array}{c} A_0 = A_1,\ A_2 = \frac{1}{4}\ A_1, \\ B_1 = 0,\ B_2 = 0, Q = \frac{1}{4}i\ A_1, \\ R = \frac{\sqrt{-16\,\nu^2 + A_1 + 16\,\kappa^2 - 16\,\sigma^2}}{4}. \end{array}\right\} \quad \textbf{(2.64)}$$

Placing values in Eqs. (2.51) and (2.32), yields the following solution

$$w_{5.1}(\xi) = \frac{A_1\,(\cosh(\xi))^2}{4(\sinh(\xi))^2}. \quad \textbf{(2.65)}$$

Then we get the following solution of Eq. (1.6):

$$L_{2_{5.1}}(x,y,z,t) = \frac{1}{2}\ln\left(\frac{A_1}{4}\frac{\cosh\left(\sigma x+\nu y+Rz-\frac{\kappa t^{\beta}}{\beta}\right)^2}{\sinh\left(\sigma x+\nu y+Rz-\frac{\kappa t^{\beta}}{\beta}\right)^2}\right). \quad \textbf{(2.66)}$$

Set 5.2:

$$\left\{\begin{array}{c} A_0 = \frac{4}{9}\ B_2,\ A_1 = 0,\ A_2 = 0, \\ B_1 = \frac{4}{3}B_2, Q = \frac{1}{9}i\ B_2, \\ R = \frac{\sqrt{36\,\kappa^2 + B_2 - 36\,\sigma^2 - 36\,\nu^2}}{6}\ . \end{array}\right\} \quad \textbf{(2.67)}$$

Placing values in Eqs. (2.51) and (2.32), yields the following solution

$$w_{5.2}(\xi) = \frac{1}{9}\frac{B_2\left(-4\cosh(\xi)\sinh(\xi)+5\,(\cosh(\xi))^2-1\right)}{-4\cosh(\xi)\sinh(\xi)+5\,(\cosh(\xi))^2-4}. \quad \textbf{(2.68)}$$

Then we get the following solution of Eq. (1.6):

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$$L_{25_{.2}}(x,y,z,t) = \frac{1}{2}\ln\left(\frac{B_2}{9}\frac{-4\cosh(\xi)\sinh(\xi)+5\left(\cosh(\xi)\right)^2-1}{-4\cosh(\xi)\sinh(\xi)+5\left(\cosh(\xi)\right)^2-4}\right). \quad \textbf{(2.69)}$$

where $\xi = \sigma x + vy + Rz - \frac{\kappa t^{\beta}}{\beta}$.

Using the values in Group 6 and (2.39) for Eq. (2.11) , we have

Set 6.1:

$$\left\{\begin{array}{c} A_0 = \frac{9}{4}A_2,\ A_1 = -3\ A_2,\ B_1 = 0, \\ B_2 = 0, Q = \frac{1}{4}i\ A_2, \\ R = \frac{\sqrt{A_2 - 16\sigma^2 + 16\kappa^2 - 16v^2}}{4}. \end{array}\right\} \quad \textbf{(2.70)}$$

Placing values in Eqs. (2.51) and (2.39), yields the following solution

$$w_{6.1}(\xi) = \frac{1}{4}\frac{A_2\left(\sinh(\xi)\right)^2}{\left(\cosh(\xi)\right)^2}. \quad \textbf{(2.71)}$$

Then we get the following solution of Eq. (1.6):

$$L_{26_{.1}}(x,y,z,t) = \frac{1}{2}\ln\left(\frac{1}{4}A_2\frac{\sinh\left(\sigma x + vy + Rz - \frac{\kappa t^{\beta}}{\beta}\right)^2}{\cosh\left(\sigma x + vy + Rz - \frac{\kappa t^{\beta}}{\beta}\right)^2}\right). \quad \textbf{(2.72)}$$

Using the values in Group 7 and (2.43) for Eq. (2.11) , we have

Set 7.1:

$$\left\{\begin{array}{c} A_0 = -\kappa^2 + v^2 + \sigma^2 + R^2, \\ A_1 = 4\sigma^2 - 4\kappa^2 + 4v^2 + 4R^2, \\ A_2 = 4\sigma^2 - 4\kappa^2 + 4v^2 + 4R^2, \\ B_1 = 0, B_2 = 0, \\ Q = i(-\kappa^2 + v^2 + \sigma^2 + R^2). \end{array}\right\} \quad \textbf{(2.73)}$$

Placing values in Eqs. (2.51) and (2.43), yields the following solution

$$w_{7.1}(\xi) = \frac{(\cosh(\xi)-1)(-\kappa^2+v^2+\sigma^2+R^2)}{1+\cosh(\xi)}. \quad \textbf{(2.74)}$$

Then we get the following solution of Eq. (1.6):

$$L_{27_{.1}}(x,y,z,t) = \ln\left(\frac{(-\kappa^2+v^2+\sigma^2+R^2)\left(\cosh\left(\sigma x + vy + Rz - \frac{\kappa t^{\beta}}{\beta}\right)-1\right)}{1+\cosh\left(\sigma x + vy + Rz - \frac{\kappa t^{\beta}}{\beta}\right)}\right). \quad \textbf{(2.75)}$$

Using the values in Group 8 and (2.47) for Eq. (2.11) , we have

Set 8.1:

$$\left\{\begin{array}{c} A_0 = \frac{9}{16}B_2, A_1 = 0, A_2 = 0, \\ B_1 = \frac{3}{2}B_2, B_2 = B_2, Q = \frac{1}{16}iB_2, \\ R = \frac{\sqrt{-16\sigma^2 + B_2 - 16v^2 + 16\kappa^2}}{4}. \end{array}\right\} \quad \textbf{(2.76)}$$

Placing values in Eqs. (2.51) and (2.47), yields the following solution

$$w_{8.1}(\xi) = 1/16\,\frac{B_2\left(-3\sinh(\xi)-4+5\cosh(\xi)\right)}{-3\sinh(\xi)+5\cosh(\xi)+4} \quad \textbf{(2.77)}$$

Then we get the following solution of Eq. (1.6):

$$L_{28_{.1}}(x,y,z,t) = \ln\left(\frac{B_2}{16}\frac{\left(-3\sinh(\xi)-4+5\cosh(\xi)\right)}{\left(-3\sinh(\xi)+5\cosh(\xi)+4\right)}\right). \quad \textbf{(2.78)}$$

where $\xi = \sigma x + vy + Rz - \frac{\kappa t^{\beta}}{\beta}$.

3. Results and Discussion

3.1 Graphical Illustrations

In this section, we show two-dimensional and three-dimensional drawings for some solutions obtained by assigning appropriate values to the parameters, to help clarify the solutions we presented. In Figure 1, 3D plots of the solution $L_{1_{1.1}}$ (2.16) with the parameters $A_2 = 0.4, R = 0.2, v = 0.1, \kappa = 0.2, \sigma = 0.3, \beta = 0.5, y = 1$ for different times, namely $t = 1,15,25$, is given. If the 3D graph is examined, the movement of the wave over time can be observed. Figure 2 shows that the density plot of $L_{1_{1.1}}$ (2.16) with $A_2 = 0.4, R = 0.2, v = 0.1, \kappa = 0.2, \sigma = 0.3, \beta = 0.5, y = 1$ for $t = 1$. In Figure 3, 2D plot of the solution $L_{1_{1.1}}$ (2.16) at $A_2 = 0.4, R = 0.2, v = 0.1, \kappa = 0.2, \sigma = 0.3, t = 1, y = 1, x = 1$. While drawing this 2D graph, different values of the β which is the conformable derivative's order were taken into account. In Figure 4, 3D and density plots of $L_{15_{.1}}$ (2.35) with $A_2 = 0.2, R = 0.15, v = 0.2, \kappa = 0.3, \sigma = 0.5, , t = 1, y = 1$ for $\beta = 0.15$ are given, respectively. Figure 5 demonstrate that 2D plot of $L_{15_{.1}}$ (2.35) with $A_2 = 0.2, R = 0.15, v = 0.2, \kappa = 0.3, \sigma = 0.5, , t = 1, y = 1, x = 1$ for different values of β.

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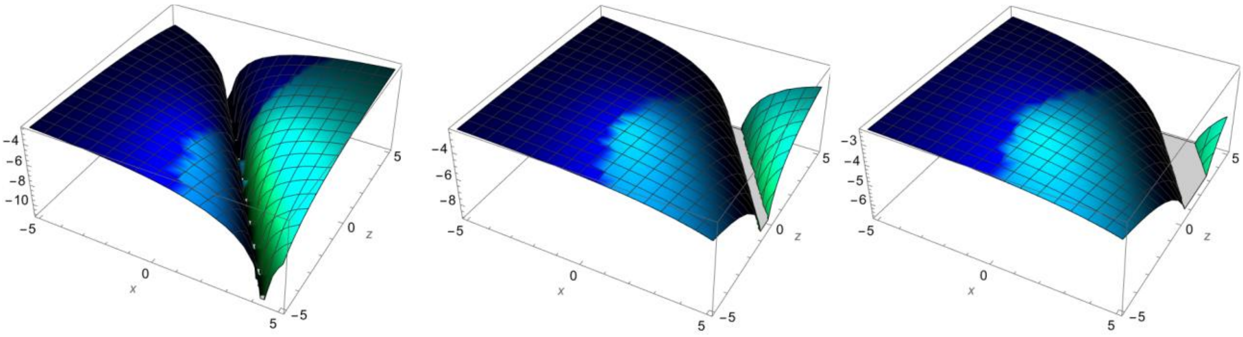

Figure 1. 3D plots of $L_{1_{1.1}}$ (2.16) with $A_2 = 0.4, R = 0.2, v = 0.1, \kappa = 0.2, \sigma = 0.3, \beta = 0.5, y = 1$ for $t = 1,15,25$, respectively.

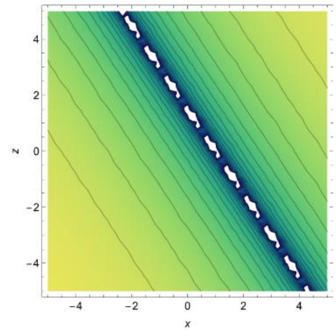

Figure 2. Density plot of $L_{1_{1.1}}$ (2.16) with $A_2 = 0.4, R = 0.2, v = 0.1, \kappa = 0.2, \sigma = 0.3, \beta = 0.5, y = 1$ for $t = 1$.

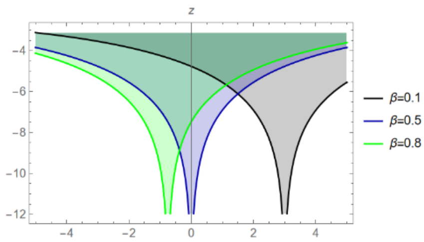

Figure 3. 2D plot of $L_{1_{1.1}}$ (2.16) with $A_2 = 0.4, R = 0.2, v = 0.1, \kappa = 0.2, \sigma = 0.3, t = 1, y = 1, x = 1$ for $\beta = 0.1,0.5,0.8$.

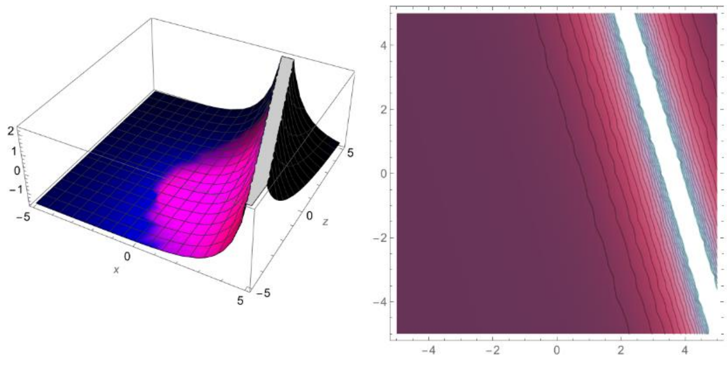

Figure 4. 3D and density plots of $L_{1_{5.1}}$ (2.35) with $A_2 = 0.2, R = 0.15, v = 0.2, \kappa = 0.3, \sigma = 0.5, , t = 1, y = 1$ for $\beta = 0.15$, respectively.

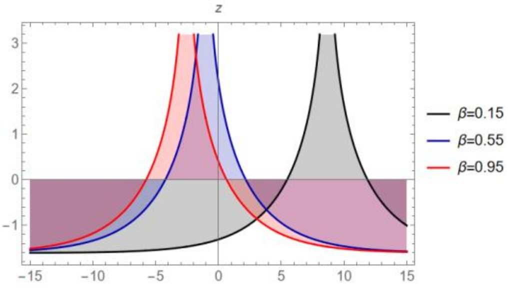

Figure 5. 2D plot of $L_{1_{5.1}}$ (2.35) with $A_2 = 0.2, R = 0.15, v = 0.2, \kappa = 0.3, \sigma = 0.5, , t = 1, y = 1, x = 1$ for $\beta = 0.15, 0.55, 0.95$, respectively.

3.2 Comparision

In this section, we compare our performed solutions with Ali et al. [25], results for Eq. (2.6) for case one. Wherein Ali et al. [25] considered the $(3+1)$ conformal time derivative generalized q-deformed Sinh-Gordon equation by using G'/G expansion method. The comparison is ascertained as follows

Table 1. Comparison of solutions

Ali et al. [25]	Our solution
For $H_0 = -\frac{\sqrt{q}\sigma^2}{\sqrt{(\sigma^2-4v)^2}}, H_1 = -\frac{4\sqrt{q}\sigma}{\sqrt{(\sigma^2-4v)^2}},$ $H_2 = -\frac{4\sqrt{q}}{\sqrt{(\sigma^2-4v)^2}}, \Upsilon = \pm\sqrt{\kappa^2 - \frac{\sqrt{q}}{\sqrt{(\sigma^2-4v)^2}} - v^2 - \varrho^2},$ $M_1 = \frac{\sqrt{q}(\sigma^2-4v)}{\sqrt{(\sigma^2-4v)^2}}$ and $\sigma^2 - 4v > 0$ the solution is	For $A_0 = \sqrt{Q}, A_1 = -2\sqrt{Q}, A_2 = \sqrt{Q}, B_1 = B_2 = 0,$ $M_1 = -\sqrt{Q}, R = \frac{\sqrt{\sqrt{Q}+4\ \ \kappa^2-4\,\sigma^2-4\,v^2}}{2}$ the solution is
$L_{1,2}(x,y,z,\tau) = \ln(H_0 + H_1\left(\frac{-\sigma}{2} + \frac{1}{2}\sqrt{\sigma^2-4v}\,\frac{g_1\sinh\frac{1}{2}\sqrt{\sigma^2-4v}\varsigma + g_2\cosh\frac{1}{2}\sqrt{\sigma^2-4v}\varsigma}{g_1\cosh\frac{1}{2}\sqrt{\sigma^2-4v}\varsigma + g_2\sinh\frac{1}{2}\sqrt{\sigma^2-4v}\varsigma}\right) + H_2\left(\frac{-\sigma}{2} + \frac{1}{2}\sqrt{\sigma^2-4v}\,\frac{g_1\sinh\frac{1}{2}\sqrt{\sigma^2-4v}\varsigma + g_2\cosh\frac{1}{2}\sqrt{\sigma^2-4v}\varsigma}{g_1\cosh\frac{1}{2}\sqrt{\sigma^2-4v}\varsigma + g_2\sinh\frac{1}{2}\sqrt{\sigma^2-4v}\varsigma}\right)^2$	$L_{1_{2.1}}(x,y,z,t) = \ln\left(\sqrt{Q}\,\frac{\left(\sinh\left(\sigma\,x+vy+Rz-\frac{\kappa\,t^\beta}{\beta}\right)\right)^2}{\left(\cosh\left(\sigma\,x+vy+Rz-\frac{\kappa\,t^\beta}{\beta}\right)\right)^2}\right)$

When the solution obtained in [25] and the solution obtained in this study are compared, it is seen that the solutions are structurally similar if the parameters are selected appropriately. When other solutions are compared, it is seen that different solutions are obtained thanks to the method applied in this study.

4. Conclusions

In this paper, soliton solutions of the $(3+1)$ conformal time derivative generalized q-deformed Sinh-Gordon equation are constructed using the generalized exponential rational function approach. The equation containing both the conformal time derivative and the generalized q-deformation was first converted to an ordinary differential equation. The GERFM, whose effectiveness and power has been proven by many studies, has been applied to the obtained ordinary differential equation and solutions have been found. Thus, the solutions of the original equation were obtained. These solutions are soliton solutions that have the extraordinary property of maintaining their uniformity in interaction with others. 3D and 2D drawings are given for some analytical solutions to show more features for the proposed model. The analytical solutions allow graphing soliton solutions of type dark and bright. In addition, the effectiveness of the applied method was emphasized with the Comparison section. We hope that the results obtained will be a guide for future research.

Author's Contributions

Yeşim Sağlam Özkan: Drafted and wrote the manuscript, performed the experiment and result analysis.

Ethics

There are no ethical issues after the publication of this manuscript.

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