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Research Article

# Fuzzifying bornivorous sets of fuzzifying bornological linear spaces

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## Abstract

The main purpose of this paper is to introduce a notion of fuzzifying bornivorous sets of fuzzifying bornological linear spaces. In particular, we provide an example of fuzzifying bornivorous sets on a fuzzifying topological linear space with its von Neumann bornology. Furthermore, the description of fuzzifying open sets of fuzzifying bornological linear spaces is showed and its equivalent illustration is discussed as well. In addition, we study the dual relationship of fuzzifying open and close sets. The fuzzifying topological space induced by fuzzifying open sets is also discussed.

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**Keywords.** fuzzifying bornivorous set, fuzzifying bornological linear space, fuzzifying open set, fuzzifying topological linear space

## 1. Introduction

As the study of general topology began with the property of families of open balls in metric spaces, the investigation of bounded sets in topological spaces has been somewhat neglected. The axiomatization of the properties of bounded sets in metric spaces was initially introduced by Hu [11, 12]. Hu's pioneering work led to the introduction of the concept of bornologies on a set. Hogle-Nled and Schaefar [10, 21] further developed the theory of bornological spaces within the context of topological linear spaces. Meyer [16] discussed the theory of smooth representations of locally compact groups on bornological linear spaces, affirming the usefulness of this theory in representation studies. Based on Almeida and Barreida's [2] results on bornological linear spaces, Meson and Vericat [17] investigated topological entropy. Later on, Beer and Levi [5] pointed out the potential application of bornological vector space theory in optimization theory. Furthermore, the theory of bornological spaces plays a fundamental role in other research fields such as convergence structures on hyperspaces [3,4,15] and topologies on function spaces [6,8,9,18].

In 2011, Abel and Šostak [1] extended the theory of bornological spaces to fuzzy sets. They introduced bornologies on complete lattices and defined the concept of an $L$-bornology. In [1], it was demonstrated that the category $L$-Born of $L$-bornological

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spaces is a topological construct. Subsequently, Paseka et al. [19,20] provided a definition of $L$-bornological vector spaces and identified the necessary and sufficient conditions on a complete lattice for the category $L$-Born to be topological. In 2016, Šostak and Uļjane [22] proposed an alternative approach to the fuzzification of the concept of bornology and developed a construction of an $L$-valued bornology on a set based on a family of crisp bornologies on the same set. Additionally, Uļjane and Šostak [24] extended $L$-valued bornologies to a completely distributive lattice. In recent years, Jin and Yan [13, 14] provided a specific description of fuzzifying bornological linear spaces and discussed the necessary and sufficient conditions for fuzzifying bornologies to be compatible with linear structures. Moreover, they studied the characterizations of Mackey-convergence and separation in fuzzifying bornological linear spaces.

Building upon [14], we continue to study the theory of fuzzifying bornological linear spaces in this paper. Our objective is to extend the important concept of bornivorous sets to fuzzifying bornological linear spaces and analyze some of its properties. The structure of the paper is as follows: In Section 2, we introduce the necessary concepts and fundamental results that will be employed throughout the paper. In Section 3, we provide an equivalent description of fuzzifying bornivorous sets in fuzzifying bornological linear spaces and present an illustrative example of fuzzifying bornivorous sets on a topological linear space with its von Neumann bornology. We then establish the description of fuzzifying open sets in fuzzifying bornological linear spaces and investigate its equivalent representation. Finally, we examine the dual relationship between fuzzifying open and closed sets, as well as the fuzzifying topological space induced by fuzzifying open sets.

## 2. Preliminaries

In this section, we recall some necessary notions and fundamental results which are used in this paper.

Throughout this paper, $X$ always denotes a universe of discourse. $2^X$ and $\mathscr{F}(X)$ denote the classes of all crisp and fuzzy subsets, respectively, of $X$. $\mathbb{K}$ represents a field of real or complex numbers, the symbol $\theta$ denotes the neutral element of a linear space and $*$ means the continuous t-norm.

**Definition 2.1** ([26]). A fuzzifying topology is a mapping $\tau : 2^X \to [0,1]$ such that

(FY1) $\tau(X) = \tau(\emptyset) = 1$;

(FY2) $\tau(U \cap V) \geq \tau(U) \wedge \tau(V)$ for all $U, V \in 2^X$;

(FY3) $\tau\left(\bigcup_{j\in J} U_j\right) \geq \bigwedge_{j\in J} \tau(U_j)$ for every family $\{U_j | j \in J\} \subseteq 2^X$.

$F : (X, \tau) \to (Y, \delta)$ is called continuous with respect to the two fuzzifying topologies $\tau$ and $\delta$ if $\delta(V) \leq \tau(F^{\leftarrow}(V))$ holds for all $V \in 2^Y$.

**Definition 2.2** ([26]). Let $(X, \tau)$ be a fuzzifying topological space. For any $x \in X$, $N_x \in \mathscr{F}\left(2^X\right)$, called a fuzzifying neighborhood system of $x$, is defined as follows. For any $A \in 2^X$,

$$A \in N_x := (\exists B \in 2^X)\left((x \in B \subseteq A) \wedge (B \in \tau)\right).$$

According to the terminology adopted in [10], a subset $B \subseteq X$ in a linear space $X$ is said to be balanced if $\lambda B \subseteq B$ whenever $\lambda \in \mathbb{K}$ and $|\lambda| \leq 1$.

**Theorem 2.3** ([25]). *Let $(X, \tau)$ be a fuzzifying topological linear space on $\mathbb{K}$ and $N_\theta(\cdot)$ be its corresponding fuzzifying neighborhood system of the neutral element. Then it has the following properties:*

*(P1)* $N_\theta(X) = 1$;
*(P2)* $\forall U \subseteq X, N_\theta(U) > 0 \Rightarrow \theta \in U$;
*(P3)* $\forall U, V \subseteq X, N_\theta(U \cap V) = N_\theta(U) \wedge N_\theta(V)$;
*(P4)* $\forall W \subseteq X, N_\theta(W) \leq \bigvee\limits_{U+V \subseteq W} N_\theta(U) \wedge N_\theta(V)$;
*(P5)* $\forall U \subseteq X, x \in X, N_\theta(U) > 0 \Rightarrow \exists \varepsilon > 0$ *such that* $kx \in U$ *for all* $|k| < \varepsilon$;
*(P6)* $\forall U \subseteq X, N_\theta(U) > a$ *implies there exists a balanced set* $V \subseteq U$ *such that* $N_\theta(V) > a$.

*Conversely, let* $X$ *be a linear space over* $\mathbb{K}$ *and consider a function* $N_\theta(\cdot) : 2^X \to [0,1]$ *which satisfies the conditions (P1)-(P6). Then there exists a fuzzifying topology* $\tau_N$ *on* $X$ *such that* $(X, \tau_N)$ *be a fuzzifying topological linear space and* $N_\theta(\cdot)$ *is a fuzzifying neighborhood system of the neutral element.*

**Definition 2.4** ([7])**.** A cl-monoid is a tuple $(L, \leq, \wedge, \vee, *)$ where $(L, \leq, \wedge, \vee)$ is a complete lattice and operation $* : L \times L \to L$ satisfies conditions:

(0t) $*$ is monotone: $a \leq b \Longrightarrow a * c \leq b * c$ for all $c \in L$,

(1t) $*$ is commutative: $a * b = b * a$ for all $a, b \in L$,

(2t) $*$ is associative: $a * (b * c) = (a * b) * c$ for all $a, b, c \in L$,

(3t) $a * 1_L = a, a * 0_L = 0_L$ for all $a \in L$,

(4t) operation $*$ distributes over arbitrary joins: $a * (\bigvee\limits_{i \in \Gamma} b_i) = \bigvee\limits_{i \in \Gamma} (a * b_i)$ for every $a \in L$ and for all $\{b_i : i \in \Gamma\} \subseteq L$.

If the operation $* : L \times L \to L$ satisfies conditions (0t)-(3t), it ia also said to be a triangular norm or $t$-norm on $L$. For a triangular norm $*$, an implication operator determined by $*$ is denoted $\to_*$, i.e., $a \to_* b = \bigvee\{c \in L : a * c \leq b\}$, $\forall\, a, b \in L$. Specially, if we restrict triangular norm $*$ is Łukasiewicz norm, i.e., $a *_L b = \max\{a + b - 1, 0\}$, then Łukasiewicz implication operator $a \to_L b = \min\{1 - a + b, 1\}$.

According to the terminology [28], a triangular norm $*$ on a complete lattice $L$ is called left (right) continuous if and only if for each $a \in L, \{a_t\}_{t \in \Gamma} \subseteq L$, $a * (\bigvee\limits_{t \in \Gamma} a_t) = \bigvee\limits_{t \in \Gamma} (a * a_t) (a * (\bigwedge\limits_{t \in \Gamma} a_t) = \bigwedge\limits_{t \in \Gamma} (a * a_t))$. Moreover, a triangular norm $*$ is said to be continuous if it is left continuous and right continuous. Specially, if $L = [0,1]$ and $*$ is a left continuous on $[0,1]$, then

$$a * b = a * \left(\bigvee\{b_t \ : \ b_t < b\}\right) = \bigvee\{a * b_t \ : \ b_t < b\} \text{ for all } a, b \in [0,1].$$

**Definition 2.5** ([23])**.** Let $(X, \tau)$ be a fuzzifying topological linear space. Then the unary fuzzy predicates $Bd \in \mathscr{F}\left(2^X\right)$, called fuzzy boundedness, is defined in terms of mathematical logics as follows:

$$Bd(A) := \left(\forall V \in 2^X\right)(V \in N_\theta \to_L (\exists \lambda \in \mathbb{K})(A \subseteq \lambda V))$$

for any $A \in 2^X$.

In our work we base on the Łukasiewicz fuzzy logic, and therefore this logical formula actually means that degree to which $A$ is bounded is

$$[Bd(A)] = \bigwedge_{U \subseteq X} \{1 - N_\theta(U) \,|\forall \lambda \in \mathbb{K}, A \nsubseteq \lambda U\}.$$

**Definition 2.6** ([22])**.** Given a cl-monoid $(L, \leq, \wedge, \vee, *)$, an $(L, *)$-valued bornology on a set $X$ is a mapping $\mathscr{B} : 2^X \to L$ satisfying the following conditions:

(B1) $\forall x \in X \Rightarrow \mathscr{B}(\{x\}) = 1_L$;

(B2) If $U \subseteq V \subseteq X$ then $\mathscr{B}(V) \leq \mathscr{B}(U)$;

(B3) $\forall U, V \subset X \Rightarrow \mathscr{B}(U \cup V) \geq \mathscr{B}(U) * \mathscr{B}(V)$.

**Definition 2.7** ([22])**.** A mapping $f : (X, \mathscr{B}_X) \to (Y, \mathscr{B}_Y)$ of $(L,*)$-valued bornological spaces is called bounded if $\mathscr{B}_X(A) \leq \mathscr{B}_Y(f(A))$ for every $A \in 2^X$. The degree to which $f$ is bounded as

$$[Bd(f)] = \bigwedge_{A \subseteq X} (1 - \mathscr{B}_X(A) + \mathscr{B}_Y(f^{\to}(A))).$$

for every $A \in 2^X$. Where the notation $f^{\to}(A)$ is the image of a set $A$.

If $[Bd(f)] = 1$, or equivalently, $\mathscr{B}_X(A) \leq \mathscr{B}_Y(f^{\to}(A)))$ for all $A \subseteq X$, we say that $f$ is bounded.

**Theorem 2.8** ([14])**.** *Let $(X, \mathscr{B})$ be a fuzzifying bornological space. Then $(X, \mathscr{B})$ is a fuzzifying bornological linear space if and only if $\mathscr{B}$ satisfies the following conditions:*

*(B4)* $\mathscr{B}(U + V) \geq \mathscr{B}(U) * \mathscr{B}(V)$;

*(B5)* $\mathscr{B}(\lambda U) \geq \mathscr{B}(U)$, *for all* $\lambda \in \mathbb{K}$;

*(B6)* $\mathscr{B}\left(\bigcup_{|\alpha| \leq 1} \alpha U\right) \geq \mathscr{B}(U)$.

By Example 3.2 in [14], $(X, Bd(\cdot))$ is a fuzzifying bornological linear space for any fuzzifying topological linear space. As we have known, the family of all bounded sets in classical topological linear spaces forms a linear bornology, and this linear bornology is called von Neumann bornology [10]. According to the notation of ordinary bornological linear spaces, we also say that $Bd(\cdot)$ is a fuzzifying von Neumann bornology.

**Definition 2.9** ([14])**.** Let $(X, \mathscr{B})$ be a fuzzifying bornological linear space and let $\{x_n\}$ be a sequence in $X$. The degree to which $x_n$ converges bornologically to a point $x \in X$ is

$$[x_n \overset{M}{\to} x] = \bigvee_{\substack{A \in Bal(X) \\ \lambda_n \to 0}} \{\mathscr{B}(A) : \forall n \in \mathbb{N}, x_n - x \in \lambda_n A\},$$

where $Bal(X)$ means the family of all balanced sets on $X$.

In classical bornological linear spaces, bornological convergence of sequences is also called Mackey convergence. In order to distinguish topological convergence from sequence, throughout this paper, we always denote $\{x_n\}$ is convergent to $x$ bornologically as $x_n \overset{M}{\to} x$.

**Definition 2.10** ([14])**.** Let $(X, \mathscr{B})$ be a fuzzifying bornological linear space. Then a unary predicate $BC \in \mathscr{F}(2^X)$ called bornologically closed is defined as follows:

$$A \in BC := (\forall \{x_n\} \subseteq A)(x_n \overset{M}{\to} x) \to_L (x \in A).$$

Where the notation $\to_L$ means the Łukasiewicz residuum.

Intuitively, the logic formula of $A \in BC$ actually means that the degree to which $(X, \mathscr{B})$ is bornologically closed is

$$[BC(A)] = \bigwedge_{\substack{\{x_n\} \subseteq A \\ x \notin A}} \bigwedge_{\substack{B \in Bal(X) \\ \lambda_n \to 0}} \{1 - \mathscr{B}(B) : \forall n \in \mathbb{N}, x_n - x \in \lambda_n B\}.$$

**Definition 2.11** ([14])**.** Let $\Sigma$ be the family of all fuzzifying bornological linear spaces. Then a unary predicate $T \in \mathscr{F}(\Sigma)$ called separation is defined in terms of mathematical logics as follows:

$$(X, \mathscr{B}) \in T := (\forall M \in Svec(X)) \rightarrow_L (M \in \mathscr{B}) \wedge (M = \{\theta\}),$$

where $Svec(X)$ denotes the family of all linear subspaces of $X$ and $\rightarrow_L$ means the Łukasiewicz residuum.

Intuitively, the logic formula of $(X, \mathscr{B}) \in T$ actually means that the degree to which $(X, \mathscr{B})$ is separated is

$$[T\,(X, \mathscr{B})] = \bigwedge_{\substack{M \neq \{\theta\} \\ M \in Svec(X)}} \{1 - \mathscr{B}\,(M)\}.$$

**Definition 2.12** ([27]). Let $(X, \tau)$ be a fuzzifying topological space. The value $[C_I(X, \tau)] = \bigwedge_{x \in X} \bigvee_{\mathscr{U}_x \vdash N_x} FC(\mathscr{U}_x)$ is called the degree to which $(X, \tau)$ is first countable, where $\mathscr{U}_x \vdash N_x$ means that $\mathscr{U}_x$ is a mapping from $2^X \rightarrow [0, 1]$ satisfying $N_x(U) = \bigvee_{V \subseteq U} \mathscr{U}_x(V)$, and $FC(\mathscr{U}_x) = 1 - \bigwedge \{r \mid C((\mathscr{U}_x)_r)\}$, where $(\mathscr{U}_x)_r = \{A \subseteq X \mid \mathscr{U}_x(A) > r\}$ and the notation $C((\mathscr{U}_x)_r$ means that the set $(\mathscr{U}_x)_r$ is at most countable.

## 3. Main results

The purpose of this section is to introduce the concept of fuzzifying bornivorous sets in fuzzifying bornological linear spaces. We provide a comprehensive description of fuzzifying open sets in such linear spaces and explore the dual relationship between fuzzifying open and closed sets. Additionally, we delve into the discussion of the fuzzifying topological space induced by fuzzifying open sets.

**Definition 3.1.** Let $(X, \mathscr{B})$ be a fuzzifying bornological linear space. Then the mapping $BV : 2^X \rightarrow [0, 1]$ is called fuzzifying bornivorous if it defined as follows:

$$P \in BV := (\forall A \subseteq X)\,(A \in \mathscr{B}) \rightarrow_L (A \in Abs(P))\,,$$

where $Abs(P) = \{A :\ \exists \delta > 0, \forall \lambda \in \mathbb{K}, |\lambda| \geq \delta, A \subseteq \lambda P\}$ and $\rightarrow$ means the Łukasiewicz residuum.

Intuitively, the logic formula of $P \in BV$ actually means that the degree to which $P$ is a bornivorous set is

$$BV(P) = \bigwedge_{A \subseteq X} \{1 - \mathscr{B}\,(A) : A \notin Abs(P)\}.$$

**Example 3.2.** (Every neighborhood of $\theta$ of a fuzzifying topological linear space with its von Neumann bornology is bornivorous) Let $(X, \tau)$ be a fuzzifying topological linear space and $(X, Bd)$ be a fuzzifying bornological linear space (see [14]). Suppose that $N_\theta(\cdot)$ is defined as Definition 2.2 and $BV$ is given as Definition 3.1. Then

$$N_\theta(U) \leq BV(U)$$

for all $U \subseteq X$.

***Proof.*** At first we may prove that

$$[Bd\,(A)] = \bigwedge_{V \subseteq X} \{1 - N_\theta\,(V)\,|\forall \lambda \in \mathbb{K},\ A \nsubseteq \lambda V\} = \bigwedge_{V \subseteq X} \{1 - N_\theta\,(V)\,|A \notin Abs(V)\}.$$

In fact, let $\bigwedge_{V \subseteq X} \{1 - N_\theta\,(V)\,|A \notin Abs(V)\} < a$, then we have $V \subseteq X$ satisfying $1 - N_\theta\,(V) < a$ and $A \notin Abs(V)$. By Theorem 2.3 (P6), there exists a balanced set $W \subseteq V$ such that $N_\theta(W) > 1 - a$. This implies $A \nsubseteq \lambda W$ for all $\lambda \in \mathbb{K}$. However, if $A \subseteq \lambda_0 W$ for some $\lambda_0 \in \mathbb{K}$, where $\lambda_0 \neq 0$, then we examine the case where $A \subseteq 0W = \theta$. In this scenario, we have $[Bd\,(A)] = [Bd(\theta)] = 1$. Since $N_\theta\,(V) > 1-a$, it follows that $N_\theta\,(V) \neq 0$. According to Theorem 2.3 (P5), we conclude that $\theta \in V$. This deduction implies $A = \theta \in$

$Abs(V)$, which contradicts the fact that $A \notin Abs(V)$. Let $\delta = |\lambda_0| > 0$, for all $|\lambda| \geq \delta$, $\lambda \in \mathbb{K}$, it holds that $A \subseteq \lambda_0 W \subseteq \lambda W \subseteq \lambda V$. This signifies that $A \in Abs(V)$, which contradicts the fact that $A \notin Abs(V)$. Therefore $[Bd\,(A)] = \bigwedge_{V \subseteq X} \{1 - N_\theta\,(V)\,|\forall \lambda \in \mathbb{K},\ A \nsubseteq \lambda V\} \leq 1 - N_\theta(W) < a$. So,

$$[Bd\,(A)] = \bigwedge_{V \subseteq X} \{1 - N_\theta\,(V)\,|\forall \lambda \in \mathbb{K},\ A \nsubseteq \lambda V\} \leq \bigwedge_{V \subseteq X} \{1 - N_\theta\,(V)\,|A \notin Abs(V)\}.$$

On the contrary, supposed that $\bigwedge_{V \subseteq X} \{1 - N_\theta\,(V)\,|\forall \lambda \in \mathbb{K},\ A \nsubseteq \lambda V\} < a$, then there exists $V \subseteq X$ such that $1 - N_\theta(V) < a$ with $A \nsubseteq \lambda V$ for all $\lambda \in \mathbb{K}$. By Theorem 2.3 (P6), there is a balanced set $W \subseteq V$ such that $1 - N_\theta(W) < a$. In this case, it is easy to prove that $A \notin Abs(W)$. Thus, $\bigwedge_{V \subseteq X} \{1 - N_\theta\,(V)\,|A \notin Abs(V)\} \leq 1 - N_\theta(W) < a$. Hence $\bigwedge_{V \subseteq X} \{1 - N_\theta\,(V)\,|A \notin Abs(V)\} \leq [Bd(A)]$.

Finally, let $BV(U) < t$, then there exists $A \subseteq X$ such that $\mathscr{B}(A) > 1 - t$ with $A \notin Abs(U)$. Thus, $\mathscr{B}(A) = [Bd\,(A)] = \bigwedge_{V \subseteq X} \{1 - N_\theta\,(V)\,|A \notin Abs(V)\} > 1 - t$. Therefore, we have $N_\theta\,(V) < t$ whenever $A \notin Abs(V)$. Without loss of generality, let $V = U$. It deduces that $N_\theta\,(U) < t$ and hence, $N_\theta\,(U) \leq BV(U)$, which completes the proof. □

**Theorem 3.3.** *Let $(X, \tau)$ be a fuzzifying topological linear space, $\mathscr{B}$ be a Von Neumann fuzzifying bornology induced by $\tau$. Then $C_I(X, \tau) *_L BV(P) \leq N_\theta(P)$ for any balanced set $P \subseteq X$. Where $*_L$ denotes Łukasiewicz t-norm.*

***Proof.*** For any balanced set $P \subseteq X$, if there exists $a \in (0, BV(P) + [C_I(X, \tau)] - 1)$ such that $N_\theta(P) \leq a$. It follows that $a + 1 - [C_I(X, \tau)] < BV(P)$. Then there exist $\mathscr{U} \vdash N_\theta$ such that $a + 1 - FC(\mathscr{U}) < BV(P)$. Let $t \in (a + 1 - FC(\mathscr{U}), BV(P))$, by Definition 2.12, we have $a + \bigwedge\{r\ :\ C((\mathscr{U})_r)\} < t$. Thus there is $r_0 \in (0, 1)$ with $a + r_0 < t$ such that the set $\{U \subseteq X:\ \mathscr{U}(U) > r_0\}$ is at most countable. Further, the set $\{U \subseteq X:\ \mathscr{U}(U) > a + r_0\}$ is countable. We may assume that $\{U \subseteq X:\ \mathscr{U}(U) > a + r_0\} = \{U_1, U_2, \cdots,\}$ and $U_n \supseteq U_{n+1}$. Clearly, the family of sets $\{U_n\}$ is a countable balanced neighborhood base of $\theta$ in classical topological linear space $(X, \tau_{a+r_0})$, where $\tau_{a+r_0}$ denotes the topology generated by the family of $\{U \mid \tau(U) > a + r_0\}$ as a basis. Further, $\{\frac{1}{n}U_n\}$ is also a countable balanced neighborhood base of $\theta$. It follows that $\frac{1}{n}U_n \nsubseteq P$ for all $n \in \mathbb{N}$. Then there exists $x_n \in U_n$ such that $x_n \notin nP$ for all $n \in \mathbb{N}$. Since $P$ is a balanced set, we have $\{x_n\} \notin Abs(P)$. It is easy to check the sequence $\{x_n\}$ converges to $\theta$ with respect to $\tau_{a+r_0}$. Thus $\{x_n\}$ is a bounded set in $(X, \tau_{a+r_0})$. So $[Bd(\{x_n\})] \geq 1 - (a + t_0)$. Hence $BV(P) = \bigwedge_{A \subseteq X} \{1 - \mathscr{B}\,(A) : A \notin Abs(P)\} \leq 1 - \mathscr{B}(\{x_n\}) \leq a + r_0 < t$. It contradicts with the assumption $BV(P) > t$. Hence we have $C_I(X, \tau) *_L BV(P) \leq N_\theta(P)$. □

**Theorem 3.4.** *Let $(X, \mathscr{B})$ be a fuzzifying bornological linear space. For all $P, Q \subseteq X$, the following statements holds:*

*(1)* $\forall P \in 2^X, BV(P) > 0 \Longrightarrow \theta \in P$;

*(2)* $BV(P \bigcap Q) \geq BV(P) \wedge BV(Q)$;

*(3) if* $P \subseteq Q$, *then* $BV(P) \leq BV(Q)$;

*(4) for all* $\alpha \in \mathbb{K} \setminus \{0\}$, $BV(\alpha P) = BV(P)$;

*(5)* $BV\left(\bigcup_{|\alpha| \leq 1} \alpha P\right) \geq BV\,(P)$.

***Proof.*** (1) If $\theta \notin P$, then for all $\lambda \neq 0$, $\{\theta\} \nsubseteq \lambda P$, which means $P$ does not absorb $\{\theta\}$. Since $\mathscr{B}(\{\theta\}) = 1$, it deduces that $BV(P) = 0$. So we have the conclusion $BV(P) > 0 \Longrightarrow \theta \in P$ for all $P \subseteq X$.

(2) Let $BV(P\bigcap Q) < t$, then there exists $A \subseteq X$ such that $\mathscr{B}(A) > 1-t$ satisfying $A \notin Abs(P\bigcap Q)$, i.e., for all $\delta > 0$, there exists $|\lambda| \geq \delta$ such that $A \nsubseteq \lambda(P\bigcap Q)$. Therefore, we obtain $A \nsubseteq \lambda P$ or $A \nsubseteq \lambda Q$ for above $\lambda \in \mathbb{K}$. Hence, we have $BV(P) \wedge BV(Q) < t$, from which it follows that $BV(P\bigcap Q) \geq BV(P) \wedge BV(Q)$.

(3) Let $BV(Q) < t$. Then there exists $A \subseteq X$ such that $\mathscr{B}(A) > 1-t$ and $A \notin Abs(Q)$. Since $P \subseteq Q$, it is clear that $A \notin Abs(P)$. Thus, we get $BV(P) < t$, which means $BV(P) \leq BV(Q)$.

(4) Let $BV(\alpha P) < t$ for all $\alpha \in \mathbb{K}\backslash\{0\}$. Then there exists $A \subseteq X$ such that $\mathscr{B}(A) > 1-t$ and $A \notin Abs(\alpha P)$. It follows that $A \notin Abs(P)$ and $BV(P) < t$. So $BV(\alpha P) \geq BV(P)$. Similarly, the inequality $BV(\alpha P) \leq BV(P)$ holds. This completes the proof.

(5) Let $BV\left(\bigcup\limits_{|\alpha|\leq 1} \alpha P\right) < t$. Then there exists $A \subseteq X$ such that $\mathscr{B}(A) > 1-t$ and $A \notin Abs(\bigcup\limits_{|\alpha|\leq 1} \alpha P)$. Clearly, $A \notin Abs(P)$. It follows that $BV(P) < t$ and $BV\left(\bigcup\limits_{|\alpha|\leq 1} \alpha P\right) \geq BV(P)$. □

**Definition 3.5.** Let $(X, \mathscr{B})$ be a fuzzifying bornological linear space. Then the mapping $BO : 2^X \to [0,1]$ is called bornologically open if it defined as follows:

$$P \in BO := (\forall a \in P) \to (P - a \in BV),$$

where $P - a = \{p - a : \ p \in P\}$ and $\to$ means the Łukasiewicz residuum.

Intuitively, the logic formula of $P \in BO$ actually means that the degree to which $P$ is bornologically open is

$$BO(P) = \bigwedge_{a\in P} BV(P - a).$$

**Theorem 3.6.** *Let $(X, \mathscr{B})$ be a fuzzifying bornological linear space. Then the mapping of fuzzifying bornological open is a fuzzifying topology.*

***Proof.*** By Definition 2.1 and Definition 3.5, we only need to show that the mapping $BO : 2^X \to [0,1]$ satisfies the three conditions (FY1) to (FY3) in Definition 2.1.

(FY1) It is obvious that (FY1) holds since $BO(X) = BO(\emptyset) = 1$.

(FY2) Let $BO(U) \wedge BO(V) > t$. For any $c \in U\bigcap V$, we have $BV(U - c) > t$ and $BV(V - c) > t$. From Theorem 3.4, $BV(U \cap V - c) \geq BV(U - c) \wedge BV(V - c) > t$. It follows that $BO(U \wedge V) \geq t$. It deduces that $BO(U) \wedge BO(V) \leq BO(U \cap V)$.

(FY3) Suppose that $BO(\bigcup\limits_{j\in J} U_j) < t$, there exists $a \in \bigcup\limits_{j\in J} U_j$ such that $BV(\bigcup\limits_{j\in J} U_j - a) < t$. Further, there is $j_0 \in J$, $a \in U_{j_0}$. By Theorem 3.4, $BV(U_{j_0} - a) \leq BV(\bigcup\limits_{j\in J} U_j - a) < t$. Thus

$$\bigwedge_{j\in J} BO(U_j) \leq BO(U_{j_0}) = \bigwedge_{b\in U_{j_0}} (BV(U_{j_0} - b)) \leq BV(U_{j_0} - a) < t.$$

By the arbitrariness of $t$, $\bigwedge\limits_{j\in J} BO(U_j) \leq BO(\bigcup\limits_{j\in J} U_j)$. The proof is completed. □

**Remark 3.7.** As a consequence of the (Theorem 3.6), the mapping of fuzzifying bornological open $BO : 2^X \to [0,1]$ is a fuzzifying topology. It is called a fuzzifying topology induced by fuzzifying bornology $\mathscr{B}$. Here it is denoted by $\tau_{\mathscr{B}}$. We also call $(X, \tau_{\mathscr{B}})$ a fuzzifying topological space.

**Theorem 3.8.** *Let $(X, \mathscr{B}_X)$, $(Y, \mathscr{B}_Y)$ be fuzzifying bornological linear spaces and a linear mapping $f : X \to Y$ is bounded. Then the following inequality holds:*

$$BV_X(f^{\leftarrow}(P)) \geq BV_Y(P), \text{ for all } P \subseteq Y.$$

*Where the notation $f^{\leftarrow}(P)$ is the pre-image of a set $P$.*

***Proof.*** Let $BV_X(f^{\leftarrow}(P)) < t$. Then there exists $A \subseteq X$ such that $\mathscr{B}_X(A) > 1-t$ and $A \notin Abs(f^{\leftarrow}(P))$. Since $f$ is bounded, it follows that $1 - t < \mathscr{B}_X(A) \leq \mathscr{B}_Y(f^{\rightarrow}(A))$. Suppose that $f^{\rightarrow}(A) \in Abs(P)$, then there exists $\delta > 0$ such that $f^{\rightarrow}(A) \subseteq \lambda P$ for all $|\lambda| \geq \delta$. Thus $A \subseteq f^{\leftarrow}(f^{\rightarrow}(A)) \subseteq f^{\leftarrow}(\lambda P) \subseteq \lambda f^{\leftarrow}(P)$. It contradicts to the fact $A \notin Abs(f^{\leftarrow}(P))$. So, $f^{\rightarrow}(A) \notin Abs(P)$, this implies that $BV_Y(P) < t$, i.e., $BV_X(f^{\leftarrow}(P)) \geq BV_Y(P)$. □

**Remark 3.9.** For each $A \in 2^Y$, noting that the following inequality:

$$\begin{aligned}\tau_{\mathscr{B}_Y}(A) = \bigwedge_{a\in A} BV_Y(A-a) &\leq \bigwedge_{a\in A} BV_X(f^{\leftarrow}(A-a)) \\ &\leq \bigwedge_{b\in f^{\leftarrow}(A)} BV_X(f^{\leftarrow}(A)-b) = \tau_{\mathscr{B}_X}(f^{\leftarrow}(A)).\end{aligned}$$

We may obtain $f$ is continuous.

**Theorem 3.10.** *Let $(X, \mathscr{B}_X)$, $(Y, \mathscr{B}_Y)$ be fuzzifying bornological linear spaces, $f$ be a linear mapping from $X$ to $Y$ and $BV_X(P) \leq \mathscr{B}_Y(f^{\rightarrow}(P))$ for all $P \in 2^X$. Then $[T(Y, \mathscr{B}_Y)] \leq [f = \theta]$, where the notation $[f = \theta]$ denotes the true value of $f$ is trivial functional.*

***Proof.*** If $[f = \theta] = 1$, then it is trivial. Suppose that $[f = \theta] = 0$, then there exists $x_0 \in X$ such that $f(x_0) \neq \theta$. Since $f$ is linear, it follows $f^{\rightarrow}(X)$ is a linear subspace of $Y$. Moreover, from Definition 3.1, we have $BV_X(X) = 1$, it follows that $\mathscr{B}_Y(f^{\rightarrow}(X)) = 1$. Hence,

$$[T(Y, \mathscr{B}_Y)] = \bigwedge_{\substack{M\neq\{\theta\}\\ M\in Svec(Y)}} \{1 - \mathscr{B}(M)\} \leq 1 - \mathscr{B}_Y(f^{\rightarrow}(X)) = 0.$$

Thus $[T(Y, \mathscr{B}_Y)] \leq [f = \theta]$. This completes the proof. □

**Theorem 3.11.** *Let $(X, \mathscr{B})$ be a fuzzifying bornological linear space. Then*

$$P \in BV \Longleftrightarrow (\forall \{x_n\} \subseteq P^c)(x_n \overset{M}{\nrightarrow} \theta), \tag{3.1}$$

*where $P^c$ means the complement of $P$.*

***Proof.*** We need to show

$$\bigwedge_{A\subseteq X} \{1 - \mathscr{B}(A) : A \notin Abs(P)\} = \bigwedge_{\{x_n\}\subseteq P^c} \bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}} \{1 - \mathscr{B}(A) : \forall n \in \mathbb{N}, x_n \in \alpha_n A\}.$$

Let $\bigwedge_{A\subseteq X} \{1 - \mathscr{B}(A) : A \notin Abs(P)\} < t$. Then there exists $A \subseteq X$ such that $1 - \mathscr{B}(A) < t$ and $A \notin Abs(P)$. By (B6) of Theorem 2.8, we may consider $A \in Bal(X)$. Put $\delta_n = \frac{1}{n} > 0, n \in \mathbb{N}$, there is $\lambda_n \in \mathbb{K}$ with $|\lambda_n| \leq \delta_n$ such that $\lambda_n A \nsubseteq P$. Take $y_n \in A$ such that $\lambda_n y_n \notin P$ for all $n \in \mathbb{N}$. Denote $x_n = \lambda_n y_n$ and $\alpha_n = \lambda_n$. Clearly $\{x_n\} \subseteq P^c$, $\alpha_n \to 0$ and $x_n \in \alpha_n A$ for all $n \in \mathbb{N}$. Thus

$$\bigwedge_{\{x_n\}\subseteq P^c} \bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}} \{1 - \mathscr{B}(A) : \forall n \in \mathbb{N}, x_n \in \alpha_n A\} < t.$$

From which it follows that

$$\bigwedge_{A\subseteq X} \{1 - \mathscr{B}(A) : \ A \notin Abs(P)\} \geq \bigwedge_{\{x_n\}\subseteq P^c} \bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}} \{1 - \mathscr{B}(A) : \forall n \in \mathbb{N}, x_n \in \alpha_n A\}.$$

Conversely, let $\bigwedge_{\{x_n\}\subseteq P^c}\bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N},x_n\in\alpha_nA\}<t$. Then there exists $\{x_n\}\subseteq P^c$, $A\in Bal(X)$ and $\alpha_n\to 0$ such that $1-\mathscr{B}(A)<t$ and $x_n\in\alpha_nA$ for all $n\in\mathbb{N}$. Clearly, we obtain $A\nsubseteq\alpha_n^{-1}P$ for all $n\in\mathbb{N}$. It is claimed that $A$ does not absorb the set $P$. Otherwise, if there is $\lambda_0\neq 0$ such that $A\subseteq\lambda_0P$. Since $\alpha_n\to 0$, we have $|\alpha_{n_0}|<\frac{1}{|\lambda_0|}$. It follows that $\alpha_{n_0}A\subseteq\frac{1}{\lambda_0}A\subseteq P$. This deduces a contradiction. Hence $\bigwedge_{A\subseteq X}\{1-\mathscr{B}(A):A\notin Abs(P)\}<t$, which means $\bigwedge_{A\subseteq X}\{1-\mathscr{B}(A):A\notin Abs(P)\}\leq\bigwedge_{\{x_n\}\subseteq P^c}\bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N},x_n\in\alpha_nA\}$. This completes the proof of the formula (3.1). □

**Theorem 3.12.** *Let $(X,\mathscr{B})$ be a fuzzifying bornological vector space and BO is given by Definition 3.5, then for all $P\subseteq X$,*

$$P\in BO\Longleftrightarrow\Big((\forall a\in P)(\forall\{x_n\}\subseteq X)(x_n\overset{M}{\to}a)\to(\{x_n\}\sqsubseteq P)\Big).\tag{3.2}$$

*Where the notation $\to$ stands for the Łukasiewicz residuum, and the notation $\{x_n\}\sqsubseteq P$ means that $\{x_n\}$ " almost in" $P$, that is, there is $n_0\in\mathbb{N}$ such that $x_n\in P$ for any $n\geq n_0$.*

***Proof.*** By Łukasiewicz fuzzy logic, the right side of formula (3.2) actually means that $\bigwedge_{a\in P}\bigwedge_{\{x_n\}\subseteq X}\bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N},x_n-a\in\alpha_nA,\{x_n\}\not\sqsubseteq P\}$. From Theorem 3.11, Definition 2.10 and Definition 3.1, we only need to show

$$\begin{aligned}&\bigwedge_{a\in P}\bigwedge_{\{x_n\}\subseteq(P-a)^c}\bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N},x_n\in\alpha_nA\}\\ =&\bigwedge_{a\in P}\bigwedge_{\{x_n\}\subseteq X}\bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N},x_n-a\in\alpha_nA,\{x_n\}\not\sqsubseteq P\}.\end{aligned}$$

First, let the left side $<t$. Then there exist $a\in P$, $\{x_n\}\subseteq(P-a)^c$, $A\in Bal(X)$ and $\alpha_n\to 0$ such that $1-\mathscr{B}(A)<t$ and $x_n\in\alpha_nA$ for all $n\in\mathbb{N}$. Setting $y_n=x_n+a$. It is clear that $y_n\notin P$ and $y_n-a=x_n\in\alpha_nA$ for all $n\in\mathbb{N}$, from which int follows that

$$\begin{aligned}t\ &>\bigwedge_{a\in P}\bigwedge_{\{x_n\}\subseteq(P-a)^c}\bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N},x_n\in\alpha_nA\}\\ &\geq\bigwedge_{a\in P}\bigwedge_{\{x_n\}\subseteq X}\bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N},x_n-a\in\alpha_nA,\{x_n\}\not\sqsubseteq P\}.\end{aligned}$$

Next, let the right side $<t$. Then there exit $a\in P$, $\{x_n\}\subseteq X$, $A\in Bal(X)$ and $\alpha_n\to 0$ such that $1-\mathscr{B}(A)<t$, $x_n-a\in\alpha_nA$ and $\{x_n\}\not\sqsubseteq P$. Since $\{x_n\}\not\sqsubseteq P$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\notin P$ for all $k\in\mathbb{N}$. Setting $z_{n_k}=x_{n_k}-a$. It is obvious that $\{z_{n_k}\}\subseteq(P-a)^c$ and $z_{n_k}\in\alpha_{n_k}A$ for all $k\in\mathbb{N}$, which leads to the result that

$$\bigwedge_{a\in P}\bigwedge_{\{x_n\}\subseteq(P-a)^c}\bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N},x_n\in\alpha_nA\}<t.$$

Hence

$$\begin{aligned}&\bigwedge_{a\in P}\bigwedge_{\{x_n\}\subseteq(P-a)^c}\bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N},x_n\in\alpha_nA\}\\ \leq&\bigwedge_{a\in P}\bigwedge_{\{x_n\}\subseteq X}\bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N},x_n-a\in\alpha_nA,\{x_n\}\not\sqsubseteq P\}\end{aligned}$$

.

This completes the proof of the formula (3.2). □

**Theorem 3.13.** *Let $(X, \mathscr{B})$ be a fuzzifying bornological vector space and $BC, BO$ given by Definition* 2.9 *and Definition* 3.5 *respectively. Then for all $P \subseteq X$,*

$$P \in BO \Longleftrightarrow P^c \in BC. \tag{3.3}$$

***Proof.*** From Theorem 3.12, we only need to show

$$\begin{aligned}&\bigwedge_{a\in P}\bigwedge_{\{x_n\}\subseteq X}\bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N}, x_n-a\in\alpha_nA, \{x_n\}\nsubseteq P\}\\ &=\bigwedge_{\substack{\{x_n\}\subseteq P^c\\ a\notin P^c}}\bigwedge_{\substack{A\in Bal(X)\\ \lambda_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N}, x_n-a\in\lambda_nA\}.\end{aligned}$$

On one hand, let the left side $< t$. Then there exit $a \in P$, $\{x_n\} \subseteq X$, $A \in Bal(X)$ and $\alpha_n \to 0$ such that $1 - \mathscr{B}(A) < t$, $x_n - a \in \alpha_n A$ and $\{x_n\} \nsubseteq P$. Thus we have $\{x_{n_k}\} \subseteq P^c$ and $a \notin P^c$. Therefore, it is clear that

$$\begin{aligned}t &> \bigwedge_{a\in P}\bigwedge_{\{x_n\}\subseteq X}\bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N}, x_n-a\in\alpha_nA, \{x_n\}\nsubseteq P\}\\ &\geq \bigwedge_{\substack{\{x_n\}\subseteq P^c\\ a\notin P^c}}\bigwedge_{\substack{A\in Bal(X)\\ \lambda_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N}, x_n-a\in\lambda_nA\}.\end{aligned}$$

Similarly, we can get

$$\begin{aligned}&\bigwedge_{a\in P}\bigwedge_{\{x_n\}\subseteq X}\bigwedge_{\substack{A\in Bal(X)\\ \alpha_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N}, x_n-a\in\alpha_nA, \{x_n\}\nsubseteq P\}\\ &\leq \bigwedge_{\substack{\{x_n\}\subseteq P^c\\ a\notin P^c}}\bigwedge_{\substack{A\in Bal(X)\\ \lambda_n\to 0}}\{1-\mathscr{B}(A):\forall n\in\mathbb{N}, x_n-a\in\lambda_nA\}\\ &< t,\end{aligned}$$

which completes the proof of the equivalent relation (3.3). □

## 4. Conclusion

Motivated by [14], this paper introduces a notion of fuzzifying bornivorous sets in fuzzifying bornological linear spaces. An example of fuzzifying bornivorous sets on a fuzzifying topological linear space is presented, along with its von Neumann bornology. Additionally, the paper examines the description and equivalent representation of fuzzifying open sets in fuzzifying bornological linear spaces. Furthermore, the dual relationship between fuzzifying open and closed sets is studied. The paper also discusses the fuzzifying topological space induced by fuzzifying open sets.

In future research, we will explore the following aspects of fuzzifying bornivorous sets and fuzzifying bornological linear spaces:

1. The Mackey-completeness of fuzzifying bornological linear spaces and their interaction with fuzzifying bornivorous sets.

2. The duality between fuzzifying bornologies and fuzzifying topologies.

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