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On the Depth of Independence Complexes

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Abstract

Let G be a graph and $I(G)$ be its edge ideal so we call $k[\text{Ind}(G)] = k[x_1,...,x_n]/I(G)$ Stanley-Reisner ring of G. The depth of a ring is a well-studied and important algebraic invariant in commutative algebra. In this paper we give some results on the depth of Stanley-Reisner rings of graphs and simplicial complexes. By depth Lemma we reduce the computing depth of a codismantlable graph into its induced subgraphs.

Introduction

Let G be a simple undirected graph on the vertex set $V(G) = \{x_1,...,x_n\}$. Let $R = k[x_1,...,x_n]$ be a polynomial ring on n variables corresponding to $V(G)$. If we define set $I(G) = \{x_i x_j : \{x_i, x_j\} \in E(G)\}$ such that $E(G)$ is the edge set of G and indeterminates are from $V(G) = \{x_1,...,x_n\}$, then this $I(G)$ is called the edge ideal of graph G VILLARREAL (1990). The independence complex of G is a simplicial complex with vertex set $V(G)$ and with faces are independet sets of G and denoted by $\text{Ind}(G)$. The Stanley-Reisner ring of a simplicial complex over a field k provides a link between commutative algebra and combinatorial structures such as graphs and simplicial complexes. For a field k, the Stanley-Reisner ring of a simplicial complex Δ is denoted by $k[\Delta]$.

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If our complex is an independence complex of a graph G on $V(G)$, then it is denoted by $\text{k}[\text{Ind}(G)]$ and equals to quotient ring $R/I(G)$ with $R = k[x_1,...,x_n]$. Krull dimension of $\text{k}[\text{Ind}(G)]$ is the supremum of the longest chain of the strict inclusions of prime ideals of $\text{k}[\text{Ind}(G)]$ and denoted by $\dim(\text{Ind}(G))$. $\text{depth}(\text{Ind}(G))$ is the longest homogeneous sequence $f_1, f_2,..., f_k$ such that f_i is not a zero-divisor of $k[x_1,...,x_n]/(I, f_1, f_2,..., f_k)$ for all $1 \le i \le k$.

If a graph G has $\text{depth}(\text{Ind}(G))$ equals $\dim(\text{Ind}(G))$ then we call $\text{Ind}(G)$ Cohen-Macaulay complex and G Cohen-Macaulay graph VILLARREAL (1990). The maximum dimensional Cohen-Macaulay skeleton of a complex determines the depth of its Stanley-Reisner ring FRÖBERG (1990). By Auslander-Buchsbaum Formula AUSLANDER AND BUCHSBAUM (1957), computing depth of a ring gives rise to computing its projective dimension so depth of a ring is an important algebraic invariant. Finding bounds for projective dimension of a complexes is recently well-studied object in commutative algebra DAO AND SCHWEIG (2013) AND KHOSH-AHANG AND MORADI (2014). For this purpose many authors studied depth of simplicial comlexes MOREY (2010) AND KUMMINI (2009) AND GITLER AND VALENCIA (2005). Beyond algebraic results, computing the depth of a complex also gives nice results about its homology and combinatorial properties DAO AND SCHWEIG (2013).

In this paper we give some results about depth of Stanley-Reisner rings of complexes. And we determine depth of codismantlable graphs in terms of its induced subgraphs.

Preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $x \in V(G)$ the open and closed neighborhoods of x are denoted by $N_G(x)$ and $N_G[x]$ respectively.For further notions and definitions see

CHARTRAND AND ZHANG (2008) AND VILLARREAL (2015).

Definition 1. Let Δ be simplicial complex and σ a face of Δ. Then we have that,

$$\mathrm{lk}_{\Delta}(\sigma)=\{\tau\in\Delta:\tau\cap\sigma=\varnothing,\tau\cup\sigma\in\Delta\}$$

$$\mathrm{del}_{\Delta}(\sigma)=\{\tau\in\Delta:\tau\cap\sigma=\varnothing\}$$

If Δ is an independence complex of a graph G and x is a vertex of G then,

$\mathrm{lk}_{\mathrm{Ind}(G)}(x)=\mathrm{Ind}(G-N_G[x])$ and $\mathrm{del}_{\mathrm{Ind}(G)}(x)=\mathrm{Ind}(G-x)$.

Teorem 2 (See REISNER (1976)) Let Δ be a simplicial complex and σ be a face of Δ. If k is a field, then the following conditions are equivalent:

(a) Δ is a Cohen-Macaulay over k.

(b) $\tilde{H}_i(\mathrm{lk}_{\Delta}(\sigma);k)=0$ for all $\sigma\in\Delta$ and $i<\dim(\mathrm{lk}_{\Delta}(\sigma))$.

Definition 3. The i-skeleton of a simplicial complex Δ is the simplicial complex consists of all j-simplices of Δ with $i<j$ and denoted by Δ^i.

Teorem 4. [*] Let Δ be a simplicial complex. Then

$\mathrm{depth}(k[\Delta])=\max\{i:k[\Delta^i]\text{ is Cohen-Macaulay}\}+1$.

Lemma 5. *(Depth Lemma)* If $0\to A\to B\to C\to 0$ is a short exact sequence of finitely generated R modules with R is a local ring then,

(a) If $\mathrm{depth}(A)\geq\mathrm{depth}(B)$ *then* $\mathrm{depth}(B)=\mathrm{depth}(C)$.

(b) If $\mathrm{depth}(B)\geq\mathrm{depth}(A)$ *then* $\mathrm{depth}(A)=\mathrm{depth}(C)+1$.

(c) If $\mathrm{depth}(C)\geq\mathrm{depth}(A)$ *then* $\mathrm{depth}(A)=\mathrm{depth}(B)$.

Lemma 6. (See VILLARREAL (2015)) Let R be a ring and I is an ideal of R. If x is an element of R then,

$$0\to R/(I:x)\to R/I\to R/(I,x)\to 0$$

is a short exact sequence.

Remark 7. (See DAO AND SCHWEIG (2013)) Let G be a graph on $V(G)=\{x_1,\dots,x_n\}$. If we assume $R=k[x_1,\dots,x_n]$ as a polynomial ring over $V(G)$ and $I(G)$ as its edge ideal. Then $(I(G),x_i)=(I(G-x_i),x_i)$ and $(I(G):x_i)=(I(G-N_G[x_i]),N(x_i))$.

Lemma 8. (See MOREY (2010)) Let I be an ideal in a polynomial ring R, let x be an indeterminate over R, and let $S=R[x]$. Then $\mathrm{depth}(S/IS)=\mathrm{depth}(R/I)+1$.

In their paper, authors BIYIKOĞLU AND CIVAN (2014) introduced a new graph class called codismantlable graphs as follows:

Definition 9. (See BIYIKOĞLU AND CIVAN (2014)) A vertex x of G is called *codominated* if there exists a vertex $y\in N(x)$ such that $N_G[y]\subseteq N_G[x]$.

Definition 10. (See BIYIKOĞLU AND CIVAN (2014)) Let G and H be graphs. If there exist graphs $G_0,G_1,\dots,G_{k+1}$ satisfying $G\cong G_0$, $H\cong G_{k+1}$ and $G_{i+1}=G_{i+1}-x_i$ for each $0\leq i\leq k$, where x_i is codominated in G_i. A graph G is called *codismantlable* if either it is an edgeless graph or it is codismantlable to an edgeless graph.

Main Results

Lemma 11. Let G be a graph and x be its vertex. If $N_G^{\circ}(x)$ is a set of degree one neighbors of x, then $\mathrm{depth}(R/(I(G),x))=\mathrm{depth}(k[\mathrm{del}_{\mathrm{Ind}(G)}(x)])+|N_G^{\circ}(x)|$.

Proof. By Remark 7 we have, $\mathrm{depth}(R/(I(G),x))=\mathrm{depth}(R/(I(G-x),x))$ and from Lemma 8 one can derive that $\mathrm{depth}(R/(I(G),x))=\mathrm{depth}(R'/(I(G-x)))\otimes k[N_G^{\circ}(x)]$

Since the quotient ring $R'/(I(G-x))$ is exactly $k[\mathrm{del}_{\mathrm{Ind}(G)}(x)]$. And with the equallity $\mathrm{depth}(R/(I(G-x)))\otimes k[N_G^{\circ}(x)]=\mathrm{depth}(R'/(I(G-x)))+|N_G^{\circ}(x)|$ the argument gives us that $\mathrm{depth}(R/(I(G),x))=\mathrm{depth}(k[\mathrm{del}_{\mathrm{Ind}(G)}(x)])+|N_G^{\circ}(x)|$.

Lemma 12. Let G be a graph and x be its vertex. Then $\mathrm{depth}(R/(I(G):x))=\mathrm{depth}(k[\mathrm{lk}_{\mathrm{Ind}(G)}(x)])+1$.

Proof. Since from Remark 7 we have the equallity $\mathrm{depth}(R/(I(G):x))=\mathrm{depth}(R/(I(G-N_G[x]),N(x)))$ by using Lemma 8 one can conclude that $\mathrm{depth}(R/(I(G):x))=\mathrm{depth}(R'/(I(G-N_G[x])))\otimes k[x]$.

Since the quotient ring $R'/(I(G-N_G[x]))$ is exactly $k[\mathrm{lk}_{\mathrm{Ind}(G)}(x)]$, we have that

$\mathrm{depth}(R/(I(G):x))=\mathrm{depth}(k[\mathrm{lk}_{\mathrm{Ind}(G)}(x)])+1$.

The next lemma shows that how notions depth and homology related to each others.

Lemma 13. Let Δ be a simplicial complex. $\operatorname{depth}(k[\Delta])=d$ then $\tilde{H}_i(\Delta;k)=0$ for all $i<d-1$.

Proof. By Reisner criterion if a complex Cohen-Macaulay all homology groups under top dimension vanish. If depth of complex is d then complex has d dimensional Cohen-Macaulay skeleton. This concludes the proof.

Proposition 14. Let Δ be simplicial complex and $\sigma\in\Delta$ be its face. If $\operatorname{depth}(k[\Delta])=\operatorname{depth}(k[\operatorname{del}_\Delta(\sigma)])$ then $\operatorname{depth}(k[\operatorname{lk}_\Delta(\sigma)])+1\le\operatorname{depth}(k[\operatorname{del}_\Delta(\sigma)])$.

Proof. If we assume that j -skeletons of Δ and $\operatorname{del}_\Delta(\sigma)$ are Cohen-Macaulay, then for all $i<j-1$, $\tilde{H}_i(\Delta)$ and $\tilde{H}_i(\operatorname{del}_\Delta(\sigma))$ vanish. The exactness of the sequence,

$$\ldots\to\tilde{H}_{j+1}(\Delta)\to\tilde{H}_j(\operatorname{lk}_\Delta(\sigma))\to\tilde{H}_j(\operatorname{del}_\Delta(\sigma))\to\tilde{H}_j(\Delta)\to\tilde{H}_{j-1}(\operatorname{lk}_\Delta(\sigma))\to\ldots$$

gives us, $\tilde{H}_{j-1}(\operatorname{lk}_\Delta(\sigma))\neq 0$, and for all $i<j-2$ the groups $\tilde{H}_i(\operatorname{lk}_\Delta(\sigma))=0$. This concludes the proof.

Teorem 15 Let G be a codismantlable graph and $\operatorname{Ind}(G)$ be its independence complex. If x is a codominated vertex, then

$$\operatorname{depth}(\mathrm{k}[\operatorname{Ind}(G)])=\operatorname{depth}(k[\operatorname{lk}_{\operatorname{Ind}(G)}(x)])+1.$$

Proof. Let G be codismantlable graph. If x codominated vertex then there exist some $y\in N_G(x)$ such that $N_G[y]\subseteq N_G[x]$. If $y\in N_G^{\circ}(x)$ then by Lemma 11 and Lemma 12 we can say that $\operatorname{depth}(R/(I(G),x))\ge\operatorname{depth}(R/(I(G):x))$. Otherwise by considering Proposition 14 we stil have $\operatorname{depth}(R/(I(G),x))\ge\operatorname{depth}(R/(I(G):x))$. If we combine this argument with depth lemma and the exactness of the sequence:

$$0\to R/(I(G):x)\to R/I(G)\to R/(I(G),x)\to 0$$

then we conclude that $\operatorname{depth}(R/I(G))=\operatorname{depth}(R/(I(G):x))$. Therefore, considering Lemma 12, we get that,

$$\operatorname{depth}(R/I(G))=\operatorname{depth}(R/(I(G):x))$$
$$=\operatorname{depth}(k[\operatorname{lk}_{\operatorname{Ind}(G)}(x)])+1.$$

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