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Research Article

Curious harmony in asymmetric & nonlinear variant of Filbert and Lilbert matrices

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Abstract

In this paper, we present a new analogue of the Filbert and Lilbert matrices whose indices have different asymmetric and nonlinear rules according to their row numbers. Explicit formulæ are derived for the LU-decompositions, their inverses and the inverse of the main matrix as well as its determinant. To prove the claimed results we use backward induction method. The asymmetric variants of the Filbert and Lilbert matrices are obtained from our results for a particular q value.

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1. Introduction

For $n > 1$, define the second order linear recurrences $\{U_n, V_n\}$ by

$$U_n = pU_{n-1} + U_{n-2} \text{ and } V_n = pV_{n-1} + V_{n-2}$$

with $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = p$, respectively. If $p = 1$, then $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number).

The Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q} \text{ and } V_n = \alpha^n + \beta^n = \alpha^n\left(1+q^n\right),$$

where α, $\beta = \left(p \pm \sqrt{\Delta}\right)/2$ with $q = \beta/\alpha = -\alpha^{-2}$ and $\Delta = p^2 + 4$, so that $\alpha = \mathbf{i}/\sqrt{q}$.

Nowadays interesting combinatorial matrices whose entries include q-binomial coefficients or well-known integer sequences such as natural numbers, Pochhammer symbol, q-integers, Fibonacci and Lucas numbers have been studied by many authors. They have found some algebraic properties of these matrices. For these studies, we refer to [1–15].

- Chu and Di Claudio [6] defined the matrix $\left[\frac{(a)_{j+\lambda_i}}{(c)_{j+\lambda_i}}\right]_{0\le i,j\le n}$, where a, c and $\{\lambda_i\}_{i=0}^n$ are complex numbers. They also worked out some versions of the matrix above.

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- Filbert matrix $H_n = (h_{i,j})_{i,j=1}^n$ is defined by $h_{i,j} = \frac{1}{F_{i+j-1}}$ as analogue of the Hilbert matrix, where F_n is the nth Fibonacci number. It has been defined and worked by Richardson [15].
- Fonseca and Anđelić [2] studied some results on the determinant of a pentadiagonal Toeplitz matrix.
- The authors of [10] worked nonlinear generalizations of Filbert and Lilbert matrices with entries
$$\frac{1}{U_{\lambda(i+r)^k+\mu(j+s)^r+c}} \quad \text{and} \quad \frac{1}{V_{\lambda(i+r)^k+\mu(j+s)^r+c}},$$
where U_n and V_n are nth generalized Fibonacci and Lucas numbers, respectively. This is the first study on nonlinear generalization of Filbert and Lilbert matrices.
- Kılıç and Ersanlı [11] constructed a combinatorial matrix whose row entries are changed according to their row numbers as follows
$$\mathcal{T} = [\mathcal{T}_{m,n}] = \begin{cases} \dfrac{1}{1+xq^{am+bn}} & \text{if } m \text{ is odd,} \\[2ex] \dfrac{1}{1+yq^{am+bn}} & \text{if } m \text{ is even,} \end{cases}$$
where any reals q, x, y and arbitrary integers a, b such that $1 + xq^{am+bn} \neq 0$, $1 + yq^{am+bn} \neq 0$.

In the mentioned works above, some algebraic properties of these matrices such as explicit formulas for LU-decomposition, Cholesky decomposition, determinants and inverses of them were evaluated. The authors of these works converted the entries of the matrices into q-form and then used the q-Zeilberger algorithm to prove the claims. In cases where the algorithm didn't work, they used backward induction method.

In this work, inspired by the matrix defining idea of [11] and by transferring the idea there to the powers of the shifted indices, by the index functions Φ and Ψ, we were able to carry it to a very different non-linear analog structure. For any reals x, y, q, λ, μ, a, b, p, r, v, w such that $1 + xq^{a+\lambda(m+p)^v+\mu(n+r)^w} \neq 0$, $1 + yq^{b+\lambda(m+p)^v+\mu(n+r)^w} \neq 0$, we define the matrix $\mathcal{G} = [\mathcal{G}_{m,n}]_{m,n>0}$ as follows
$$\mathcal{G} = [\mathcal{G}_{m,n}] = \begin{cases} \dfrac{1}{1+xq^{a+\Phi_m+\Psi_n}} & \text{if } m \text{ is odd,} \\[2ex] \dfrac{1}{1+yq^{b+\Phi_m+\Psi_n}} & \text{if } m \text{ is even,} \end{cases}$$
with
$$\Phi_i := \lambda\,(i+p)^v \quad \text{and} \quad \Psi_i := \mu\,(i+r)^w.$$

So consecutive row elements of $\mathcal{G}$ will have the forms of elements of the nonlinear and asymmetric Filbert and Lilbert matrices with the choice of q; x and y. In this respect, the matrix $\mathcal{G}$ has a very special and interesting harmony.

When $x = y = \mp 1$, our results included the results of [10]. When also $a = b = 0$, $\Phi_m = cm$ and $\Psi_n = dn$ for arbitrary integers c, d such that $1 + xq^{cm+dn} \neq 0$, $1 + yq^{cm+dn} \neq 0$ we get the results of [11].

1.1. Our contribution

We will even see this harmony when we obtain the algebraic properties of the matrix. Let's briefly summarize what we will do:

- We give explicit formulæ for L and U matrices and their inverses come from LU-decomposition. Cholesky decomposition is not given as the inputs of $\mathcal{G}$ are asymmetric. The size of $\mathcal{G}$ isn't important, except for its determinant and inverse. We don't need the dimension of $\mathcal{G}$ while deriving properties of $\mathcal{G}$ such as LU-decomposition, L^{-1}, U^{-1} as the formulæ for them are independent from the dimension. Thus, the matrix $\mathcal{G}$ can be thought of as infinitely dimensional so that we restrict it to the first N rows representing rows when needed, and use the $\mathcal{G}_N$ notation.
- While the results we need seem to be q-hypergeometric sums, the q-Zeilberger algorithm doesn't work as they aren't Gosper summable. So we prove them by backward induction. We prove only one claimed result as a showcase.
- By the q-forms of the Binet formulas of $\{U_n, V_n\}$, according to the choice of any real x and y, the matrix $\mathcal{G}$ have the forms of Filbert and Lilbert matrices. Clearly;
  i. If $x = y = -1$, then the consecutive rows of matrix $\mathcal{G}$ are in the form of elements of the Filbert matrix.

  ii. If $x = y = 1$, then the entries of matrix $\mathcal{G}$ are in the form of elements of the Lilbert matrix.

  iii. If $x = -y = -1$, then the consecutive rows of matrix $\mathcal{G}$ are in the form of Filbert-Lilbert, respectively.

  iv. If $x = -y = 1$, then the consecutive rows of matrix $\mathcal{G}$ are in the form of Lilbert-Filbert, respectively.

- All identities we will derive are valid for general q. Thus, as an application of our results, we derive results for general Fibonacci and Lucas numbers by choosing q specifically.

2. The main results

We will give the results related with the LU-decomposition of the matrix $\mathcal{G}$ and the L^{-1}, U^{-1} matrices and the inverse matrix $\mathcal{G}^{-1}$.

For LU-decomposition, we have the following results.

Theorem 2.1. *Let $1 \leq d \leq n$.*
For odd n;
(i) if d is odd,

$$L_{n,d} = \prod_{t=1}^{d} \frac{1 + xq^{a+\Phi_d+\Psi_t}}{1 + xq^{a+\Phi_n+\Psi_t}} \prod_{t=1}^{(d-1)/2} \frac{\left(1 - xy^{-1}q^{a-b+\Phi_n-\Phi_{2t}}\right)\left(1 - q^{\Phi_n-\Phi_{2t-1}}\right)}{\left(1 - xy^{-1}q^{a-b+\Phi_d-\Phi_{2t}}\right)\left(1 - q^{\Phi_d-\Phi_{2t-1}}\right)},$$

(ii) if d is even,

$$L_{n,d} = \prod_{t=1}^{d} \frac{1 + yq^{b+\Phi_d+\Psi_t}}{1 + xq^{a+\Phi_n+\Psi_t}} \prod_{t=1}^{(d-2)/2} \frac{1 - xy^{-1}q^{a-b+\Phi_n-\Phi_{2t}}}{1 - q^{\Phi_d-\Phi_{2t}}} \prod_{t=1}^{d/2} \frac{1 - q^{\Phi_n-\Phi_{2t-1}}}{1 - x^{-1}yq^{b-a+\Phi_d-\Phi_{2t-1}}}.$$

For even n;
(iii) if d is odd,

$$L_{n,d} = \prod_{t=1}^{d} \frac{1 + xq^{a+\Phi_d+\Psi_t}}{1 + yq^{b+\Phi_n+\Psi_t}} \prod_{t=1}^{(d-1)/2} \frac{\left(1 - x^{-1}yq^{b-a+\Phi_n-\Phi_{2t-1}}\right)\left(1 - q^{\Phi_n-\Phi_{2t}}\right)}{\left(1 - xy^{-1}q^{a-b+\Phi_d-\Phi_{2t}}\right)\left(1 - q^{\Phi_d-\Phi_{2t-1}}\right)},$$

(iv) if d is even,

$$L_{n,d}=\prod_{t=1}^{d}\frac{1+yq^{b+\Phi_d+\Psi_t}}{1+yq^{b+\Phi_n+\Psi_t}}\prod_{t=1}^{d/2}\frac{1-x^{-1}yq^{b-a+\Phi_n-\Phi_{2t-1}}}{1-x^{-1}yq^{b-a+\Phi_d-\Phi_{2t-1}}}\prod_{t=1}^{(d-2)/2}\frac{1-q^{\Phi_n-\Phi_{2t}}}{1-q^{\Phi_d-\Phi_{2t}}}.$$

Theorem 2.2. *Let $1\leq d\leq n$;*
(i) if d is odd,

$$U_{d,n}=\frac{\left(xq^{a+\Phi_d}\right)^{d-1}}{1+xq^{a+\Phi_d+\Psi_n}}\prod_{t=1}^{d-1}\frac{q^{\Psi_t}\left(1-q^{\Psi_n-\Psi_t}\right)}{1+xq^{a+\Phi_d+\Psi_t}}$$
$$\times\prod_{t=1}^{(d-1)/2}\frac{\left(1-x^{-1}yq^{b-a+\Phi_{2t}-\Phi_d}\right)\left(1-q^{\Phi_{2t-1}-\Phi_d}\right)}{(1+yq^{b+\Phi_{2t}+\Psi_n})\,(1+xq^{a+\Phi_{2t-1}+\Psi_n})},$$

(ii) if d is even,

$$U_{d,n}=\left(yq^{b+\Phi_d}\right)^{d-1}\prod_{t=1}^{d-1}\frac{q^{\Psi_t}\left(1-q^{\Psi_n-\Psi_t}\right)}{1+yq^{b+\Phi_d+\Psi_t}}$$
$$\times\prod_{t=1}^{d/2}\frac{1-xy^{-1}q^{a-b+\Phi_{2t-1}-\Phi_d}}{(1+xq^{a+\Phi_{2t-1}+\Psi_n})\,(1+yq^{b+\Phi_{2t}+\Psi_n})}\prod_{t=1}^{(d-2)/2}\left(1-q^{\Phi_{2t}-\Phi_d}\right).$$

Now we will present formulations for L^{-1} and U^{-1}.
For later use we recall the Iverson notation defined as

$$[P]=\begin{cases}1 & \text{if } P \text{ is true,}\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 2.3. *Let $1\leq d\leq n$.*
For odd n;
(i) if d is odd,

$$L^{-1}_{n,d}=(-1)^{[n\neq d]}\prod_{t=1}^{(n-1)/2}\frac{1-xy^{-1}q^{a-b+\Phi_n-\Phi_{2t}}}{1-xy^{-1}q^{a-b+\Phi_d-\Phi_{2t}}}$$
$$\times\prod_{\substack{t=1\\ t\neq(d+1)/2}}^{(n-1)/2}\frac{1-q^{\Phi_n-\Phi_{2t-1}}}{1-q^{\Phi_d-\Phi_{2t-1}}}\prod_{t=1}^{n-1}\frac{1+xq^{a+\Phi_d+\Psi_t}}{1+xq^{a+\Phi_n+\Psi_t}}$$

with $L^{-1}_{n,d}=0$ for $n<d$,
(ii) if d is even,

$$L^{-1}_{n,d}=-\prod_{t=1}^{(n-1)/2}\frac{1-q^{\Phi_n-\Phi_{2t-1}}}{1-x^{-1}yq^{b-a+\Phi_d-\Phi_{2t-1}}}$$
$$\times\prod_{\substack{t=1\\ t\neq d/2}}^{(n-1)/2}\frac{1-q^{\Phi_d-\Phi_{2t}}}{1-xy^{-1}q^{a-b+\Phi_n-\Phi_d}}\prod_{t=1}^{n-1}\frac{1+yq^{b+\Phi_d+\Psi_t}}{1+xq^{a+\Phi_n+\Psi_t}}$$

with $L^{-1}_{n,d}=0$ for $n<d$.
For even n;

(iii) if d is odd,

$$L_{n,d}^{-1} = -\prod_{t=1}^{(n-2)/2} \frac{1-q^{\Phi_n-\Phi_{2t}}}{1-xy^{-1}q^{a-b+\Phi_d-\Phi_{2t}}} \times \prod_{\substack{t=1\\ t\neq (d+1)/2}}^{n/2} \frac{1-x^{-1}yq^{b-a+\Phi_n-\Phi_{2t-1}}}{1-q^{\Phi_d-\Phi_{2t-1}}} \prod_{t=1}^{n-1} \frac{1+xq^{a+\Phi_d+\Psi_t}}{1+yq^{b+\Phi_n+\Psi_t}}$$

with $L_{n,d}^{-1}=0$ for $n<d$,
(iv) if d is even,

$$L_{n,d}^{-1} = (-1)^{[n\neq d]} \prod_{t=1}^{n/2} \frac{1-x^{-1}yq^{b-a+\Phi_n-\Phi_{2t-1}}}{1-x^{-1}yq^{b-a+\Phi_d-\Phi_{2t-1}}} \times \prod_{\substack{t=1\\ t\neq d/2}}^{(n-2)/2} \frac{1-q^{\Phi_n-\Phi_{2t}}}{1-q^{\Phi_d-\Phi_{2t}}} \prod_{t=1}^{n-1} \frac{1+yq^{b+\Phi_n+\Psi_t}}{1+yq^{b+\Phi_d+\Psi_t}}$$

with $L_{n,d}^{-1}=0$ for $n<d$.

Theorem 2.4. *Let $1\le d\le n$;*
(i) if n is odd,

$$U_{d,n}^{-1} = \frac{(-1)^{d+1}q^{(d-n)\Psi_d}}{\left(xq^{a+\Phi_n}\right)^{n-1}} \frac{\prod\limits_{t=1}^{n}\left(1+xq^{a+\Phi_n+\Psi_t}\right)}{\prod\limits_{t=1}^{n-d}(1-q^{\Psi_{t+d}-\Psi_d})\prod\limits_{t=1}^{d-1}q^{\Psi_t}(1-q^{\Psi_d-\Psi_t})} \times \prod_{t=1}^{(n-1)/2} \frac{\left(1+xq^{a+\Phi_{2t-1}+\Psi_d}\right)\left(1+yq^{b+\Phi_{2t}+\Psi_d}\right)}{(1-q^{\Phi_{2t-1}-\Phi_n})(1-x^{-1}yq^{b-a+\Phi_{2t}-\Phi_n})}$$

with $U_{d,n}^{-1}=0$ for $n<d$,

(ii) if n is even,

$$U_{d,n}^{-1} = \frac{(-1)^{d}q^{(d-n)\Psi_d}}{\left(yq^{b+\Phi_n}\right)^{n-1}} \frac{\prod\limits_{t=1}^{n}\left(1+yq^{b+\Phi_n+\Psi_t}\right)}{\prod\limits_{t=1}^{n-d}(1-q^{\Psi_{t+d}-\Psi_d})\prod\limits_{t=1}^{d-1}q^{\Psi_t}(1-q^{\Psi_d-\Psi_t})} \times \prod_{t=1}^{(n-2)/2} \frac{1+yq^{b+\Phi_{2t}+\Psi_d}}{1-q^{\Phi_{2t}-\Phi_n}} \prod_{t=1}^{n/2} \frac{1+xq^{a+\Phi_{2t-1}+\Psi_d}}{1-xy^{-1}q^{a-b+\Phi_{2t-1}-\Phi_n}}$$

with $U_{d,n}^{-1}=0$ for $n<d$.

Now we compute the inverse of $\mathcal{G}$ which depends on its dimension N.

Theorem 2.5. *For $1\le m,n\le N$;*
(i) if n is odd,

$$(\mathcal{G}_N)_{m,n}^{-1} = \frac{(-1)^{N+1}}{\left(xq^{a+\Phi_n+\Psi_m}\right)^{N-1}} \prod_{\substack{t=1\\ t\neq m}}^{N} \frac{1+xq^{a+\Phi_n+\Psi_t}}{1-q^{\Psi_t-\Psi_m}} \prod_{\substack{t=1\\ t\neq (n+1)/2}}^{\lfloor (N+1)/2\rfloor} \frac{1}{1-q^{\Phi_{2t-1}-\Phi_n}} \times \prod_{t=1}^{\lfloor N/2\rfloor} \frac{1+yq^{b+\Phi_{2t}+\Psi_m}}{1-x^{-1}yq^{b-a+\Phi_{2t}-\Phi_n}} \prod_{t=1}^{\lfloor (N+1)/2\rfloor}\left(1+xq^{a+\Phi_{2t-1}+\Psi_m}\right),$$

(ii) if n is even,

$$(\mathcal{G}_N)^{-1}_{m,n} = \frac{(-1)^{N+1}}{\left(yq^{b+\Phi_n+\Psi_m}\right)^{N-1}} \prod_{\substack{t=1\\ t\neq m}}^{N} \frac{1+yq^{b+\Phi_n+\Psi_t}}{1-q^{\Psi_t-\Psi_m}} \prod_{\substack{t=1\\ t\neq n/2}}^{\lfloor N/2\rfloor} \frac{1}{1-q^{\Phi_{2t}-\Phi_n}}$$
$$\times \prod_{t=1}^{\lfloor (N+1)/2\rfloor} \frac{1+xq^{a+\Phi_{2t-1}+\Psi_m}}{1-xy^{-1}q^{a-b+\Phi_{2t-1}-\Phi_n}} \prod_{t=1}^{\lfloor N/2\rfloor} \left(1+yq^{b+\Phi_{2t}+\Psi_m}\right).$$

Now we present our last result for computing the determinant of $\mathcal{G}_N$ which is evaluated as the product of the diagonal entries of the matrix U.

Theorem 2.6. *For $N \geq 1$;*

$$\det \mathcal{G}_N = (-1)^{\lfloor N/4\rfloor} \left(yq^b\right)^{\binom{\lfloor N/2\rfloor}{2}} \prod_{t=1}^{\lfloor (N-1)/2\rfloor} \prod_{k=t}^{\lfloor (N-1)/2\rfloor} q^{\Phi_{2t-1}} \left(1-q^{\Phi_{2k+1}-\Phi_{2t-1}}\right)$$
$$\times \prod_{t=1}^{N-1}\prod_{k=t}^{N-1} q^{\Psi_t}\left(1-q^{\Psi_{k+1}-\Psi_t}\right) \prod_{t=1}^{\lfloor (N-2)/2\rfloor} \prod_{k=t}^{\lfloor (N-2)/2\rfloor} q^{\Phi_{2t}}\left(1-q^{\Phi_{2k+2}-\Phi_{2t}}\right)$$
$$\times \prod_{t=1}^{\lfloor (N+1)/2\rfloor} \prod_{k=1}^{N} \frac{1}{1+xq^{a+\Phi_{2t-1}+\Psi_k}} \prod_{t=1}^{\lfloor N/2\rfloor}\prod_{k=1}^{N} \frac{1}{1+yq^{b+\Phi_{2t}+\Psi_k}}$$
$$\times \prod_{t=1}^{\lfloor N/2\rfloor} \prod_{k=1}^{\lfloor (N+1)/2\rfloor} q^{\Phi_{2k-1}}\left(1-x^{-1}yq^{b-a+\Phi_{2t}-\Phi_{2k-1}}\right)$$
$$\times \begin{cases} (xq^a)^{3\binom{(N+1)/2}{2}} & \text{if } N \text{ is odd,}\\ (-1)^{N/2}(xq^a)^{\binom{3N/2}{2}/3} & \text{if } N \text{ is even.}\end{cases}$$

3. Proofs

Although our results look like q-hypergeometric summations, the q-Zeilberger's algorithm doesn't work as they aren't Gosper summable. For this reason, we prove the claimed results by the backward induction method. Since the operations in this method are long and time consuming, we only give a proof for LU-decomposition of $\mathcal{G}$. The proofs of other results can be done in a similar way.

For the LU-decomposition of $\mathcal{G}$, we need to prove that

$$\sum_{1\leq d\leq \min(m,n)} L_{m,d}U_{d,n} = \mathcal{G}_{m,n}.$$

Here, since two consecutive rows and columns of L are defined by four different formula and U are defined by two different formula we must consider four cases to prove the LU-decomposition of $\mathcal{G}$. Before this, we will consider more general case. We can assume without loss of generality that $m \geq n$, and also prove a general formula depending on an extra variable K:

$$\mathtt{SUM}_K := \sum_{K\leq d\leq n} L_{m,d}U_{d,n}. \tag{3.1}$$

To prove the claimed result for the LU-decomposition of $\mathcal{G}$, we need the case $K=1$ in the above sum or $\mathtt{SUM}_1$. Before this, we have to consider parities of m and K. There are four subcases of $\mathtt{SUM}_K$ as follows

$$\sum_{K\le d\le n} L_{m,d}U_{d,n} = \begin{cases} \mathtt{SUM}_K^{(1)} & \text{if } m \text{ and } n \text{ are odd,}\\ \mathtt{SUM}_K^{(2)} & \text{if } m \text{ is even and } n \text{ is odd,}\\ \mathtt{SUM}_K^{(3)} & \text{if } m \text{ is odd and } n \text{ is even,}\\ \mathtt{SUM}_K^{(4)} & \text{if } m \text{ and } n \text{ are even,}\end{cases}$$

where,

(i) For m and K are odd,

$$\mathtt{SUM}_K^{(1)} := \sum_{d=K}^{\min(m,n)} \frac{\left(xq^{a+\Phi_m}\right)^{d-1}\left(1+xq^{a+\Phi_d+\Psi_d}\right)}{\left(1+xq^{a+\Phi_d+\Psi_n}\right)\left(1+xq^{a+\Phi_m+\Psi_d}\right)} \times \prod_{t=1}^{d-1} \frac{q^{\Psi_t}\left(1-q^{\Psi_n-\Psi_t}\right)}{1+xq^{a+\Phi_m+\Psi_t}} \prod_{t=1}^{\lfloor d/2\rfloor} \frac{\left(1-q^{\Phi_{2t-1}-\Phi_m}\right)\left(1-x^{-1}yq^{b-a+\Phi_{2t}-\Phi_m}\right)}{\left(1+xq^{a+\Phi_{2t-1}+\Psi_n}\right)\left(1+yq^{b+\Phi_{2t}+\Psi_n}\right)},$$

(ii) For m is even and K is odd,

$$\mathtt{SUM}_K^{(2)} := \sum_{d=K}^{\min(m,n)} \frac{\left(yq^{b+\Phi_m}\right)^{d-1}\left(1+xq^{a+\Phi_d+\Psi_d}\right)}{\left(1+xq^{a+\Phi_d+\Psi_n}\right)\left(1+yq^{b+\Phi_m+\Psi_d}\right)} \times \prod_{t=1}^{d-1} \frac{q^{\Psi_t}\left(1-q^{\Psi_n-\Psi_t}\right)}{1+yq^{b+\Phi_m+\Psi_t}} \prod_{t=1}^{\lfloor d/2\rfloor} \frac{\left(1-q^{\Phi_{2t}-\Phi_m}\right)\left(1-xy^{-1}q^{a-b+\Phi_{2t-1}-\Phi_m}\right)}{\left(1+xq^{a+\Phi_{2t-1}+\Psi_n}\right)\left(1+yq^{b+\Phi_{2t}+\Psi_n}\right)},$$

(iii) For m is odd and K is even,

$$\mathtt{SUM}_K^{(3)} := \sum_{d=K}^{\min(m,n)} \frac{\left(xq^{a+\Phi_m}\right)^{d-1}\left(1+yq^{b+\Phi_d+\Psi_d}\right)}{\left(1+xq^{a+\Phi_m+\Psi_d}\right)\left(1-x^{-1}yq^{b-a+\Phi_d-\Phi_m}\right)} \times \prod_{t=1}^{d-1} \frac{q^{\Psi_t}\left(1-q^{\Psi_n-\Psi_t}\right)}{1+xq^{a+\Phi_m+\Psi_t}} \prod_{t=1}^{\lfloor d/2\rfloor} \frac{\left(1-q^{\Phi_{2t-1}-\Phi_m}\right)\left(1-x^{-1}yq^{b-a+\Phi_{2t}-\Phi_m}\right)}{\left(1+xq^{a+\Phi_{2t-1}+\Psi_n}\right)\left(1+yq^{b+\Phi_{2t}+\Psi_n}\right)},$$

(iv) For m and K are even,

$$\mathtt{SUM}_K^{(4)} := \sum_{d=K}^{\min(m,n)} \frac{\left(yq^{b+\Phi_m}\right)^{d-1}\left(1+yq^{b+\Phi_d+\Psi_d}\right)}{1+yq^{b+\Phi_m+\Psi_d}} \prod_{t=1}^{d-1} \frac{q^{\Psi_t}\left(1-q^{\Psi_n-\Psi_t}\right)}{1+yq^{b+\Phi_m+\Psi_t}} \times \prod_{t=1}^{\lfloor d/2\rfloor} \frac{1-xy^{-1}q^{a-b+\Phi_{2t-1}-\Phi_m}}{\left(1+xq^{a+\Phi_{2t-1}+\Psi_n}\right)\left(1+yq^{b+\Phi_{2t}+\Psi_n}\right)} \prod_{t=1}^{\lfloor (d-1)/2\rfloor} \left(1-q^{\Phi_{2t}-\Phi_m}\right).$$

Now we present the following Lemma to evaluate $\mathtt{SUM}_K^{(\mathbf{i})}$ for $1\le i\le 4$.

Lemma 3.1. *(i) For both odd m, K,*

$$\mathtt{SUM}_K^{(1)} = \frac{\left(xq^{a+\Phi_m}\right)^{K-1}}{1+xq^{a+\Phi_m+\Psi_n}} \prod_{t=1}^{K-1} \frac{q^{\Psi_t}\left(1-q^{\Psi_n-\Psi_t}\right)}{1+xq^{a+\Phi_m+\Psi_t}} \times \prod_{t=1}^{(K-1)/2} \frac{\left(1-q^{\Phi_{2t-1}-\Phi_m}\right)\left(1-x^{-1}yq^{b-a+\Phi_{2t}-\Phi_m}\right)}{\left(1+xq^{a+\Phi_{2t-1}+\Psi_n}\right)\left(1+yq^{b+\Phi_{2t}+\Psi_n}\right)}. \tag{3.2}$$

(ii) For even m, odd K,

$$\mathtt{SUM}_K^{(2)} = \frac{\left(yq^{b+\Phi_m}\right)^{K-1}}{1+yq^{b+\Phi_m+\Psi_n}} \prod_{t=1}^{K-1} \frac{q^{\Psi_t}\left(1-q^{\Psi_n-\Psi_t}\right)}{1+yq^{b+\Phi_m+\Psi_t}} \times \prod_{t=1}^{(K-1)/2} \frac{\left(1-q^{\Phi_{2t}-\Phi_m}\right)\left(1-xy^{-1}q^{a-b+\Phi_{2t-1}-\Phi_m}\right)}{\left(1+xq^{a+\Phi_{2t-1}+\Psi_n}\right)\left(1+yq^{b+\Phi_{2t}+\Psi_n}\right)}. \tag{3.3}$$

(iii) For odd m, even K,

$$\mathtt{SUM}_K^{(3)} = \frac{\left(xq^{a+\Phi_m}\right)^{K-1}\left(1+yq^{b+\Phi_K+\Psi_n}\right)}{\left(1+xq^{a+\Phi_m+\Psi_n}\right)\left(1-x^{-1}yq^{b-a+\Phi_K-\Phi_m}\right)} \prod_{t=1}^{K-1} \frac{q^{\Psi_t}\left(1-q^{\Psi_n-\Psi_t}\right)}{1+xq^{a+\Phi_m+\Psi_t}} \times \prod_{t=1}^{K/2} \frac{\left(1-q^{\Phi_{2t-1}-\Phi_m}\right)\left(1-x^{-1}yq^{b-a+\Phi_{2t}-\Phi_m}\right)}{\left(1+xq^{a+\Phi_{2t-1}+\Psi_n}\right)\left(1+yq^{b+\Phi_{2t}+\Psi_n}\right)}. \tag{3.4}$$

(iv) For both even m, K,

$$\mathtt{SUM}_K^{(4)} = \frac{\left(yq^{b+\Phi_m}\right)^{K-1}\left(1+yq^{b+\Phi_K+\Psi_n}\right)}{1+yq^{b+\Phi_m+\Psi_n}} \prod_{t=1}^{K-1} \frac{q^{\Psi_t}\left(1-q^{\Psi_n-\Psi_t}\right)}{1+yq^{b+\Phi_m+\Psi_t}} \times \prod_{t=1}^{K/2} \frac{1-xy^{-1}q^{a-b+\Phi_{2t-1}-\Phi_m}}{\left(1+xq^{a+\Phi_{2t-1}+\Psi_n}\right)\left(1+yq^{b+\Phi_{2t}+\Psi_n}\right)} \prod_{t=1}^{(K-2)/2}\left(1-q^{\Phi_{2t}-\Phi_m}\right). \tag{3.5}$$

Proof. To prove these results, we consider odd or even cases of both m and K. First when m is odd, consider the results given in (3.2) and (3.4) as these two results will be thought together. Similarly, for the case m is even, the results in (3.3) and (3.5) will be considered together.

For the case m and K are odd, we have to prove that

$$\mathtt{SUM}_{K-1}^{\mathbf{(3)}} = \mathtt{SUM}_K^{\mathbf{(1)}} + S_{K-1}^{(3)},$$

where $S_d^{(t)}$ is the summand of $\mathtt{SUM}_{\mathbf{K}}^{\mathbf{(t)}}$ $(1 \le t \le 4)$.

Now we prove the following equation in which these two relations are common in the case where m is odd. In general we assume that $m \ge n$ because the case $n > m$ is similar. Note the claim is obvious when $K = n$. Since $K-1$ is even, the backward induction step amounts to show that

$$\begin{aligned}
\mathtt{SUM}^{(\mathbf{3})}_{K-1} &= \mathtt{SUM}^{(\mathbf{1})}_{K} + S^{(3)}_{K-1} \\
&= \frac{\left(xq^{a+\Phi_m}\right)^{K-1}}{1+xq^{a+\Phi_m+\Psi_n}} \prod_{t=1}^{K-1} \frac{q^{\Psi_t}\left(1-q^{\Psi_n-\Psi_t}\right)}{1+xq^{a+\Phi_m+\Psi_t}} \\
&\quad \times \prod_{t=1}^{(K-1)/2} \frac{\left(1-q^{\Phi_{2t-1}-\Phi_m}\right)\left(1-x^{-1}yq^{b-a+\Phi_{2t}-\Phi_m}\right)}{\left(1+xq^{a+\Phi_{2t-1}+\Psi_n}\right)\left(1+yq^{b+\Phi_{2t}+\Psi_n}\right)} \\
&\quad + \frac{\left(xq^{a+\Phi_m}\right)^{K-2}\left(1+yq^{b+\Phi_{K-1}+\Psi_{K-1}}\right)}{\left(1+xq^{a+\Phi_m+\Psi_{K-1}}\right)\left(1-x^{-1}yq^{b-a+\Phi_{K-1}-\Phi_m}\right)} \prod_{t=1}^{K-2} \frac{q^{\Psi_t}\left(1-q^{\Psi_n-\Psi_t}\right)}{1+xq^{a+\Phi_m+\Psi_t}} \\
&\quad \times \prod_{t=1}^{(K-1)/2} \frac{\left(1-q^{\Phi_{2t-1}-\Phi_m}\right)\left(1-x^{-1}yq^{b-a+\Phi_{2t}-\Phi_m}\right)}{\left(1+xq^{a+\Phi_{2t-1}+\Psi_n}\right)\left(1+yq^{b+\Phi_{2t}+\Psi_n}\right)} \\
&= \frac{\left(xq^{a+\Phi_m}\right)^{K-2}}{1+xq^{a+\Phi_m+\Psi_{K-1}}} \prod_{t=1}^{K-2} \frac{q^{\Psi_t}\left(1-q^{\Psi_n-\Psi_t}\right)}{1+xq^{a+\Phi_m+\Psi_t}} \\
&\quad \times \prod_{t=1}^{(K-1)/2} \frac{\left(1-q^{\Phi_{2t-1}-\Phi_m}\right)\left(1-x^{-1}yq^{b-a+\Phi_{2t}-\Phi_m}\right)}{\left(1+xq^{a+\Phi_{2t-1}+\Psi_n}\right)\left(1+yq^{b+\Phi_{2t}+\Psi_n}\right)} \\
&\quad \times \left(\frac{xq^{a+\Phi_m+\Psi_{K-1}}\left(1-q^{\Psi_n-\Psi_{K-1}}\right)}{1+xq^{a+\Phi_m+\Psi_n}} + \frac{1+yq^{b+\Phi_{K-1}+\Psi_{K-1}}}{1-x^{-1}yq^{b-a+\Phi_{K-1}-\Phi_m}} \right).
\end{aligned}$$

After some simplifications, we get

$$\begin{aligned}
\mathtt{SUM}^{(\mathbf{3})}_{K-1} &= \frac{\left(xq^{a+\Phi_m}\right)^{K-2}\left(1+yq^{b+\Phi_{K-1}+\Psi_n}\right)}{\left(1+xq^{a+\Phi_m+\Psi_n}\right)\left(1-x^{-1}yq^{b-a+\Phi_{K-1}-\Phi_m}\right)} \prod_{t=1}^{K-2} \frac{q^{\Psi_t}\left(1-q^{\Psi_n-\Psi_t}\right)}{1+xq^{a+\Phi_m+\Psi_t}} \\
&\quad \times \prod_{t=1}^{(K-1)/2} \frac{\left(1-q^{\Phi_{2t-1}-\Phi_m}\right)\left(1-x^{-1}yq^{b-a+\Phi_{2t}-\Phi_m}\right)}{\left(1+xq^{a+\Phi_{2t-1}+\Psi_n}\right)\left(1+yq^{b+\Phi_{2t}+\Psi_n}\right)},
\end{aligned}$$

which completes the proof.
The remaining cases could be proven by a similar fashion. □

Especially, we derive the claimed results for the matrix $\mathcal{G}$, we need. When $K = 1$ in (3.2)-(3.4) and (3.3)-(3.5), we obtain the results for the LU-decomposition of $\mathcal{G}$ as shown

$$\frac{1}{1+xq^{a+\Phi_m+\Psi_n}} = \sum_{1\le d\le \min(m,n)} L_{m,d}U_{d,n} = \begin{cases} \mathtt{SUM}^{(\mathbf{1})}_{1} & \text{if } n \text{ is odd,} \\ \mathtt{SUM}^{(\mathbf{3})}_{1} & \text{if } n \text{ is even} \end{cases}$$

and

$$\frac{1}{1+yq^{b+\Phi_m+\Psi_n}} = \sum_{1\le d\le \min(m,n)} L_{m,d}U_{d,n} = \begin{cases} \mathtt{SUM}^{(\mathbf{2})}_{1} & \text{if } n \text{ is odd,} \\ \mathtt{SUM}^{(\mathbf{4})}_{1} & \text{if } n \text{ is even.} \end{cases}$$

The proofs of L^{-1} and U^{-1} may be done similarly. Also, we use the equation $\mathcal{G}_N^{-1} = U^{-1}L^{-1}$ when proving the inverse of the $\mathcal{G}$ matrix. Finally the determinant of $\mathcal{G}$ is the product of the main diagonal entries of the matrix U.

4. Applications

In this section, we will define the matrix $\mathcal{H} = [\mathcal{H}_{m,n}]$ as shown

$$\mathcal{H}_{m,n} = \begin{cases} \dfrac{1}{U_{a+\lambda(p+m)^v+\mu(r+n)^w}} & \text{if } m \text{ is odd,} \\ \\ \dfrac{1}{V_{b+\lambda(p+m)^v+\mu(r+n)^w}} & \text{if } m \text{ is even} \end{cases}$$

or in closed form

$$\mathcal{H}_{m,n} = \begin{cases} \dfrac{1}{U_{a+\Phi_m+\Psi_n}} & \text{if } m \text{ is odd,} \\ \\ \dfrac{1}{V_{b+\Phi_m+\Psi_n}} & \text{if } m \text{ is even,} \end{cases}$$

where

$$\Phi_i := \lambda\,(i+p)^v \quad \text{and} \quad \Psi_i := \mu\,(i+r)^w$$

for arbitrary integers a, b, p, r, λ, μ, v, w.

For example, the matrix $\mathcal{H}$ of order 4 takes the form

$$\mathcal{H} = \begin{bmatrix} \frac{1}{U_{a+\Phi_1+\Psi_1}} & \frac{1}{U_{a+\Phi_1+\Psi_2}} & \frac{1}{U_{a+\Phi_1+\Psi_3}} & \frac{1}{U_{a+\Phi_1+\Psi_4}} \\ \frac{1}{V_{b+\Phi_2+\Psi_1}} & \frac{1}{V_{b+\Phi_2+\Psi_2}} & \frac{1}{V_{b+\Phi_2+\Psi_3}} & \frac{1}{V_{b+\Phi_2+\Psi_4}} \\ \frac{1}{U_{a+\Phi_3+\Psi_1}} & \frac{1}{U_{a+\Phi_3+\Psi_2}} & \frac{1}{U_{a+\Phi_3+\Psi_3}} & \frac{1}{U_{a+\Phi_3+\Psi_4}} \\ \frac{1}{V_{b+\Phi_4+\Psi_1}} & \frac{1}{V_{b+\Phi_4+\Psi_2}} & \frac{1}{V_{b+\Phi_4+\Psi_3}} & \frac{1}{V_{b+\Phi_4+\Psi_4}} \end{bmatrix}.$$

By the Binet formulas of U_n and V_n, we rewrite the matrix $\mathcal{H}$ as shown

$$\mathcal{H}_{m,n} = \frac{1}{\alpha^{\Phi_m+\Psi_n}} \begin{cases} \dfrac{1-q}{\alpha^{a-1}} \dfrac{1}{1+xq^{a+\Phi_m+\Psi_n}} & \text{if } m \text{ is odd,} \\ \\ \alpha^{-b} \dfrac{1}{1+yq^{b+\Phi_m+\Psi_n}} & \text{if } m \text{ is even,} \end{cases}$$

where $x=-1$, $y=1$, $q=\beta/\alpha$ and

$$\Phi_i := \lambda\,(i+p)^v \quad \text{and} \quad \Psi_i := \mu\,(i+r)^w.$$

Throughout this paper, in general, we already defined the matrix

$$\mathcal{G} = [\mathcal{G}_{m,n}] = \begin{cases} \dfrac{1}{1+xq^{a+\Phi_m+\Psi_n}} & \text{if } m \text{ is odd,} \\ \\ \dfrac{1}{1+yq^{b+\Phi_m+\Psi_n}} & \text{if } m \text{ is even,} \end{cases}$$

and derived its some properties such that LU-decomposition, L^{-1},U^{-1}, $\mathcal{G}^{-1}$ for arbitrary reals x, y and q.

Note that the matrix $\mathcal{H}$ is a special case of the matrix $\mathcal{G}$ without the constant factors. Therefore we could derive the properties (such as LU-decomposition, L^{-1},U^{-1}, $\mathcal{G}^{-1}$) of $\mathcal{H}$ from the main results of this paper given for $\mathcal{G}$ for the values $y=-x=1$ and $q=\beta/\alpha$.

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