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On the Jost Solutions of A Class of the Quadratic Pencil of the Sturm-Liouville Equation

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Abstract: In this study we construct new integral representations of Jost-type solutions of the quadratic pencil of the Sturm-Liouville equation with the piece-wise constant coefficient on the entire real line. Our aim is to express the special solutions of the Sturm-Liouville quadratic pencil in the form of some integral operators which kernels is related with the potential function of the Sturm-Liouville equation. This problem is technically diffucult due to the discontinous coefficient which causes the kernel function to also have a jump discontinuity.

Key words: Sturm-Liouville equation, operator pencil, transformation operator, diferantial equation with discontinuous cofficients, Jost solution

1. Introduction

We will focus at the Sturm-Liouville equation

$$-y'' + q(u)y + 2\tau p(u)y = \tau^2 \rho(u)y\,, u \in (-\infty, +\infty) \tag{1}$$

which has the discontinuous coefficient

$$\rho(u) = \begin{cases} 1, & u \geq 0 \\ \alpha^2, & u < 0 \end{cases} (\alpha \neq 1, \alpha > 0), \tag{2}$$

where $q(u)$ and $p(u)$ are real functions, τ is a complex parameter, $p(u)$ is absolutely continuous on every closed interval of the real axis and

$$\int_{-\infty}^{+\infty} |p(u)|du < \infty\,, \int_{-\infty}^{+\infty} (1+|u|)\left(|q(u)| + |p'(u)|\right)du < +\infty \tag{3}$$

Equation (1) arises when solving the Klein-Gordon equation with a static potential and zero charge in quantum scattering theory [7]. In addition, scattering problems arising in the theory of transmission lines, theories of electromagnetism, and the theory of elasticity are also reduced to equation (1). The transformation operators approach, which Marchenko [8,1] used to solve the inverse problems for the Sturm-Liouville operator on a finite interval and on the half line, is an important method in inverse problems theory.

When $\rho(u) = 1$, there are enough studies in the literature for the direct and inverse problems of equation (1) [7, 6, 3, 11]. Inverse scattering problems related to the discontinuous Sturm-Liouville have been considered by many authors, for details we refer to [2, 9, 4, 5]. The direct and inverse scattering problems for equation (1) with $p(u) = 0$ in various settings have been investigated [4, 5, 13] where some integral representations, similar transformations operators for the Jost solutions (Js) of the Schroedinger equation, are obtained and applied for studying the discussed problems. In this study we construct new integral representations of Jost-type solutions of equation (1) on the entire axis under conditions (2) and (3). Our aim is to express the special solutions of the Sturm-Liouville quadratic pencil in the form of some integral operators which kernel is related with the potential of the equation (1). This problem is technically diffucult due to the discontinous coefficient which causes the kernel function to also have a jump discontinuity.

2. Integral representation of the Jost solutions

Let $g_{\pm}(u,\tau)$ are the solutions of (1) with the condition at infinity

$$\lim_{u\to\pm\infty} g_{\pm}(u,\tau)\exp\big(\mu i\tau(u)\big) = 1,$$

where $\exp(\mathrm{u}) = e^{\mathrm{u}}$ and $\mu(u) = u[\rho(u)]^{\frac{1}{2}}$. $g_{+}(u,\tau)$ and $g_{-}(u,\tau)$ are called the right and the left Jost solutions (rlJs) of (1) respectively. First, let's transform the given differential equation with the above conditions at infinity into an equivalent tintegral equation. We easily have

$$g_{\pm}(u,\tau) = F_{\pm}(u,\tau) + \int_{u}^{\pm\infty} N(u,t,\tau)\,\big(q(t) + 2\tau p(t)\big) g_{\pm}(t,\tau)dt \tag{4}$$

where,

$$F_{+}(u,\tau) = F\left(u,\rho^{-\frac{1}{2}}(u),\tau\right) + F\left(u,-\rho^{-\frac{1}{2}}(u),-\tau\right)$$

$$F_{-}(u,\tau) = F\left(u,-\alpha\rho^{-\frac{1}{2}}(u),\tau\right) + F\left(u,\alpha\rho^{-\frac{1}{2}}(u),-\tau\right)$$

with

$$F(u,h(u),\tau) = \frac{1}{2}\big(1 + h(u)\big)e^{i\tau\mu(u)}$$

and

$$N(u,t,\tau) = \frac{1}{2}\left(\rho^{-\frac{1}{2}}(t) - \rho^{-\frac{1}{2}}(u)\right)\int_{0}^{\mu(t)+\mu(u)} cos\tau s ds$$
$$+ \frac{1}{2}\left(\rho^{-\frac{1}{2}}(t) - \rho^{-\frac{1}{2}}(u)\right)\int_{0}^{\mu(t)-\mu(u)} cos\tau s ds$$

Think about the solution $g_+(u,\tau)$. It is known that [6, 3] for $u>0$ and all $Im\tau \geq 0$ the solution $g_+(u,\tau)$ has the representation

$$g_+(u,\tau) = \exp(i\,\tau u + i\omega_+(u)) + \int_u^{+\infty} A^+(u,t)\exp(i\tau t)\,dt\,, \tag{5}$$

where $\omega_+(u) = \int_u^{+\infty} p(t)dt$ and the function $A^+(u,t)$ satisfies

$$\int_u^{+\infty} |A^+(u,t)| \leq C_0(exp(\sigma^+(u)) - 1) \tag{6}$$

for some constant $C_0 > 0$ and $\sigma^+(u) = \left(\int_u^{+\infty}(s-u)|q(t)| + 2|p(t)|\right)dt$. Furthermore, the kernel function $A^+(u,t)$ satisfies the condition

$$A^+(u,u) = \frac{1}{2}\left(\int_u^{+\infty}[q(t)+p^2(t)]dt - ip(u)\right)\exp\left(i\omega_+(u)\right). \tag{7}$$

For convenience afterwards we set $A^+(u,t) = F^+(w,z)$ with $w-z=u\,, w+z=t$.

Now we investigate the case $u<0$ for the solution $g_+(u,\tau)$. Let us set $\alpha_\pm = \frac{1}{2}\left(1 \pm \frac{1}{\alpha}\right)$. In this case the equation (4) takes the form of

$$\begin{aligned} g_+(u,\tau) = {} & \alpha_+\exp(i\alpha\tau u) + \alpha_-\exp(-i\alpha\tau u) \\ & - \int_u^0 \frac{sin\alpha\tau(u-s)}{\tau} q(s)g_+(s,\tau)ds \\ & + \int_0^\infty \left[\alpha^- \frac{sin\tau(\alpha u+s)}{\tau} - \alpha^+\frac{sin\tau(\alpha u - s)}{\tau}\right] \\ & * [q(t)+2\tau p(t)]g_+(s,\tau)ds \end{aligned} \tag{8}$$

We easily reveal that equation (8) has the solution

$$\begin{aligned} g_+(u,\tau) = {} & R_+(u)exp(i\alpha\tau u) + R_-(u)exp(-i\alpha\tau u) \\ & + \int_{\alpha u}^{+\infty} B^+(u,s)\exp(i\tau s)\,ds\,, Im\tau \geq 0, u<0 \end{aligned} \tag{9}$$

where

$$R_\pm(u) = \alpha_\pm \exp(i\omega_+(0) \pm \frac{i}{\alpha}\int_u^0 p(\beta)d\beta)$$

and $B^+(u,s) = G^+(w,z)$ $(\alpha u = w - z\,, s = w+z)$ is defined after replacing $g_+(u,\tau)$ in equation (8) with formulas (5), (9) and transforming some integrals of the Fourier type:

$$G^{+}(w,z)=\frac{1}{2\alpha}\int_{\frac{w}{\alpha}}^{0} q(\beta)\,R_{+}(\beta)d\beta+\frac{1}{2\alpha}\int_{\frac{-z}{\alpha}}^{0} q(\beta)\,R_{-}(\beta)d\beta-\frac{i}{2\alpha^{2}}p\left(\frac{w}{\alpha}\right)R_{+}\left(\frac{w}{\alpha}\right)$$
$$+\frac{i}{2\alpha^{2}}p\left(\frac{-z}{\alpha}\right)R_{-}\left(\frac{-z}{\alpha}\right)+\frac{\alpha_{+}}{2}\int_{0}^{+\infty} q(\beta)e^{i\omega_{+}(\beta)}\,d\beta$$
$$-\frac{\alpha_{-}}{2}\int_{0}^{z} q(\beta)e^{i\omega_{+}(\beta)}\,d\beta-\frac{i\alpha_{-}}{2}p(z)e^{i\omega_{+}(z)}$$
$$+\alpha^{+}\int_{0}^{z} ds\int_{s}^{+\infty} q(\gamma-s)Z^{+}(\gamma,s)d\gamma-\alpha^{-}\int_{0}^{z} ds\int_{s}^{z} q(\gamma-s)Z^{+}(\gamma,s)d\gamma$$
$$+\frac{1}{\alpha^{2}}\int_{0}^{z} ds\int_{w}^{s} q\left(\frac{\gamma-s}{\alpha}\right)G^{+}(\gamma,s)d\gamma+i\alpha_{+}\int_{z}^{+\infty} p(\gamma-s)Z^{+}(\gamma,s)d\gamma$$
$$-i\alpha_{-}\int_{0}^{z} p(z-\delta)A^{+}(z,\delta)d\delta+\frac{1}{i\alpha^{2}}\int_{0}^{z} p\left(\frac{w}{\alpha}-\frac{\delta}{\alpha}\right)Z^{+}(w,\delta)d\delta$$
$$-\frac{1}{i\alpha^{2}}\int_{w}^{z} p\left(\frac{\gamma}{\alpha}-\frac{z}{\alpha}\right)G^{+}(\gamma,z)d\gamma\,,\qquad z>0\,,w<0$$

$$G^{+}(w,z)=\frac{\alpha^{+}}{2}\int_{w}^{+\infty} q(\beta)e^{i\omega_{+}(\beta)}\,d\beta+\frac{\alpha^{-}}{2}\int_{z}^{+\infty} q(\beta)e^{i\omega_{+}(\beta)}\,d\beta-\frac{i\alpha_{-}}{2}p(z)e^{i\omega_{+}(z)}$$
$$-\frac{i\alpha_{+}}{2}p(w)e^{i\omega_{+}(w)}+\alpha^{+}\int_{w}^{z} ds\int_{\delta}^{+\infty} q(\gamma-s)Z^{+}(\gamma,s)d\gamma$$
$$+\alpha^{+}\int_{0}^{z} ds\int_{w}^{+\infty} q(\gamma-s)Z^{+}(\gamma,s)d\gamma+\alpha^{-}\int_{0}^{z} ds\int_{z}^{+\infty} q(\gamma-s)Z^{+}(\gamma,s)d\gamma$$
$$-\alpha^{-}\int_{w}^{z} ds\int_{s}^{z} q(\gamma-s)A^{+}(\gamma-s,\gamma+s)d\gamma$$
$$+\frac{1}{\alpha^{2}}\int_{w}^{z} ds\int_{w}^{s} q\left(\frac{\gamma-s}{\alpha}\right)G^{+}(\gamma,s)d\gamma+\ i\alpha_{+}\int_{z}^{+\infty} p(\gamma-z)Z^{+}(\gamma,z)d\gamma$$
$$-\ i\alpha_{+}\int_{0}^{w} p(w-\delta)Z^{+}(w,\delta)d\delta$$
$$-i\alpha_{-}\int_{0}^{w} p(z-\delta)Z^{+}(z,\delta)d\delta+i\alpha_{-}\int_{w}^{+\infty} p(\gamma-w)Z^{+}(\gamma,w)d\gamma \tag{10}$$
$$+\frac{1}{i\alpha^{2}}\int_{w}^{z} p\left(\frac{w}{\alpha}-\frac{s}{\alpha}\right)G^{+}(w,s)ds$$
$$-\frac{1}{i\alpha^{2}}\int_{w}^{z} p\left(\frac{\gamma}{\alpha}-\frac{z}{\alpha}\right)G^{+}(\gamma,z)d\gamma\,,0<w<z\,. \tag{11}$$

Here we assume $A^{+}(u,t)\equiv 0$ for $t<u$ and $B^{+}(u,t)\equiv 0$ for $t<\alpha u$.

Similarly, when we consider the solution $g_-(u,\tau)$, we get for $u<0$

$$g_-(u,\tau)=exp(-i\alpha\tau u+i\omega_-(u))+\int_{-\infty}^{\alpha u} A^-(u,t)\exp(-i\tau t)\,dt\,, Im\tau\geq 0\,, \qquad (12)$$

where $\omega_-(u)=\frac{1}{\alpha}\int_{-\infty}^{u} p(t)dt$ and the kernel function $A^-(u,s)=Z^-(w,z), u=w+z, s=\alpha(w-z)$ satisfies the integral equation

$$Z^-(w,z)=\frac{1}{2\alpha}\int_{-\infty}^{w} q(s)ds+\frac{1}{2i\alpha^2}p(w)e^{i\omega_-(w)}+\int_{-\infty}^{w} d\gamma\int_{0}^{z} q(\gamma+s)Z^-(\gamma,s)ds$$
$$+\frac{1}{i\alpha}\int_{0}^{z} p(w+\delta)\,A^-(w,\delta)d\delta$$
$$-\frac{1}{i\alpha}\int_{-\infty}^{w} p(\gamma+z)\,A^-(\gamma,z)d\gamma, w<-z\leq 0 \qquad (13)$$

which implies

$$\int_{-\infty}^{\alpha u} |A^-(u,t)dt|\leq C_1(exp(\sigma^-(u))-1) \qquad (14)$$

for some constant $C_1>0$ and $\sigma^-(u)=\left(\int_{-\infty}^{u}(u-s)|q(t)|+\frac{2}{\alpha}|p(t)|\right)dt.$ Here $A^-(u,t)\equiv 0$ for $t>\alpha u$. Moreover, the kernel function $A^-(u,t)$ satisfies the condition

$$A^-(u,\alpha u)=\frac{1}{2\alpha}\left(\int_{-\infty}^{u}\left[q(t)+\frac{1}{\alpha^2}p^2(t)\right]dt+\frac{1}{2i\alpha}p(u)\right)\exp\big(i\omega_-(u)\big). \qquad (15)$$

As in the case of the (rJs) we have for $u>0$

$$g_-(u,\tau)=T_+(u)exp(i\tau u)+T_-(u)exp(-i\tau u)$$
$$+\int_{-\infty}^{U} B^-(u,t)\exp(-i\tau t)\,dt\,, Im\tau\geq 0, u>0 \qquad (16)$$

where

$$T_\pm(u)=\frac{1}{2}(1\mp\alpha)e^{i\omega_-(0)\mp i\int_0^u p(s)ds}$$

and $B^-(u,s)=G^-(w,z), u=w+z, s=w-z$ with

$$G^-(w,z)=\frac{\alpha^-}{2}\int_{\frac{-z}{\alpha}}^{0}q(\beta)\,e^{i\omega_-(\beta)}d\beta+\frac{\alpha^+}{2}\int_{-\infty}^{0}q(\beta)\,e^{i\omega_-(\beta)}d\beta+\frac{1}{2}\int_{0}^{w}q(\beta)\,T_+(\beta)d\beta$$
$$+\frac{1}{2}\int_{0}^{z}q(\beta)\,T_-(\beta)d\beta+\frac{i\alpha_-}{2\alpha}p\left(\frac{-z}{\alpha}\right)e^{i\omega_-\left(\frac{-z}{\alpha}\right)}-\frac{i}{2}p(w)T_-(w)$$
$$+\frac{i}{2}p(z)T_+(z)+\alpha\alpha^-\int_{0}^{\frac{z}{\alpha}}ds\int_{\frac{-z}{\alpha}}^{-s}q(\gamma+s)Z^-(\gamma,s)d\gamma$$
$$+\alpha\alpha^+\int_{0}^{\frac{z}{\alpha}}ds\int_{-\infty}^{-s}q(\gamma+s)Z^-(\gamma,s)d\gamma+\int_{0}^{z}ds\int_{-s}^{w}q(\gamma+s)G^-(\gamma,s)d\gamma$$
$$+i\alpha_-\int_{0}^{\frac{z}{\alpha}}p\left(\frac{-z}{\alpha}+\gamma\right)Z^-\left(\frac{-z}{\alpha},\gamma\right)d\gamma$$
$$+i\alpha_+\int_{-\infty}^{\frac{-z}{\alpha}}p\left(\gamma+\frac{z}{\alpha}\right)Z^-\left(\gamma,\frac{z}{\alpha}\right)d\gamma+i\int_{-z}^{w}p(\gamma+z)G^-(\gamma,z)d\gamma$$
$$-\ i\int_{0}^{z}p(w+s)G^-(w,s)ds\ ,w>0\,,z>0 \qquad (17)$$

$$G^-(w,z)=\frac{\alpha^+}{2}\int_{-\infty}^{\frac{w}{\alpha}}q(\beta)\,e^{i\omega_-(\beta)}d\beta-\frac{\alpha^-}{2}\int_{-\infty}^{\frac{-z}{\alpha}}q(\beta)\,e^{i\omega_-(\beta)}d\beta$$
$$+\frac{i\alpha_-}{2\alpha}p(-\alpha^{-1}z)e^{i\omega_-(-\alpha^{-1}z)}-\frac{i\alpha_+}{2\alpha}p(\alpha^{-1}w)e^{i\omega_-(\alpha^{-1}w)}$$
$$+\alpha^-\alpha\int_{\frac{-w}{\alpha}}^{\frac{z}{\alpha}}ds\int_{\frac{-z}{\alpha}}^{-s}q(\gamma+s)Z^-(\gamma,s)d\gamma$$
$$-\alpha^-\alpha\int_{0}^{\frac{-w}{2}}ds\int_{-\infty}^{\frac{-z}{\alpha}}q(\gamma+s)Z^-(\gamma,s)d\gamma$$
$$+\alpha\alpha^+\int_{0}^{\frac{-w}{\alpha}}ds\int_{-\infty}^{\frac{w}{\alpha}}q(\gamma+s)Z^-(\gamma,s)d\gamma$$
$$+\alpha\alpha^+\int_{\frac{-w}{\alpha}}^{\frac{z}{\alpha}}ds\int_{-\infty}^{-s}q(\gamma+s)Z^-(\gamma,s)d\gamma+\int_{-w}^{z}ds\int_{-s}^{w}q(\gamma+s)G^-(\gamma,s)d\gamma$$

$$-i\alpha_{-}\int_{-\infty}^{\frac{w}{\alpha}} p\left(\gamma-\frac{w}{\alpha}\right)Z^{-}(\gamma,w)d\gamma+i\alpha_{-}\int_{0}^{\frac{z}{\alpha}} p\left(-\frac{z}{\alpha}+\delta\right)Z^{-}\left(-\frac{z}{\alpha},\delta\right)d\delta$$
$$+i\alpha_{+}\int_{-\infty}^{\frac{-z}{\alpha}} p\left(\gamma+\frac{z}{\alpha}\right)Z^{-}\left(\gamma,\frac{z}{\alpha}\right)d\gamma$$
$$-i\alpha_{+}\int_{0}^{\frac{-w}{\alpha}} p\left(\frac{w}{\alpha}+\delta\right)Z^{-}\left(\frac{w}{\alpha},\delta\right)d\delta+i\int_{-z}^{w} p(\gamma+z)G^{-}(\gamma,z)d\gamma$$
$$-\,i\int_{0}^{-z} p(w-\delta)G^{-}(w,\delta)d\delta\ ,-z\le w<0 \tag{18}$$

and $B^{-}(u,t)\equiv 0$ for $t>u$. Estimating (10), (11) and (17), (18) and for some $C>0$, we easily achieve

$$\pm\int_{\alpha u}^{\pm\infty} |B^{\pm}(u,t)|dt\le C\{exp(\sigma^{\pm}(\alpha u))-1\} \tag{19}$$

As a result of setting $K^{\pm}(u,t)=\begin{cases}A^{\pm}(u,t) & ,\pm u\ge 0\\ B^{\pm}(u,t) & ,\pm u<0\end{cases}$ and combining all of our results, we may derive the following theorems.

Theorem 1. *If (3) is satisfied and* $Im\tau\ge 0$ *then the (Js)* $g_{+}(u,\tau)$ *has the representation*

$$g_{+}(u,\tau)=R_{+}(u)e^{i\tau\mu(u)}+R_{-}(u)e^{i\tau\mu(u)}+\int_{\mu(u)}^{+\infty} K^{+}(u,t)\,\mathrm{e}^{i\tau t}dt\,, \tag{20}$$

where,

$$R_{+}(u)=\frac{1}{2}\left(1+\left(\rho(u)\right)^{-\frac{1}{2}}\right)exp\left(\int_{u}^{+\infty} i\left(\rho(t)\right)^{-\frac{1}{2}}p(t)dt\right),$$
$$R_{-}(u)=\frac{1}{2}\left(1-\left(\rho(u)\right)^{-\frac{1}{2}}\right)exp\left(i\int_{u}^{+\infty} \left(\rho(t)\right)^{-\frac{1}{2}}p(t)sgnt dt\right),$$

$\mu(u)=u[\rho(u)]^{-\frac{1}{2}}$ *and* $K^{+}(u,t)$ *satisfies*

$$\int_{\mu(u)}^{+\infty} |K^{+}(u,t)|dt\le C\{exp(\sigma^{+}(u))-1\}\ (C>0). \tag{21}$$

Furthermore,

$$K^{+}\big(u,\mu(u)\big)=R_{+}(u)\left\{\frac{1}{2}\int_{u}^{+\infty}\left(\rho^{-\frac{1}{2}}(s)q(s)+\rho^{-\frac{3}{2}}(s)p^{2}(s)\right)ds\right.$$
$$\left.+\frac{i}{2}\int_{u}^{+\infty}\left[\left(\frac{1}{\rho^{\frac{1}{2}}(s)}\right)^{2}+\left(1-\frac{\rho^{\frac{1}{2}}(s)}{\rho^{\frac{1}{2}}(x)}\right)^{2}\right]p^{'}(s)ds\right\}, \quad (22)$$

$$K^{+}(u,t)\big|_{t=-\mu(u)-0}^{t=-\mu(u)+0}$$
$$=R_{-}(u)\left\{\frac{1}{2}\int_{u}^{+\infty}\left(\rho^{-\frac{1}{2}}(s)q(s)+\rho^{-\frac{3}{2}}(s)p^{2}(s)\right)sgnsds\right.$$
$$\left.+\frac{i}{2}\int_{u}^{+\infty}\left[\left(\frac{1}{\rho^{\frac{1}{2}}(s)}\right)^{2}+\left(1+\frac{(sgns)\rho^{\frac{1}{2}}(s)}{\rho^{\frac{1}{2}}(x)}\right)^{2}\right]p^{'}(s)ds\right\}, \quad (23)$$

where $sgn\ s$ is the sign function.

Theorem 2. *If (3) is satisfied and $Im\tau\geq 0$ then the (Js) $g_{-}(u,\tau)$ of equation (1) has the representation*

$$g_{-}(u,\tau)=T_{+}(u)e^{i\tau\mu(u)}+T_{-}(u)e^{i\tau\mu(u)}+\int_{-\infty}^{\mu(u)}K^{-}(u,t)\,\mathrm{e}^{-\mathrm{i}\tau\mathrm{t}}dt\,, \quad (24)$$

where,

$$T_{+}(u)=\frac{1}{2}\Big(1-\alpha\big(\rho(u)\big)^{-\frac{1}{2}}\Big)exp\left(\int_{-\infty}^{u}-i\rho(t)^{-\frac{1}{2}}p(t)\,sgntdt\right),$$
$$T_{-}(u)=\frac{1}{2}\Big(1+\alpha\big(\rho(u)\big)^{-\frac{1}{2}}\Big)exp\left(\int_{-\infty}^{u}i\rho(t)^{-\frac{1}{2}}p(t)\,dt\right),$$

$\mu(u)=u[\rho(u)]^{\frac{1}{2}}$ *and $K^{-}(u,t)$ satisfies the inequality*

$$\int_{-\infty}^{\mu(u)}|K^{-}(u,t)|dt\leq C\{exp(\sigma^{-}(u))-1\}\ (C>0)\,. \quad (25)$$

Furthermore, the following expressions are fulfilled:

$$K^{-}\big(u,\mu(u)\big)=T_{-}(u)\left\{\frac{1}{2}\int_{-\infty}^{u}\left(\rho^{-\frac{1}{2}}(s)q(s)+\rho^{-\frac{3}{2}}(s)p^{2}(s)\right)ds\right.$$
$$\left.-\frac{i}{2}\int_{-\infty}^{u}\left[\left(\frac{1}{\rho^{\frac{1}{2}}(s)}\right)^{2}+\left(1-\frac{\rho^{\frac{1}{2}}(x)}{\rho^{\frac{1}{2}}(s)}\right)^{2}\right]p^{'}(s)ds\right\}, \quad (26)$$

$$K^{-}(u,t)\Big|_{t=-\mu(u)-0}^{t=-\mu(u)+0} = T_{+}(u)\left\{\frac{1}{2}\int_{-\infty}^{U}\left(\rho^{-\frac{1}{2}}(s)q(s)+\rho^{-\frac{3}{2}}(s)p^{2}(s)\right)sgnsds + \frac{i}{2}\int_{-\infty}^{u}\left[1+\left(1-\frac{(sgns)\rho^{\frac{1}{2}}(s)}{\rho^{\frac{1}{2}}(x)}\right)^{2}\right]p^{'}(s)ds\right\}, \tag{27}$$

where sgns is the sign function.

4. Conclusion

In this paper, new integral representations for the quadratic pencil of Sturm-Liouville equation with discontinuous coefficients are obtained and, with their help, a connection is established between the potential functions of the equation and the kernel in the representation of the solution. This result is important in studying the properties of spectral data and in solving the inverse scattering problem.

Authorship contribution statement

A. N. Adiloğlu: Conceptualization, Methodology, Supervision.

N. D. Cücen: Data Curation, Original Draft Writing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Ethics Committee Approval and/or Informed Consent Information

As the authors of this study, we declare that we do not have any ethics committee approval and/or informed consent statement.

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