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On Generalized Darboux Frame of a Pseudo Null Curve Lying on a Lightlike Surface in Minkowski 3-Space

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

In this paper we define the generalized Darboux frame of a pseudo null curve α lying on a lightlike surface in Minkowski space $\mathbb{E}_1^3$. We prove that α has two such frames and obtain generalized Darboux frame's equations. We obtain the relations between the curvature functions of α with respect to the Darboux frame and generalized Darboux frames. We also find parameter equations of the Darboux vectors of the Frenet, Darboux and generalized Darboux frames, and give the necessary and sufficient conditions for such vectors to have the same directions. Finally, we present related examples.

Keywords: *Generalized Darboux frame, pseudo null curve, lightlike surface, Minkowski space.*
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1. Introduction

Lightlike (null, degenerate) submanifolds play an important role in General relativity, where they provide models for studying different horizon types (event, Cauchy, Kruskal). Null submanifolds have the property that their normal bundle intersects their tangent bundle and that the induced metric is degenerate, which distinguishes them from semi-Riemannian manifolds ([2]). Some characterizations of lightlike surfaces and hypersurfaces in Minkowski space can be found in [1, 3, 7]. Spacelike surfaces immersed in null hypersurfaces are of interest in singularity theory in General relativity and Cosmology, where they appear as trapped surfaces ([11]).

Pseudo null curves in Minkowski space $\mathbb{E}_1^3$ are the curves whose principal normal and binormal vector fields are null. When such a curve lies on a lightlike surface in $\mathbb{E}_1^3$, its geometric properties can be expressed in the terms of geodesic curvature, normal curvature, and geodesic torsion, determined by the Darboux frame along the curve ([5, 10, 15]). Pseudo null curves lying on non-degenerate or degenerate surfaces in Minkowski 3-space, defined in terms of the Darboux frame's vector fields, are introduced as k-type pseudo null slant helices and isophote curves in [10, 8]. Darboux frame's vector fields are also used in characterizations of different types of curves, such as pseudo-spherical Darboux images and lightcone images of principal-directional curves ([16]) and in investigations of generalized focal surfaces ([6]) and lightlike surfaces along pseudo-spherical normal Darboux images of spacelike curves ([14]).

It is known that the Frenet frame of a pseudo null curve in Minkowski space $\mathbb{E}_1^3$ can be considered as a rotating object, having an axis of the rotation whose direction is given by the *Darboux vector* (angular velocity vector, centrode). On the other hand, the Darboux frame of a pseudo null curve lying on a lightlike surface in Minkowski 3-space, also has an axis of rotation, satisfying the corresponding *Darboux equations*. The relation between Darboux vectors of the Frenet and Darboux frame of a pseudo null curve lying on a spacelike or a timelike surface in $\mathbb{E}_1^3$, is obtained in [8].

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Generalized Darboux frame of a spacelike curve with a non-null principal normal lying on a lightlike surface in Minkowski space $\mathbb{E}^3_1$ is introduced in [4]. In particular, it is shown in [4] that Cartan frame's vector fields of a null Cartan curve β generate lightlike ruled surfaces on which the tangent and the binormal indicatrices of β are the spacelike principal curvature lines having generalized geodesic torsion equal to zero.

In this paper, we define the generalized Darboux frame along a pseudo null curve α lying on a lightlike surface in Minkowski space $\mathbb{E}^3_1$ in such way that α is geodesic, asymptotic, or principal curvature line if and only if the corresponding curvature function of α with respect to a generalized Darboux frame is equal to zero. We prove that α has two frames with the mentioned property. Since they coincide with the Darboux frame in a special case, we called them *generalized Darboux frames of the first* and *the second kind*. In particular, we prove that every pseudo null curve α lying on a lightlike surface is a geodesic or an asymptotic line. We derive the generalized Darboux frame's equations and the relations between the curvature functions of α with respect to the Darboux frame and generalized Darboux frames. We also obtain parameter equations of the Darboux vectors of the Frenet, Darboux and generalized Darboux frames and find the necessary and sufficient conditions for such vectors to have the same directions. We show that only pseudo null curves lying on a lightlike surface in $\mathbb{E}^3_1$, whose Darboux vectors of the Frenet, Darboux, generalized Darboux frame of the first kind and generalized Darboux frame of the second kind have the same direction, are pseudo null circles. Finally, we give some examples.

2. Preliminaries

Minkowski space $\mathbb{E}^3_1$ is the real vector space $\mathbb{E}^3$ equipped with indefinite flat metric $\langle\cdot,\cdot\rangle$ given by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3,$$

for any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{E}^3_1$. Since $\langle\cdot,\cdot\rangle$ is an indefinite metric, a vector $x \neq 0$ in $\mathbb{E}^3_1$ can be *spacelike*, *timelike*, or *null* (*lightlike*) if $\langle x, x\rangle > 0$,$\langle x, x\rangle < 0$, or $\langle x, x\rangle = 0$ respectively ([9]). In particular, the vector $x = 0$ is said to be spacelike. The *norm* (*length*) of a vector x in $\mathbb{E}^3_1$ is given by $\|x\| = \sqrt{|\langle x, x\rangle|}$.
The *vector product* of vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{E}^3_1$ is defined by ([13])

$$u \times v = (u_3v_2 - u_2v_3,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1).$$

An arbitrary curve $\alpha : I \to \mathbb{E}^3_1$ can be *spacelike, timelike*, or *null* (*lightlike*) if all of its velocity vectors α' are spacelike, timelike, or null respectively ([9]).

A spacelike curve $\alpha(s)$ parameterized by arc length s in $\mathbb{E}^3_1$ is called a *pseudo null curve* if its principal normal vector $N(s) = \alpha''(s)$ and binormal vector $B(s)$ are linearly independent null vectors.

Frenet frame $\{T, N, B\}$ of a pseudo null curve α is a pseudo-orthonormal frame consisting of a tangential vector field T, the principal normal vector field N, and the binormal vector field B, satisfying the conditions ([13])

$$\langle T, T\rangle = 1, \quad \langle N, N\rangle = \langle B, B\rangle = 0, \quad \langle T, N\rangle = \langle T, B\rangle = 0, \quad \langle N, B\rangle = \epsilon = \pm 1, \tag{2.1}$$

$$T \times N = \epsilon N, \quad N \times B = T, \quad B \times T = \epsilon B. \tag{2.2}$$

The Frenet frame is *positively oriented*, if $[T, N, B] = det(T, N, B) = 1$. Frenet frame's equations read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & \epsilon\tau & 0 \\ -\epsilon\kappa & 0 & -\epsilon\tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.3}$$

where $\kappa(s) = 1$ is the *curvature* and $\tau(s)$ is the *torsion* of α, respectively. If $k(s) = 1$ and $\tau(s) = 0$ for each s, α is called a *pseudo null circle*. In particular, if $k(s) = 1$ and $\tau(s) = constant \neq 0$ for each s, α is called a *pseudo null helix*.

A surface M in $\mathbb{E}^3_1$ is called *lightlike* (*null, degenerate*) if each tangent plane at regular points of the surface is lightlike ([2]). A point (u_0, t_0) is a *regular* (resp. *singular*) point of the lightlike surface M in $\mathbb{E}^3_1$ with parametrization $x(u, t)$ if $x_u \times x_t|_{(u_0,t_0)} \neq 0$ (resp. $x_u \times x_t|_{(u_0,t_0)} = 0$) ([12]).

Darboux frame $\{T, \zeta, \eta\}$ of a pseudo null curve α lying on a lightlike surface with parametrization $x(u, t)$ in $\mathbb{E}^3_1$ is a pseudo orthonormal frame consisting of a spacelike tangential vector field $T = \alpha'$, the null normal vector field

$$\eta = x_u \times x_t|_\alpha$$

and the null vector field ζ satisfying the conditions ([10])

$$\langle T,T\rangle = 1, \quad \langle \eta,\eta\rangle = \langle \zeta,\zeta\rangle = \langle T,\zeta\rangle = \langle T,\eta\rangle = 0, \quad \langle \zeta,\eta\rangle = \epsilon_1 = \pm 1, \tag{2.4}$$

$$T\times\zeta = \epsilon_1\zeta, \quad \zeta\times\eta = T, \quad \eta\times T = \epsilon_1\eta. \tag{2.5}$$

Definition 2.1. *Geodesic curvature, normal curvature,* and *geodesic torsion* of a pseudo null curve α lying on a lightlike surface in $\mathbb{E}^3_1$ are respectively given by

$$k_g = \langle T',\zeta\rangle, \quad k_n = \langle T',\eta\rangle, \quad \tau_g = \langle \zeta',\eta\rangle. \tag{2.6}$$

Definition 2.2. Pseudo null curve α lying on a lightlike surface in $\mathbb{E}^3_1$ is called *geodesic*, *asymptotic*, or *principal curvature line*, if $k_g(s) = 0$, $k_n(s) = 0$, and $\tau_g(s) = 0$ respectively for each s.

Darboux frame's equations read ([10])

$$\begin{bmatrix} T' \\ \zeta' \\ \eta' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_1 k_n & \epsilon_1 k_g \\ -k_g & \epsilon_1\tau_g & 0 \\ -k_n & 0 & -\epsilon_1\tau_g \end{bmatrix} \begin{bmatrix} T \\ \zeta \\ \eta \end{bmatrix}. \tag{2.7}$$

Throughout the next sections, $\mathbb{R}\setminus\{0\}$ will be denoted by $\mathbb{R}_0$.

3. Generalized Darboux frame of a pseudo null curve lying on a lightlike surface

In this section, we introduce two new frames along a pseudo null curve lying on a lightlike surface in $\mathbb{E}^3_1$. We call them *generalized Darboux frames of the first and the second kind*, due to the fact that they coincide with the Darboux frame in a special case. We obtain the generalized Darboux frame's equations and the relations between the curvature functions with respect to the Darboux frame and generalized Darboux frames. Throughout this and the next section, let α denote a pseudo null curve parameterized by arc length s and lying on a lightlike surface in Minkowski space $\mathbb{E}^3_1$. Firstly we prove the next property.

Theorem 3.1. *Every pseudo null curve lying on a lightlike surface in $\mathbb{E}^3_1$ is a geodesic or an asymptotic line.*

Proof. By using relation (2.7) and the condition $\langle N,N\rangle = \langle T',T'\rangle = 0$, we get $k_n(s)k_g(s) = 0$. Thus $k_g(s) = 0$ and $k_n(s) \neq 0$, or $k_n(s) = 0$ and $k_g(s) \neq 0$. If $k_n(s) = k_g(s) = 0$, relations (2.3) and (2.7) imply $N(s) = T'(s) = 0$, which is a contradiction, since $N(s)$ is a null vector. □

By using relations (2.3), (2.7), and Theorem 3.1, the next two statements can be easily proved.

Theorem 3.2. *If α is a geodesic line, then:*

(i) Darboux frame's equations read

$$\begin{bmatrix} T' \\ \zeta' \\ \eta' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_1 k_n & 0 \\ 0 & \epsilon_1\tau_g & 0 \\ -k_n & 0 & -\epsilon_1\tau_g \end{bmatrix} \begin{bmatrix} T \\ \zeta \\ \eta \end{bmatrix}; \tag{3.1}$$

(ii) Frenet and Darboux frame's vector fields are related by

$$T = T, \quad N = \epsilon_1 k_n\zeta, \quad B = \frac{\epsilon_1}{k_n}\eta; \tag{3.2}$$

(iii) the curvature functions of α satisfy the equation

$$\tau = \epsilon_1\frac{k_n'}{k_n} + \tau_g, \tag{3.3}$$

where $\epsilon = \epsilon_1$ and $k_n \neq 0$.

Theorem 3.3. *If α is an asymptotic line, then:*

(i) Darboux frame's equations read

$$\begin{bmatrix} T' \\ \zeta' \\ \eta' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \epsilon_1 k_g \\ -k_g & \epsilon_1 \tau_g & 0 \\ 0 & 0 & -\epsilon_1 \tau_g \end{bmatrix} \begin{bmatrix} T \\ \zeta \\ \eta \end{bmatrix}; \tag{3.4}$$

(ii) Frenet and Darboux frame's vector fields are related by

$$T = T, \quad N = \epsilon_1 k_g \eta, \quad B = -\frac{\epsilon_1}{k_g}\zeta; \tag{3.5}$$

(iii) the curvature functions of α satisfy the equation

$$\tau = -\epsilon_1 \frac{k_g'}{k_g} + \tau_g, \tag{3.6}$$

where $\epsilon = -\epsilon_1$ and $k_g \neq 0$.

In what follows, we will define two new frames along α in such way that if α is geodesic, asymptotic, or principal curvature line if and only if the corresponding curvature function of α with respect to a new frame is equal to zero at each point of the curve. In relation to that, let us consider a new frame $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$ along α, satisfying the conditions

$$\tilde{\eta}(s) = \mu\eta, \tag{3.7}$$

$$\langle \tilde{T}, \tilde{T} \rangle = 1, \quad \langle \tilde{T}, \tilde{\zeta} \rangle = \langle \tilde{T}, \tilde{\eta} \rangle = \langle \tilde{\zeta}, \tilde{\zeta} \rangle = \langle \tilde{\eta}, \tilde{\eta} \rangle = 0, \quad \langle \tilde{\zeta}, \tilde{\eta} \rangle = \epsilon_2 = \pm 1, \tag{3.8}$$

$$\tilde{T} \times \tilde{\zeta} = \epsilon_2 \tilde{\zeta}, \quad \tilde{\zeta} \times \tilde{\eta} = \tilde{T}, \quad \tilde{\eta} \times \tilde{T} = \epsilon_2 \tilde{\eta}, \tag{3.9}$$

where $\mu \neq 0$ is some differentiable function. In the next theorem, we obtain the relation between the Darboux frame $\{T, \zeta, \eta\}$ and the introduced frame $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$.

Theorem 3.4. *The frame $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$ and the Darboux frame $\{T, \zeta, \eta\}$ of α are related by:*

(i)

$$\begin{aligned} \tilde{T}(s) &= T(s) + \lambda(s)\eta(s), \\ \tilde{\zeta}(s) &= -\epsilon_2 \frac{\lambda(s)}{\mu(s)} T(s) + \frac{1}{\mu(s)}\zeta(s) - \epsilon_2 \frac{\lambda^2(s)}{2\mu(s)}\eta(s), \\ \tilde{\eta}(s) &= \mu(s)\eta(s), \end{aligned} \tag{3.10}$$

where $\mu \neq 0$, $\lambda \neq 0$ are some differentiable functions and $\epsilon_2 = \epsilon_1 = \pm 1$;

(ii)

$$\begin{aligned} \tilde{T}(s) &= T(s), \\ \tilde{\zeta}(s) &= \frac{1}{\mu(s)}\zeta(s), \\ \tilde{\eta}(s) &= \mu(s)\eta(s), \end{aligned} \tag{3.11}$$

where $\mu \neq 0$ is some differentiable function.

Proof. Assume that α has a Darboux frame whose vector fields satisfy relations (2.4) and (2.5) and pseudo-orthonormal frame $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$, whose vector fields satisfy relations (3.7), (3.8), and (3.9). Since the vector field $\tilde{\eta}$ is lightlike, its orthogonal complement $\tilde{\eta}^{\perp}$ is also lightlike and represents a lightlike tangent plane of the surface. The conditions (2.4), (3.7) and (3.8) imply $\langle \tilde{T}, \tilde{\eta} \rangle = \langle \eta, \tilde{\eta} \rangle = \langle T, \tilde{\eta} \rangle = 0$. Hence $\tilde{T}, \eta, T \in \tilde{\eta}^{\perp}$, so the unit spacelike vector field $\tilde{T}$ can be written as

$$\tilde{T}(s) = T(s) + \lambda(s)\eta(s), \tag{3.12}$$

where $\lambda(s)$ is some differentiable function. If $\lambda(s) \neq 0$ for each s, by using (3.7), (3.8), (3.9) and (3.12), we get

$$\tilde{\zeta}(s) = -\epsilon_2 \frac{\lambda(s)}{\mu(s)} T(s) + \frac{1}{\mu(s)}\zeta(s) - \epsilon_2 \frac{\lambda^2(s)}{2\mu(s)}\eta(s), \tag{3.13}$$

where $\langle \tilde{\zeta}, \tilde{\eta} \rangle = \epsilon_2 = \langle \zeta, \eta \rangle = \epsilon_1 = \pm 1$. Relations (3.7), (3.12), and (3.13) give relation (3.10). If $\lambda(s) = 0$ for each s, by using (3.7), (3.8), (3.12), and (3.13), we find $\tilde{T}(s) = T(s)$ and $\tilde{\zeta}(s) = \frac{1}{\mu(s)}\zeta(s)$. Hence we get the frame given by relation (3.11), where $\langle \tilde{\zeta}, \tilde{\eta} \rangle = \epsilon_2 = \langle \zeta, \eta \rangle = \epsilon_1 = \pm 1$. □

Remark 3.1. The frames given by relations (3.10) and (3.11) coincide with the Darboux frame $\{T, \zeta, \eta\}$, if $\mu(s) = 1$ and $\lambda(s) = 0$ for each s.

We define the curvature functions with respect to the frames given by (3.10) and (3.11) as follows.

Definition 3.1. The curvature functions of α given by

$$\tilde{k}_n = \langle \tilde{T}', \tilde{\eta} \rangle, \quad \tilde{k}_g = \langle \tilde{T}', \tilde{\zeta} \rangle, \quad \tilde{\tau}_g = \langle \tilde{\zeta}', \tilde{\eta} \rangle. \tag{3.14}$$

are called *generalized normal curvature*, *generalized geodesic curvature*, and *generalized geodesic torsion*, respectively.

By using relation (3.14), we obtain the next theorem which can be easily proved, so we omit its proof.

Theorem 3.5. *If α has the frame given by relation (3.10) or (3.11), the frame's equations read*

$$\begin{bmatrix} \tilde{T}' \\ \tilde{\zeta}' \\ \tilde{\eta}' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2 \tilde{k}_n & \epsilon_2 \tilde{k}_g \\ -\tilde{k}_g & \epsilon_2 \tilde{\tau}_g & 0 \\ -\tilde{k}_n & 0 & -\epsilon_2 \tilde{\tau}_g \end{bmatrix} \begin{bmatrix} \tilde{T} \\ \tilde{\zeta} \\ \tilde{\eta} \end{bmatrix}. \tag{3.15}$$

Next we obtain the relation between the curvature functions k_g, k_n, τ_g of α with respect to the Darboux frame and the curvature functions $\tilde{k}_g$, $\tilde{k}_n$, and $\tilde{\tau}_g$ with respect to the introduced frames.

Theorem 3.6. *The curvature functions of α with respect to the frame given by (3.10) or (3.11) and the curvature functions with respect to Darboux frame, are related by*

$$\begin{aligned} \tilde{k}_n &= \mu k_n, \\ \tilde{\tau}_g &= -\epsilon_2 \lambda k_n - \epsilon_2 \frac{\mu'}{\mu} + \tau_g, \\ \tilde{k}_g &= \frac{1}{\mu} k_g + \epsilon_2 \frac{1}{\mu} \lambda' - \frac{1}{\mu} \lambda \tau_g + \epsilon_2 \frac{1}{2\mu} \lambda^2 k_n, \end{aligned} \tag{3.16}$$

where $\mu \neq 0$, $\lambda = 0$ if α has the frame given by (3.11), or $\mu \neq 0$, $\lambda \neq 0$ if α has the frame given by (3.10).

Proof. Differentiating the relation $\tilde{\eta} = \mu \eta$ with respect to s and using (2.7), we obtain

$$\tilde{\eta}' = (-\mu k_n) T + (\mu' - \epsilon_1 \mu \tau_g) \eta. \tag{3.17}$$

According to relations (3.10) and (3.15), we have

$$\tilde{\eta}' = -\tilde{k}_n \tilde{T} - \epsilon_2 \tilde{\tau}_g \tilde{\eta} = (-\tilde{k}_n) T + (-\lambda \tilde{k}_n - \epsilon_2 \mu \tilde{\tau}_g) \eta. \tag{3.18}$$

Relations (3.17) and (3.18) give $\tilde{k}_n = \mu k_n$ and $\tilde{\tau}_g = -\epsilon_2 \lambda k_n - \epsilon_2 \frac{\mu'}{\mu} + \tau_g$. By using (3.10) and (3.14), we obtain

$$\tilde{k}_g = \langle \tilde{T}', \tilde{\zeta} \rangle = \langle T' + \lambda' \eta + \lambda \eta', -\epsilon_2 \frac{\lambda}{\mu} T + \frac{1}{\mu} \zeta - \epsilon_2 \frac{\lambda^2}{2\mu} \eta \rangle.$$

The last relation together with relations (2.4) and (2.7) give

$$\tilde{k}_g = \frac{1}{\mu} k_g + \epsilon_2 \frac{1}{\mu} \lambda' - \frac{1}{\mu} \lambda \tau_g + \epsilon_2 \frac{1}{2\mu} \lambda^2 k_n,$$

which completes the proof. □

We define *generalized Darboux frames* of α as the frames of the form (3.10) and (3.11), where functions functions λ and μ are chosen properly.

Definition 3.2. *Generalized Darboux frame of the first kind* of α is the frame given by (3.10), where the function $\lambda \neq 0$ satisfies Riccati differential equation

$$2\epsilon_2 \lambda'(s) - 2\lambda(s)\tau_g(s) + \epsilon_2 \lambda^2(s) k_n(s) = 0, \tag{3.19}$$

and the function $\mu \neq 0$ satisfies differential equation

$$\mu'(s) + \mu(s)\lambda(s)k_n(s) = 0. \tag{3.20}$$

Definition 3.3. *Generalized Darboux frame of the second kind* of α is the frame given by (3.11), where $\mu(s) \in \mathbb{R}_0$ for each s.

By using Definitions 3.2, 3.3, and Theorem 3.6, we easily get the next statement.

Theorem 3.7. *The curvature functions of α with respect to the Darboux frame and the curvature functions with respect to generalized Darboux frame $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$, are related by:*

$$\begin{aligned} \tilde{k}_n(s) &= \mu(s)k_n(s), \\ \tilde{\tau}_g(s) &= \tau_g(s), \\ \tilde{k}_g(s) &= \frac{1}{\mu(s)}k_g(s), \end{aligned} \tag{3.21}$$

where $\mu \neq 0$ satisfies differential equation (3.20) if $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$ is the generalized Darboux frame of the first kind, or $\mu(s) \in R_0$ if $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$ is the generalized Darboux frame of the second kind.

Theorem 3.7 implies the next property.

Theorem 3.8. *The curve α is a geodesic, asymptotic or principal curvature line, if and only if $\tilde{k}_g(s) = 0$, $\tilde{k}_n(s) = 0$, and $\tilde{\tau}_g(s) = 0$ respectively for each s.*

4. Darboux vectors of Frenet, Darboux and generalized Darboux frame

In this section, we obtain parameter equations of Darboux vectors of the Frenet, Darboux and generalized Darboux frame of α. We also find the necessary and sufficient conditions for such vectors to have the same directions. Denote by D Darboux vector of the Darboux frame. It satisfies *Darboux equations* of the form

$$T' = D \times T, \quad N' = D \times N, \quad B' = D \times B. \tag{4.1}$$

By using (2.3) and (4.1), it follows that D has parameter equation

$$D(s) = \tau(s)T(s) - \epsilon N(s), \tag{4.2}$$

where $\epsilon = \pm 1 = \langle N, B \rangle$. Darboux vectors $\bar{D}$ and $\tilde{D}$ of Darboux and generalized Darboux frame respectively, satisfy analogous *Darboux equations* of the form

$$T' = \bar{D} \times T, \quad \zeta' = \bar{D} \times \zeta, \quad \eta' = \bar{D} \times \eta \tag{4.3}$$

and

$$\tilde{T}' = \tilde{D} \times \tilde{T}, \quad \tilde{\zeta}' = \tilde{D} \times \tilde{\zeta}, \quad \tilde{\eta}' = \tilde{D} \times \tilde{\eta}. \tag{4.4}$$

Relations (2.5), (2.7), (3.9), (3.15), (4.3) and (4.4) yield

$$\bar{D}(s) = \tau_g(s)T(s) - k_n(s)\zeta(s) + k_g(s)\eta(s), \tag{4.5}$$

and

$$\tilde{D}(s) = \tilde{\tau}_g(s)\tilde{T}(s) - \tilde{k}_n(s)\tilde{\zeta}(s) + \tilde{k}_g(s)\tilde{\eta}(s). \tag{4.6}$$

In what follows, we find the necessary and sufficient conditions for vectors D, $\bar{D}$, and $\tilde{D}$ to have the same directions. According to Theorem 3.1, we may consider two cases:

(A) α is a geodesic curve. By Theorem 3.7, it holds $k_g(s) = \tilde{k}_g(s) = 0$, $\tau_g(s) = \tilde{\tau}_g(s)$, $\tilde{k}_n(s) = \mu(s)k_n(s)$. By using (3.2), (4.2), and (4.5), we get the next theorem.

Theorem 4.1. *Darboux vectors D and $\bar{D}$ of the Frenet and Darboux frame respectively of geodesic curve α have the same directions if and only if $\tau(s) = \tau_g(s)$ for each s.*

If $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$ is the generalized Darboux frame of the second kind of α, by using (3.11) we easily get

$$\tilde{D}(s) = \tilde{\tau}_g(s)\tilde{T}(s) - \tilde{k}_n(s)\tilde{\zeta}(s) = \tau_g(s)T(s) - \mu_0 k_n(s)\frac{1}{\mu_0}\zeta(s) = \bar{D}(s).$$

Hence the following statement holds.

Theorem 4.2. *Darboux vectors $\tilde{D}$ and $\bar{D}$ of the generalized Darboux frame of the second kind and Darboux frame respectively of geodesic curve α have the same directions for each s.*

A straightforward calculation shows that relations (3.3), (3.10), (3.19), (4.2), and (4.6) imply the next property.

Theorem 4.3. *Darboux vectors $\tilde{D}$ and D of the generalized Darboux frame of the first kind and Frenet frame respectively of geodesic curve α have the same directions if and only if*

$$\tau(s) = -\tau_g(s) = constant \neq 0, \quad \lambda(s) = -2\epsilon_2 \frac{\tau_g(s)}{k_n(s)}. \tag{4.7}$$

(B) α is an asymptotic curve. According to Theorem 3.7, it holds $\tilde{k}_g(s) = \frac{1}{\mu(s)} k_g(s)$, $\tau_g(s) = \tilde{\tau}_g(s)$, $\tilde{k}_n(s) = k_n(s) = 0$. By using (3.2), (4.2), and (4.5), we get the next theorem.

Theorem 4.4. *Darboux vectors D and $\bar{D}$ of the Frenet and Darboux frame respectively of an asymptotic curve α have the same directions if and only if $\tau(s) = \tau_g(s)$ for each s.*

If $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$ is the generalized Darboux frame of the second kind of α, by using (3.11) we easily get

$$\tilde{D}(s) = \tilde{\tau}_g(s)\tilde{T}(s) + \tilde{k}_g(s)\tilde{\eta}(s) = \tau_g(s)T(s) + \frac{1}{\mu_0} k_g(s)\mu_0\eta(s) = \bar{D}(s).$$

This proves the next theorem.

Theorem 4.5. *Darboux vectors $\tilde{D}$ and $\bar{D}$ of the generalized Darboux frame of the second kind and Darboux frame respectively of an asymptotic curve α have the same directions for each s.*

In particular, the relations (3.10), (3.19), (4.2), and (4.6) imply the next statement.

Theorem 4.6. *Darboux vectors $\tilde{D}$ and D of the generalized Darboux frame of the first kind and Frenet frame respectively of an asymptotic curve α have the same directions if and only if $\tau(s) = \tau_g(s) = 0$ for each s.*

By using Theorems 4.4, 4.5, and 4.6, we obtain the next corollary.

Corollary 4.1. *The only pseudo null asymptotic curves lying on a lightlike surface in $\mathbb{E}^3_1$, whose all four Darboux vectors of the Frenet, Darboux, generalized Darboux frame of the first kind and generalized Darboux frame of the second kind have the same directions, are pseudo null circles.*

5. Some examples

In this section, we give some examples of the generalized Darboux frames of pseudo null curves lying on the lightlike ruled surfaces in $\mathbb{E}^3_1$. In the mentioned examples, we obtain parameter equations of the Darboux vectors of the Frenet, Darboux, and generalized Darboux frames and show that under the corresponding conditions, they have the same directions.

Example 5.1. Let us consider a ruled surface M in $\mathbb{E}^3_1$ with parametrization (see Figure 1)

$$x(s,t) = \alpha(s) + tB(s),$$

where $\alpha(s)$ is a pseudo null helix with parameter equation $\alpha(s) = (e^s, e^s, s)$, $B(s)$ is its binormal vector, $s, t \in R$ and $t \neq 1$.

The Frenet frame of α has the form

$$T(s) = (e^s, e^s, 1), \quad N(s) = (e^s, e^s, 0), \quad B(s) = (-\frac{e^{2s}+1}{2e^s}, \frac{1-e^{2s}}{2e^s}, -1). \tag{5.1}$$

The Frenet curvatures of α read

$$k(s) = 1, \quad \tau(s) = 1.$$

Since $T \times N = \epsilon N = N$, it follows $\epsilon = 1$. A straightforward calculation shows that the normal vector field on M is given by

$$U(s,t) = x_s \times x_t = (t-1)B(s).$$

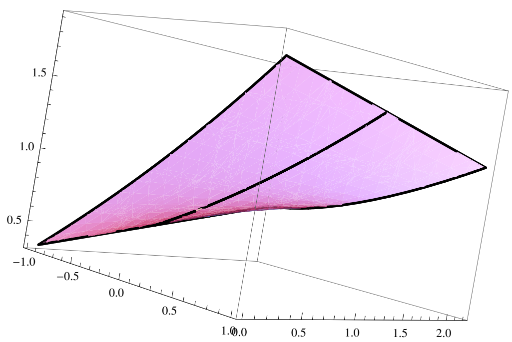

Figure 1. Pseudo null helix α lying on lightlike ruled surface M

Since $t \neq 1$, $U(s,t)$ is a lightlike vector field and M is a lightlike ruled surface. Darboux frame along α reads

$$T(s) = \alpha'(s), \quad \zeta(s) = -N(s), \quad \eta(s) = x_s \times x_t|_\alpha = -B(s), \tag{5.2}$$

and the curvature functions of α have the form

$$k_g(s) = 0, \quad k_n(s) = -1, \quad \tau_g(s) = 1. \tag{5.3}$$

Hence α is a geodesic line. Substituting (5.1) in (3.19) and (3.20) and using the relation $\epsilon_2 = \epsilon_1 = 1$, we get

$$\lambda(s) = -\frac{2e^s}{e^s - 1}, \quad \mu(s) = \frac{1}{(1-e^s)^2}. \tag{5.4}$$

According to Definition 3.2, we find that the generalized Darboux frame of the first kind of α reads

$$\tilde{T}(s) = T(s) - \frac{2e^s}{e^s - 1}\eta(s), \tag{5.5}$$

$$\tilde{\zeta}(s) = 2e^s(e^s-1)T(s) + (1-e^s)^2\zeta(s) - 2e^{2s}\eta(s), \tag{5.6}$$

$$\tilde{\eta}(s) = \frac{1}{(1-e^s)^2}\eta(s). \tag{5.7}$$

Relations (3.14), (5.3), (5.4) and (5.5) imply that the curvature functions of α with respect to generalized Darboux frame of the first kind have to form

$$\tilde{k}_g(s) = 0, \quad \tilde{k}_n(s) = -\frac{1}{(1-e^s)^2}, \quad \tilde{\tau}_g(s) = 1. \tag{5.8}$$

According to Definition 3.3, the generalized Darboux frame of the second kind of α is given by

$$\tilde{T}(s) = T(s), \quad \tilde{\zeta}(s) = \frac{1}{\mu_0}\zeta(s), \quad \tilde{\eta}(s) = \mu_0\eta(s), \tag{5.9}$$

where $\mu(s) = \mu_0 \in \mathbb{R}_0$. By using (3.14) and (5.7), it can be easily checked that the curvature functions with respect to the generalized Darboux frame of the second kind have to form

$$\tilde{k}_g(s) = 0, \quad \tilde{k}_n(s) = -\mu_0, \quad \tilde{\tau}_g(s) = 1. \tag{5.10}$$

Relations (5.3), (5.4), (5.8) and (5.10) imply that the statements of Theorems 3.7 and 3.8 hold.

By using relations (5.1), (5.2), (5.3), (5.9), (5.10), and $\tau(s) = \epsilon = 1$, it follows that Darboux vectors D, $\bar{D}$, and $\tilde{D}$ of the Frenet, Darboux, and generalized Darboux frame of the second kind have parameter equations of the form

$$\begin{aligned} D(s) &= \tau(s)T(s) - \epsilon N(s) = (0,0,1), \\ \bar{D}(s) &= \tau_g(s)T(s) - k_n(s)\zeta(s) = (0,0,1), \\ \tilde{D}(s) &= \tilde{\tau}_g(s)\tilde{T}(s) - \tilde{k}_n(s)\tilde{\zeta}(s) = (0,0,1). \end{aligned}$$

Therefore, they have the same directions, which means that Theorems 4.1 and 4.2 hold.

Example 5.2. Let us consider a ruled surface M in $\mathbb{E}^3_1$ with parametrization (see Figure 2)

$$x(s,t) = \alpha(s) + t(T(s) + 2N(s)),$$

where α is a pseudo null circle with parameter equation $\alpha(s) = (\frac{s^2}{2}, \frac{s^2}{2}, s)$, T and N are the tangent and the principal normal vector fields of α, $s, t \in R$ and $t \neq 2$.

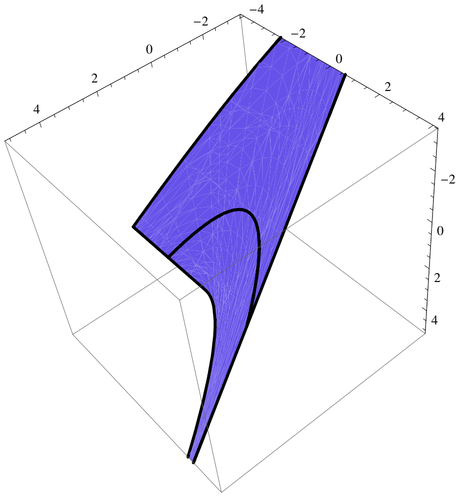

Figure 2. Pseudo null circle α lying on lightlike plane

The Frenet frame of α reads

$$T(s) = (s, s, 1), \quad N(s) = (1, 1, 0), \quad B(s) = (-\frac{1+s^2}{2}, \frac{1-s^2}{2}, -s). \tag{5.11}$$

The Frenet curvatures of α read

$$k(s) = 1, \quad \tau(s) = 0. \tag{5.12}$$

Since $T \times N = \epsilon N = N$, we get $\epsilon = 1$. The normal vector field on M is given by

$$U(s,t) = x_s \times x_t = (2-t)N(s).$$

Since $t \neq 2$, $U(s,t)$ is a lightlike vector field and M is a lightlike plane with parameter equation $x_1 = x_2$. The Darboux frame along α reads

$$T(s) = \alpha'(s), \quad \zeta(s) = -\frac{1}{2}B(s), \quad \eta(s) = x_s \times x_t|_\alpha = 2N(s). \tag{5.13}$$

By using (2.6), it follows that the curvature functions of α have the form

$$k_g(s) = -\frac{1}{2}, \quad k_n(s) = 0, \quad \tau_g(s) = 0. \tag{5.14}$$

Consequently, α is an asymptotic and a principal curvature line. Substituting (5.14) in (3.19) and (3.20) and using the relation $\epsilon_2 = \epsilon_1 = -1$, we get

$$\lambda(s) = \lambda_0 \in R_0, \quad \mu(s) = \mu_0 \in R_0. \tag{5.15}$$

By using Definition 3.2 and relations (3.10) and (5.15), we find that generalized Darboux frame of the first kind of α reads

$$\tilde{T}(s) = T(s) + \lambda_0\eta(s), \tag{5.16}$$

$$\tilde{\zeta}(s) = \frac{\lambda_0}{\mu_0}T(s) + \frac{1}{\mu_0}\zeta(s) + \frac{\lambda^2}{2\mu_0}\eta(s), \tag{5.17}$$

$$\tilde{\eta}(s) = \mu_0\eta(s). \tag{5.18}$$

Hence relations (3.14), (5.16), (5.17) and (5.18) imply that the curvature functions of α with respect to generalized Darboux frame of the first kind have to form

$$\tilde{k}_g(s) = -\frac{1}{2\mu_0}, \quad \tilde{k}_n(s) = 0, \quad \tilde{\tau}_g(s) = 0. \tag{5.19}$$

By Definition 3.3, the generalized Darboux frame of the second kind of α is given by

$$\tilde{T}(s) = T(s), \quad \tilde{\zeta}(s) = \frac{1}{\mu_0}\zeta(s), \quad \tilde{\eta}(s) = \mu_0\eta(s). \tag{5.20}$$

Thus the curvature functions of α with respect to the generalized Darboux frame of the second kind, have the form

$$\tilde{k}_g(s) = -\frac{1}{2\mu_0}, \quad \tilde{k}_n(s) = 0, \quad \tilde{\tau}_g(s) = 0. \tag{5.21}$$

By using (5.14), (5.15), (5.19) and (5.20), we obtain that the statements of Theorems 3.7 and 3.8 hold.

By using relations (5.11), (5.12), (5.16), (5.18), and (5.19), it follows that Darboux vectors D and $\tilde{D}$ of the Frenet and generalized Darboux frame of the first kind have parameter equations of the form

$$\begin{aligned} D(s) &= \tau(s)T(s) - \epsilon N(s) = (-1, -1, 0), \\ \tilde{D}(s) &= \tilde{\tau}_g(s)\tilde{T}(s) + \tilde{k}_g(s)\tilde{\eta}(s) = (-1, -1, 0). \end{aligned}$$

Therefore, they have the same direction, which means that Theorem 4.6 holds. In particular, relations (5.11), (5.13), (5.14), (5.20) and (5.21) imply that Darboux vectors $\bar{D}$ and $\tilde{D}$ of Darboux and generalized Darboux frame of the second kind respectively are given by $\bar{D}(s) = \tilde{D}(s) = (-1, -1, 0)$. Consequently, the statement of Corollary 4.1 holds.

Example 5.3. Let us consider ruled surface M in $\mathbb{E}^3_1$ with parametrization (see Figure 3)

$$x(s,t) = \alpha(s) + tN(s),$$

where the pseudo null base curve α has parameter equation

$$\alpha(s) = (\frac{s^3}{12}, \frac{s^3 + 12s}{12\sqrt{2}}, \frac{s^3 - 12s}{12\sqrt{2}}),$$

N is the principal normal vector field of α, s, $t \in R$ and $s \neq 0$. A straightforward calculation shows that the Frenet frame of α has the form

$$\begin{aligned} T(s) &= (\frac{s^2}{4}, \frac{s^2+4}{4\sqrt{2}}, \frac{s^2-4}{4\sqrt{2}}), \\ N(s) &= s(\frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}), \\ B(s) &= (\frac{s^3}{16} + \frac{1}{s}, \frac{s^3}{16\sqrt{2}} + \frac{s\sqrt{2}}{4} - \frac{1}{s\sqrt{2}}, \frac{s^3}{16\sqrt{2}} - \frac{s\sqrt{2}}{4} - \frac{1}{s\sqrt{2}}). \end{aligned}$$

The Frenet curvatures of α read

$$k(s) = 1, \quad \tau(s) = -\frac{1}{s}.$$

Since $T \times N = \epsilon N = -N$, we get $\epsilon = -1$. The normal vector field on M is given by

$$U(s,t) = x_s \times x_t = -N(s).$$

Since $s \neq 0$, $U(s,t)$ is a lightlike vector field and M is a lightlike ruled surface. Darboux frame along α reads

$$T(s) = \alpha'(s), \quad \zeta(s) = B(s), \quad \eta(s) = x_s \times x_t|_\alpha = -N(s). \tag{5.22}$$

Thus the curvature functions of α have the form

$$k_g(s) = -1, \quad k_n(s) = 0, \quad \tau_g(s) = -\frac{1}{s}. \tag{5.23}$$

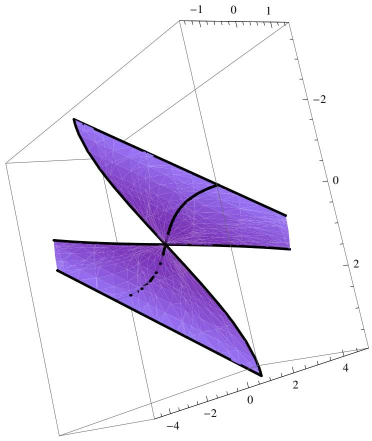

Figure 3. Pseudo null curve α lying on lightlike ruled surface M

Therefore, α is an asymptotic line. Substituting (5.23) in (3.19) and (3.20) and using the relation $\epsilon_2 = \epsilon_1 = 1$, we get

$$\lambda(s) = s, \quad \mu(s) = \mu_0 \in R_0. \tag{5.24}$$

Definition 3.3 and relations (3.10) and (5.24) imply that the generalized Darboux frame of the first kind of α reads

$$\begin{aligned} \tilde{T}(s) &= T(s) + s\eta(s), \\ \tilde{\zeta}(s) &= -\frac{s}{\mu_0}T(s) + \frac{1}{\mu_0}\zeta(s) - \frac{s^2}{2\mu_0}\eta(s), \\ \tilde{\eta}(s) &= \mu_0\eta(s). \end{aligned}$$

The curvature functions of α, with respect to the generalized Darboux frame of the first kind, have the form

$$\tilde{k}_g(s) = -\frac{1}{\mu_0}, \quad \tilde{k}_n(s) = 0, \quad \tilde{\tau}_g(s) = -\frac{1}{s}. \tag{5.25}$$

According to Definition 3.3, the generalized Darboux frame of the second kind of α is given by

$$\tilde{T}(s) = T(s), \quad \tilde{\zeta}(s) = \frac{1}{\mu_0}\zeta(s), \quad \tilde{\eta}(s) = \mu_0\eta(s). \tag{5.26}$$

Hence the curvature functions of α, with respect to the generalized Darboux frame of the second kind, read

$$\tilde{k}_g(s) = -\frac{1}{\mu_0}, \quad \tilde{k}_n(s) = 0, \quad \tilde{\tau}_g(s) = -\frac{1}{s}. \tag{5.27}$$

By using relations (5.23), (5.24), (5.25) and (5.27), it can be easily verified that the statements of Theorems 3.7 and 3.8 hold. By using relations (5.22), (5.23), (5.26), and (5.27), it follows that Darboux vectors $\bar{D}$ and $\tilde{D}$ of the Frenet and generalized Darboux frame of the second kind have parameter equations of the form

$$\begin{aligned} \bar{D}(s) &= \tau_g(s)T(s) + k_g(s)\eta(s) = (\frac{s}{4}, \frac{s^2-4}{4s\sqrt{2}}, \frac{s^2+4}{4s\sqrt{2}}), \\ \tilde{D}(s) &= \tilde{\tau}_g(s)\tilde{T}(s) + \tilde{k}_g(s)\tilde{\eta}(s) = (\frac{s}{4}, \frac{s^2-4}{4s\sqrt{2}}, \frac{s^2+4}{4s\sqrt{2}}). \end{aligned}$$

Therefore, they have the same directions, which means that Theorem 4.5 holds.

Example 5.4. Let us consider ruled surface M in $\mathbb{E}^3_1$ with parametrization (see Figure 4)

$$x(s,t) = \alpha(s) + t(-\frac{1}{2}e^{6s} - \frac{1}{8}e^{2s}, -\frac{e^{4s}}{2}, \frac{1}{8}e^{2s} - \frac{1}{2}e^{6s}),$$

where the pseudo null helix α has parameter equation $\alpha(s) = (e^{2s}, s, e^{2s})$, $s,\ t \in R$ and $t \neq e^{-4s}$. A straightforward calculation shows that the Frenet frame of α has the form

$$T(s) = (2e^{2s}, 1, 2e^{2s}), \tag{5.28}$$

$$N(s) = 4e^{2s}(1,0,1), \tag{5.29}$$

$$B(s) = (\frac{1}{2}e^{2s} + \frac{1}{8}e^{-2s}, \frac{1}{2}, -\frac{1}{8}e^{-2s} + \frac{1}{2}e^{2s}).$$

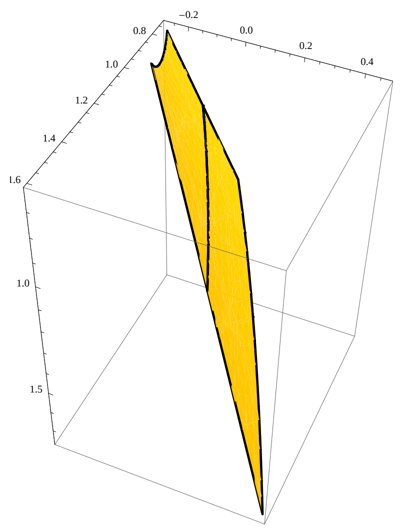

Figure 4. Pseudo null helix α lying on lightlike ruled surface M

The Frenet curvatures of α read

$$k(s) = 1, \quad \tau(s) = -2. \tag{5.30}$$

Since $T \times N = \epsilon N = -N$, we obtain $\epsilon = -1$. The normal vector field on M is given by

$$U(s,t) = x_s \times x_t = (1 - te^{4s})\eta(s).$$

Since $t \neq e^{-4s}$, $U(s,t)$ is a lightlike vector field and M is a lightlike ruled surface. The Darboux frame along α reads

$$T(s) = \alpha'(s),$$

$$\zeta(s) = -4e^{-2s}(1,0,1),$$

$$\eta(s) = x_s \times x_t|_\alpha = (-\frac{1}{2}e^{6s} - \frac{1}{8}e^{2s}, -\frac{e^{4s}}{2}, \frac{1}{8}e^{2s} - \frac{1}{2}e^{6s}).$$

Thus the curvature functions of α have the form

$$k_g(s) = 0, \quad k_n(s) = e^{4s}, \quad \tau_g(s) = 2. \tag{5.31}$$

Therefore, α is a geodesic line. Substituting (5.31) in (3.19) and (3.20) and using the relation $\epsilon_2 = \epsilon_1 = -1$, we get

$$\lambda(s) = \frac{4}{e^{4s}}, \quad \mu(s) = e^{-4s}. \tag{5.32}$$

Definition 3.2 and relations (3.10) and (5.32) imply that the generalized Darboux frame of the first kind of α reads

$$\tilde{T}(s) = T(s) + \frac{4}{e^{4s}}\eta(s), \tag{5.33}$$

$$\tilde{\zeta}(s) = 4T(s) + e^{4s}\zeta(s) + \frac{8}{e^{4s}}\eta(s), \tag{5.34}$$

$$\tilde{\eta}(s) = e^{-4s}\eta(s).$$

The curvature functions of α, with respect to the generalized Darboux frame of the first kind, have the form

$$\tilde{k}_g(s) = 0, \quad \tilde{k}_n(s) = 1, \quad \tilde{\tau}_g(s) = 2. \tag{5.35}$$

According to Definition 3.3, the generalized Darboux frame of the second kind of α is given by

$$\tilde{T}(s) = T(s), \quad \tilde{\zeta}(s) = \frac{1}{\mu_0}\zeta(s), \quad \tilde{\eta}(s) = \mu_0\eta(s),$$

where $\mu_0 \in \mathbb{R}_0$. Hence the curvature functions of α, with respect to generalized Darboux frame of the second kind, read

$$\tilde{k}_g(s) = 0, \quad \tilde{k}_n(s) = \mu_0 e^{4s}, \quad \tilde{\tau}_g(s) = 2. \tag{5.36}$$

By using relations (5.31), (5.32), (5.35) and (5.36), it can be easily verified that the statements of Theorems 3.7 and 3.8 hold.

In particular, from the relations (5.28), (5.29), (5.30), (5.33), and (5.34), it follows that Darboux vectors D and $\tilde{D}$ of the Frenet and generalized Darboux frame of the first kind have parameter equations of the form

$$\begin{aligned} D(s) &= -2T(s) + N(s) = (0, -2, 0), \\ \tilde{D}(s) &= 2\tilde{T}(s) - \tilde{C}(s) = (0, -2, 0). \end{aligned}$$

Hence the statement of Theorem 4.3 holds.

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Competing interests

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Carlsen, B., Clelland, J. N.: *The geometry of lightlike surfaces in Minkowski space.* Journal of Geometry and Physics. **74**, 43-55 (2013). https://doi.org/10.1016/j.geomphys.2013.07.005

[2] Duggal, K. L., Jin, D. H.: *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds.* Hackensack, NJ, USA: World Scientific Publishing, 2007.

[3] Inoguchi, J. I., Lee, S.: *Lightlike surfaces in Minkowski 3-space.* International Journal of Geometric Methods in Modern Physics. **6** (2), 267-283 (2009). https://doi.org/10.1142/S0219887809003552

[4] Djordjević, J., Nešović, E., Özturk, U.: *On generalized Darboux frame of a spacelike curve lying on a lightlike surface in Minkowski space* $\mathbb{E}^3_1$. Turkish Journal of Mathematics. **47**, 883-897 (2023). http://doi:10.55730/1300-0098.3399

[5] Liu, H., *Curves in the lightlike cone.* Beiträge zur Algebra und Geometrie. **45** (1), 291-303 (2004).

[6] Liu, S., Wang, Z.: *Generalized focal surfaces of spacelike curves lying in lightlike surfaces.* Mathematical Methods in the Applied Sciences. **44**, 7501–7525 (2021). https://doi.org/10.1002/mma.6296

[7] Navarro. M., Palmas, O., Solis, D. A.: *On the geometry of null hypersurfaces in Minkowski space.* Journal of Geometry and Physics. **75**, 199-212 (2014). https://doi.org/10.1016/j.geomphys.2013.10.005

[8] Nešović, E., Özturk, U., Koç Öztürk, E. B.: *Some characterizations of pseudo null isophotic curves in Minkowski 3-space.* Journal of Geometry. **122** (29), 1-13 (2021). https://doi.org/10.1007/s00022-021-00593-4

[9] O'Neill, B.: Semi-Riemannian Geometry with Applications to Relativity. Academic Press. London (1983).

[10] Öztürk, U., Nešović, E., Koç Öztürk, E. B.: *On k-type spacelike slant helices lying on lightlike surfaces.* Filomat. **33** (9), 2781-2796 (2019). http://dx.doi.org/10.2298/FIL1909781O
[11] Senovilla, J. M. M.: *Singularity theorems and their consequences.* General Relativity and Gravitation. **30** (5), 701-848 (1998). https://doi.org/10.1023/A:1018801101244
[12] Umehara, M., Yamada, K.: *Hypersurfaces with light-like points in a Lorentzian manifold.* The Journal of Geometric Analysis. textbf29, 3405-3437 (2019). https://doi.org/10.1007/s12220-018-00118-7
[13] Walrave, J.: Curves and surfaces in Minkowski space. Ph.D. thesis. Leuven University (1995).
[14] Wang, Y., Pei, D., Cui, X.: *Pseudo-spherical normal Darboux images of curves on a lightlike surface.* Mathematical Methods in the Applied Sciences. **40** (18), 7151-7161 (2017). https://doi.org/10.1002/mma.4519
[15] Yakıcı, Topbas E. S., Gök, I., Ekmekci, N., Yayli, Y.: *Darboux frame of a curve lying on a lightlike surface.* Mathematical Sciences and Applications E-Notes. **4** (2), 121-130 (2016). https://doi.org/10.36753/mathenot.421465
[16] Zhou, K., Wang, Z.: *Pseudo-spherical Darboux images and lightcone images of principal-directional curves of nonlightlike curves in Minkowski 3-space.* Mathematical Methods in the Applied Sciences. **43** (1), 35-77 (2020). https://doi.org/10.1002/mma.5374

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