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Research Article

Soliton Solutions of the Generalized Dullin-Gottwald-Holm Equation with Parabolic Law Nonlinearity

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Abstract

In this paper, soliton solutions of the generalized Dullin-Gottwald-Holm (gDGH) equation with parabolic law nonlinearity are investigated. The gDGH describes the behavior of waves in shallow water with surface tension. There are only a few studies in the literature regarding gDGH equation with parabolic law nonlinearity, and to our best knowledge, the unified Riccati equation expansion method (UREEM) has not been applied to this equation before. Many soliton solutions of the considered gDGH equation are successfully attained using the UREEM, which is a powerful technique for solving nonlinear partial differential equations. We verify that the obtained analytical solutions satisfy the gDGH equation using Mathematica. Furthermore, some plots of the acquired solitons are demonstrated with the aid of Matlab to examine the properties of the soliton solutions. The obtained results show that the considered gDGH equation admits dark, bright, singular, and periodic solutions. This study may contribute to a comprehensive investigation of the soliton solutions of the gDGH equation, which has practical applications in fields such as oceanography and nonlinear optics.

Keywords: Soliton solutions, shallow water, nonlinear wave dynamics, dark soliton.

Parabolik Doğrusal olmayan Kanunlu Genelleştirilmiş Dullin-Gottwald-Holm Denkleminin Soliton Çözümleri

Öz

Bu makalede, parabolik yasalı genelleştirilmiş Dullin-Gottwald-Holm (gDGH) denkleminin soliton çözümleri incelenmiştir. İlgili denklem yüzey gerilimli sığ sularda dalga davranışını modellemektedir. Literatürde doğrusal olmayan parabolik kanuna sahip gDGH denklemi ile ilgili sadece birkaç çalışma vardır ve literatür araştırmalarından görüldüğü üzere bu makalede kullanılacak olan birleştirilmiş Riccati denklemi genişletme (BRDG) yöntemi daha önce bu denkleme uygulanmamıştır. Bu çalışmada gDGH denkleminin birçok soliton çözümü, doğrusal olmayan kısmi diferansiyel denklemleri çözmek için güçlü bir teknik olan BRDG yöntemi kullanılarak başarılı bir şekilde elde edilmiştir. Elde edilen analitik çözümlerin gDGH denklemini sağladığı Mathematica kullanarak doğrulanmıştır. Ayrıca, soliton çözümlerinin özelliklerini ve davranışını incelemek için elde edilen solitonların bazı grafikleri Matlab yardımıyla çizdirilmiştir. Elde edilen sonuçlar, gDGH denkleminin karanlık, parlak, tekil ve periyodik gibi farklı türlerde çözümler içerdiğini göstermektedir. Bu çalışma, okyanus bilimi ve doğrusal olmayan optik gibi alanlarda uygulamaları olan gDGH denkleminin soliton çözümlerinin kapsamlı bir şekilde incelenmesine katkı sunabilir.

Anahtar Kelimeler: Soliton çözümleri, sığ su, doğrusal olmayan dalga dinamiği, karanlık soliton.

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1. Introduction

Nonlinear partial differential equations (NLPDEs) are mathematical equations that describe physical phenomena in a wide range of scientific fields, including physics, engineering, chemistry, biology, and economics (Farlow, 2012), (Levy & Shearer 2015), (Braun, 1983b), (Zachmanoglou & Thoe, 1986). The ability to model complex systems and predict their behavior is critical in modern science and engineering and NLPDEs are a powerful tool for achieving this goal. NLPDEs can be used to model a variety of physical phenomena, such as fluid flow, heat transfer, wave propagation, and electromagnetism (Farlow, 2012), (Levy & Shearer 2015), (Braun, 1983b), (Zachmanoglou & Thoe, 1986).

In 2001, Dullin et al. introduced an equation that models shallow water waves with surface tension, later called the DGH equation (Dullin et al., 2001). Due to advances in physics and mathematics, various extensions of the DGH equation have been generated and investigated. For example; the dimensionless form of the generalized DGH (gDGH) equation was studied by (Biswas & Kara, 2010). Zhang, Y., & Xia, Y. (2021) dealt with the gDGH equation with parabolic law nonlinearity.

This paper aims to obtain the soliton solutions of the gDGH equation with parabolic law nonlinearity. The equation can be given by (Zhang & Xia, 2021b):

$$z_t - \alpha z_{xxt} + 2\beta z_x + (\gamma z + \sigma z^2) z_x + \mu z_{xxx} = \alpha(2 z_x z_{xx} + z z_{xxx}), \tag{1}$$

where $z = z(x,t)$. Here, x, t denote spatial and temporal variables, respectively. $\alpha, \beta, \gamma, \sigma$, and μ are constants with real values. The term $\gamma z + \sigma z^2$ stands for parabolic law nonlinearity. Besides, γ and σ are coefficients of the parabolic law nonlinearity form that occurs in optical fibers.

Yang et al. (2022) studied exact soliton solutions of the gDGH equation with cubic power law nonlinearity. They also investigated the bifurcations of the solutions. Yin et al. (2013) presented painleve analysis of gDGH and KdV equations. In a study by Leta and Li (2017), the authors investigated the soliton solutions of the various exact soliton solutions and bifurcations. These studies highlight the importance of the gDGH equation and its soliton solutions in understanding nonlinear wave dynamics in various physical systems and exploring their practical applications in fields such as oceanography and nonlinear optics.

In the literature, to obtain soliton solution of the NLPDEs, there are powerful methods such as new Kudryashov (Kudryashov, 2020), auxiliary equation technique (Ozisik et al., 2022b), sine-Gordon equation (Yıldırım, 2021), F-expansion method (Zhou et al., 2003b), modified F-expansion method (Ozisik et al., 2023), the generalized unified method (Osman & Wazwaz, 2018), modified extended tanh-function method (El-Wakil et al., 2002), generalized Kudryashov (Cakicioglu et al., 2023) and Kudryashov's integrability approaches (Ozisik et al., 2022). In this paper, UREEM (Sirendaoreji, 2017), (Zayed et al., 2020) is used to extract soliton solutions of gDGH equation with parabolic law nonlinearity.

The rest of the paper is structured as follows: In Section 2, we provide a wave transformation and apply UREEM to the considered model. In section 3, we present some plots of the soliton solutions of the gDGH equation and give explanations of the figures. In conclusion, we summarize our main findings and contributions.

2. Method

2.1. Wave Transformation

The following transformation in eq. (2) is employed to convert the gDGH with parabolic law nonlinearity into an ordinary differential equation (ODE) that can be solved.

$$z(x,t) = Z(\xi), \;\; \xi = x - v\, t, \tag{2}$$

in which v denotes the speed of the wave. So, we obtain:

$$(\mu - \alpha Z + \alpha v)\, Z^{(3)} + (2\beta - 2\alpha Z'' + Z(\gamma + \sigma Z) - v) Z' = 0, \tag{3}$$

where $Z = \mathrm{Z}(\xi)$. Here, Z', Z'', and $Z^{(3)}$ are the first, second, and third order derivatives with respect to ξ, respectively.

2.2. Unified Riccati Equation Expansion Method

Assume that the eq. (3) possesses the solutions as follows (Sirendaoreji, 2017), (Zayed et al., 2020):

$$Z(\xi) = A_0 + \sum_{i=1}^{\mathrm{N}} A_i \varphi^i(\xi), \tag{4}$$

in which $A_N \neq 0$ and N denotes a balance number. Balancing the terms $Z\, Z^{(3)}$ and $Z^2 Z'$ in Eq. (3), we get $N = 2$. So, Eq. (4) is converted into:

$$Z(\xi) = A_0 + A_1\, \varphi(\xi) + A_2\, \varphi(\xi)^2, \tag{5}$$

where $A_2 \neq 0$ and $\varphi(\xi)$ represents the solution of the following equation:

$$\varphi'(\xi) = c_0 + c_1 \varphi(\xi) + c_2 \varphi^2(\xi), \tag{6}$$

in which Eq. (6) has the following solutions (Sirendaoreji, 2017), (Zayed et al., 2020):

$$\varphi_1(\xi) = -\frac{c_1}{2c_2} - \frac{\sqrt{\Delta}\left(k_1 \tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right) + k_2\right)}{2c_2\left(k_1 + k_2 \tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right)}, \quad \text{if } \Delta > 0,$$

$$\varphi_2(\xi) = -\frac{c_1}{2c_2} - \frac{\sqrt{\Delta}\left(k_1 \coth\left(\frac{\sqrt{\Delta}}{2}\xi\right) + k_2\right)}{2c_2\left(k_1 + k_2 \coth\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right)}, \quad \text{if } \Delta > 0,$$

$$\varphi_3(\xi) = -\frac{c_1}{2c_2} - \frac{1}{c_2\xi + k_3}, \quad \text{if } \Delta = 0,$$

$$\varphi_4(\xi) = -\frac{c_1}{2c_2} + \frac{\sqrt{-\Delta}\left(k_4 \tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right) - k_5\right)}{2c_2\left(k_4 + k_5 \tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right)}, \quad \text{if } \Delta < 0,$$

$$\varphi_5(\xi) = -\frac{c_1}{2c_2} - \frac{\sqrt{-\Delta}\left(k_4 \cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right) - k_5\right)}{2c_2\left(k_4 + k_5 \cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right)}, \quad \text{if } \Delta < 0,$$

where $\Delta = c_1^2 - 4c_0c_2$ and k_1, k_2, k_3, k_4, k_5 are arbitrary constants. The solutions $\varphi_1(\xi)$ and $\varphi_2(\xi)$ are non-trivial and nondegenerate if and only if $k_2 = \pm k_1, k_1^2 + k_2^2 \neq 0$. The solutions $\varphi_4(\xi)$ and $\varphi_5(\xi)$ are non-trivial and nondegenerate if and only if $k_4^2 + k_5^2 \neq 0$. By collecting all terms with the same power of $\varphi(\xi)$ in Eq. (3) and setting all coefficients to zero, the following system of equations is obtained:

The coefficient of $\varphi^0(\xi)$:

$$\begin{aligned} &c_0\big(-2\alpha A_1^2 c_0 c_1 + A_1(A_0(-\alpha c_1^2 - 2\alpha c_0 c_2 + \gamma) - 4\alpha A_2 c_0^2 + A_0^2\sigma + 2\beta + c_1^2\mu + 2c_0c_2\mu + \alpha c_1^2 v + 2\alpha c_0 c_2 v - v) \\ &\quad + 6A_2c_0c_1(-\alpha A_0 + \mu + \alpha v)\big) = 0. \end{aligned} \tag{7}$$

The coefficient of $\varphi^1(\xi)$:

$$\begin{aligned} &A_1^2 c_0(2A_0\sigma - 5\alpha c_1^2 - 6\alpha c_0 c_2 + \gamma) \\ &+ A_1c_1(A_0(-\alpha c_1^2 - 8\alpha c_0c_2 + \gamma) - 26\alpha A_2 c_0^2 + A_0^2\sigma + 2\beta + c_1^2\mu + 8c_0c_2\mu + \alpha c_1^2 v + 8\alpha c_0c_2 v - v) \\ &+ 2A_2c_0(A_0(-7\alpha c_1^2 - 8\alpha c_0c_2 + \gamma) - 4\alpha A_2c_0^2 + A_0^2\sigma + 2\beta + 7c_1^2\mu + 8c_0c_2\mu + 7\alpha c_1^2 v + 8\alpha c_0c_2 v - v) = 0. \end{aligned} \tag{8}$$

The coefficient of $\varphi^2(\xi)$:

$$\begin{aligned} &A_1^2c_1(2A_0\sigma - 3\alpha c_1^2 - 20\alpha c_0c_2 + \gamma) + A_1^3c_0\sigma \\ &+ A_1\big(A_2c_0(6A_0\sigma - 43\alpha c_1^2 - 46\alpha c_0c_2 + 3\gamma) \\ &+ c_2(A_0(-7\alpha c_1^2 - 8\alpha c_0c_2 + \gamma) + A_0^2\sigma + 2\beta + 8c_0c_2\mu + 7c_1^2(\mu + \alpha v) + 8\alpha c_0c_2 v - v)\big) \\ &+ 2A_2c_1(A_0(-4\alpha c_1^2 - 26\alpha c_0c_2 + \gamma) - 19\alpha A_2c_0^2 + A_0^2\sigma + 2\beta + 4c_1^2\mu + 26c_0c_2\mu + 4\alpha c_1^2 v + 26\alpha c_0c_2 v - v) = 0. \end{aligned} \tag{9}$$

The coefficient of $\varphi^3(\xi)$:

$$\begin{aligned} &A_1^2(c_2(2A_0\sigma - 15\alpha c_1^2 - 16\alpha c_0c_2 + \gamma) + 4A_2c_0\sigma) + A_1^3c_1\sigma \\ &+ 2A_2\big(A_2c_0(2A_0\sigma - 27\alpha c_1^2 - 28\alpha c_0c_2 + \gamma) \\ &+ c_2(A_0(-19\alpha c_1^2 - 20\alpha c_0c_2 + \gamma) + A_0^2\sigma + 2\beta + 20c_0c_2\mu + 19c_1^2(\mu + \alpha v) + 20\alpha c_0c_2 v - v)\big) \\ &+ 3A_1c_1\big(A_2(2A_0\sigma - 7\alpha c_1^2 - 44\alpha c_0c_2 + \gamma) + 4c_2^2(-\alpha A_0 + \mu + \alpha v)\big) = 0. \end{aligned} \tag{10}$$

The coefficient of $\varphi^4(\xi)$:

$$\begin{aligned} &2A_1^2c_1(2A_2\sigma - 11\alpha c_2^2) + A_1^3c_2\sigma + A_1\big(A_2c_2(6A_0\sigma - 89\alpha c_1^2 - 92\alpha c_0c_2 + 3\gamma) + 5A_2^2c_0\sigma + 6c_2^3(-\alpha A_0 + \mu + \alpha v)\big) \\ &\quad + 2A_2c_1\big(A_2(2A_0\sigma - 12\alpha c_1^2 - 74\alpha c_0c_2 + \gamma) + 27c_2^2(-\alpha A_0 + \mu + \alpha v)\big) = 0. \end{aligned} \tag{11}$$

The coefficient of $\varphi^5(\xi)$:

$$A_2^2(2c_2(2A_0\sigma - 47\alpha c_1^2 - 48\alpha c_0 c_2 + \gamma) + 5A_1 c_1 \sigma) - 10\alpha A_1^2 c_2^3 + 2A_2^3 c_0 \sigma + 2A_2 c_2(-59\alpha A_1 c_1 c_2 + 12c_2^2(-\alpha A_0 + \mu + \alpha v) + 2A_1^2\sigma) = 0. \tag{12}$$

The coefficient of $\varphi^6(\xi)$:

$$A_2(A_2 c_2(5A_1\sigma - 118\alpha c_1 c_2) - 50\alpha A_1 c_2^3 + 2A_2^2 c_1 \sigma) = 0. \tag{13}$$

The coefficient of $\varphi^7(\xi)$:

$$2A_2^2 c_2(A_2\sigma - 24\alpha c_2^2) = 0. \tag{14}$$

When the system of algebraic equations that is given above is solved utilizing a computer algebra system, the following set is obtained:

SET 1$^{\pm}$.

$$\mu = -\frac{\alpha\left(\gamma + 2\sigma v + 3\sqrt{\gamma^2 - 8\beta\sigma - 8\alpha^2 c_1^4 + 64\alpha^2 c_0 c_2 c_1^2 - 128\alpha^2 c_0^2 c_2^2 + 4\sigma v}\right)}{2\sigma},$$

$$A_0 = \pm\frac{4\alpha c_1^2 + 32\alpha c_0 c_2 - \gamma + \sqrt{-8\beta\sigma + \gamma^2 - 8\alpha^2 c_1^4 + 64\alpha^2 c_0 c_2 c_1^2 - 128\alpha^2 c_0^2 c_2^2 + 4\sigma v}}{2\sigma},$$

$$A_1 = \frac{24\alpha c_1 c_2}{\sigma},$$

$$A_2 = \frac{24\alpha c_2^2}{\sigma},$$

where

$$\gamma^2 - 8\beta\sigma - 8\alpha^2 c_1^4 + 64\alpha^2 c_0 c_2 c_1^2 - 128\alpha^2 c_0^2 c_2^2 + 4\sigma v > 0.$$

Substituting the set above to Eq. (4) along with Eq. (2), we derive the following solutions of the gDGH equation in Eq. (1):

$$z_1^{\pm}(x,t) = A_0 + \frac{24\alpha c_1 c_2\left(-\frac{\sqrt{\Delta}\left(k_1\tanh\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)+k_2\right)}{2c_2\left(k_2\tanh\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)+k_1\right)} - \frac{c_1}{2c_2}\right) + 24\alpha c_2^2\left(-\frac{\sqrt{\Delta}\left(k_1\tanh\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)+k_2\right)}{2c_2\left(k_2\tanh\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)+k_1\right)} - \frac{c_1}{2c_2}\right)^2}{\sigma},$$

$$z_2^{\pm}(x,t) = A_0 + \frac{24\alpha c_1 c_2\left(-\frac{\sqrt{\Delta}\left(k_1\tan\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)+k_2\right)}{2c_2\left(k_2\tan\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)+k_1\right)} - \frac{c_1}{2c_2}\right) + 24\alpha c_2^2\left(-\frac{\sqrt{\Delta}\left(k_1\tan\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)+k_2\right)}{2c_2\left(k_2\tan\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)+k_1\right)} - \frac{c_1}{2c_2}\right)^2}{\sigma},$$

$$z_3^{\pm}(x,t) = A_0 + \frac{24\alpha c_1 c_2\left(-\frac{1}{c_2(x-tv)+k_3} - \frac{c_1}{2c_2}\right) + 24\alpha c_2^2\left(-\frac{1}{c_2(x-tv)+k_3} - \frac{c_1}{2c_2}\right)^2}{\sigma},$$

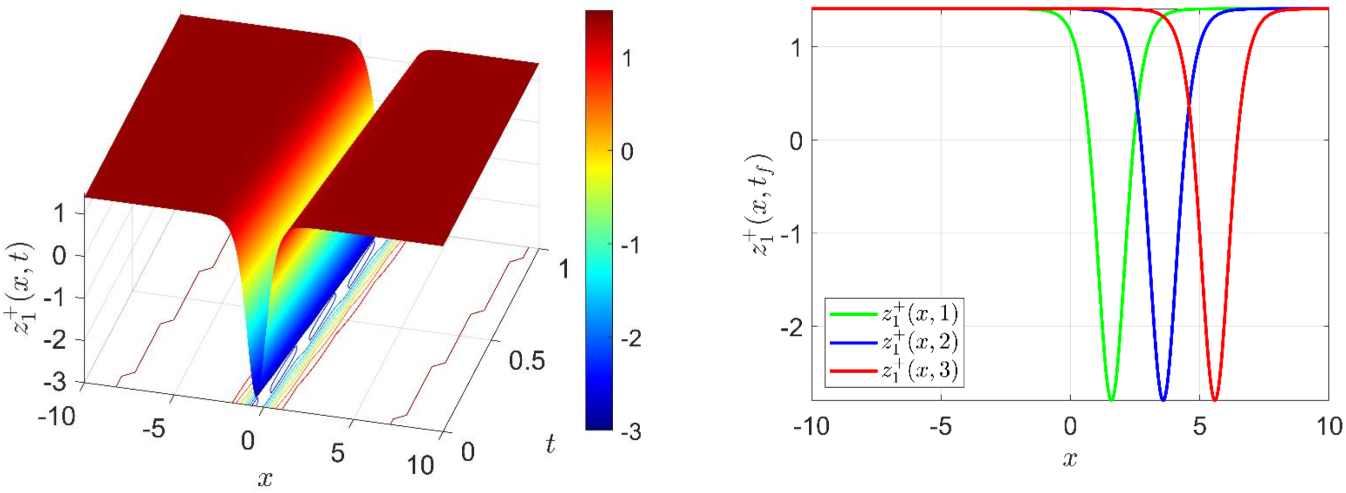
$$z_4^{\pm}(x,t) = A_0 + \frac{24\alpha c_1 c_2\left(\frac{\sqrt{\Delta}\left(k_1\tan\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)-k_2\right)}{2c_2\left(k_2\tan\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)+k_1\right)} - \frac{c_1}{2c_2}\right) + 24\alpha c_2^2\left(\frac{\sqrt{\Delta}\left(k_1\tan\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)-k_2\right)}{2c_2\left(k_2\tan\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)+k_1\right)} - \frac{c_1}{2c_2}\right)^2}{\sigma},$$

$$z_5^{\pm}(x,t) = A_0 + \frac{24\alpha c_1 c_2\left(-\frac{\sqrt{\Delta}\left(k_1\tan\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)-k_2\right)}{2c_2\left(k_2\tan\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)+k_1\right)} - \frac{c_1}{2c_2}\right) + 24\alpha c_2^2\left(-\frac{\sqrt{\Delta}\left(k_1\tan\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)-k_2\right)}{2c_2\left(k_2\tan\left(\frac{1}{2}\sqrt{\Delta}(x-tv)\right)+k_1\right)} - \frac{c_1}{2c_2}\right)^2}{\sigma}.$$

3. Results and Discussion

We successfully derived the soliton solutions of the gDGH equation with parabolic law nonlinearity using the UREEM. The results of our study show that the gDGH equation admits dark, bright, singular, and periodic solutions.

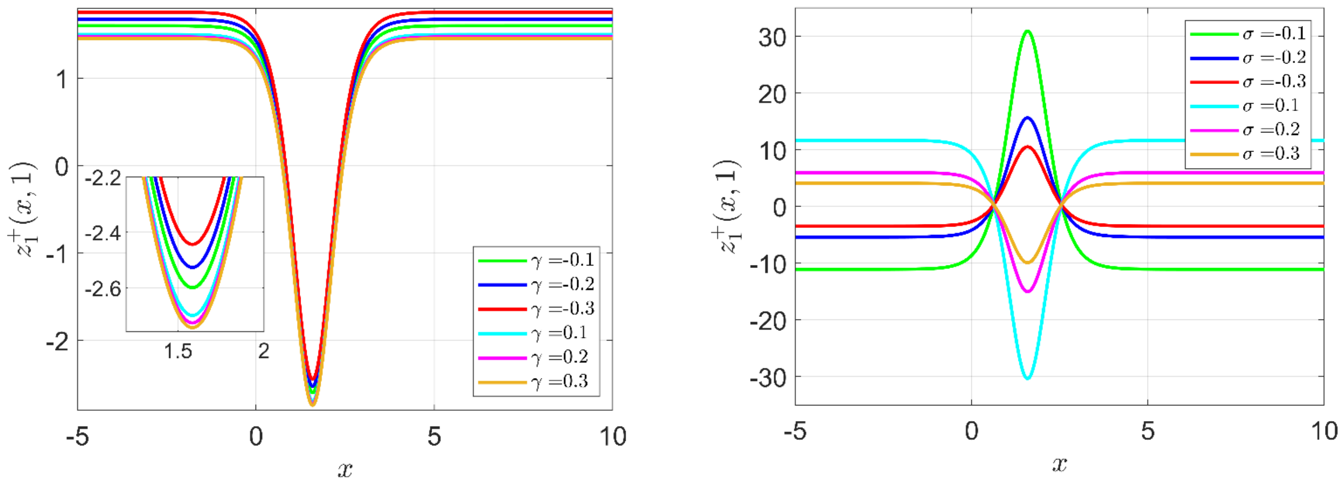
Figure 1 presents diverse plots of the ${z_1}^{+}(x,t)$ using the parameters $v = 2, \alpha = \frac{1}{10}, \beta = \frac{1}{2}, \gamma = 4, \mu = \frac{1}{2}\left(-\frac{3}{25}\sqrt{\frac{201}{2}} - \frac{4}{5}\right)$, $\sigma = 1, c_0 = 1, c_1 = 3, c_2 = \frac{1}{2}, k_1 = 2,$ and $k_2 = 1$. It admits a dark soliton. Fig. (1-a) shows 3D demonstration of the soliton while Fig. (1-b) illustrates 2D plot of the soliton for $t_f = 1,2,$ and 3. As can be deduced from Fig. (1-b), the soliton goes to the right on the horizontal axis.



(a) (b)

Figure 1. Three and two-dimensional plots of ${z_1}^{+}(x,t)$ (dark soliton)

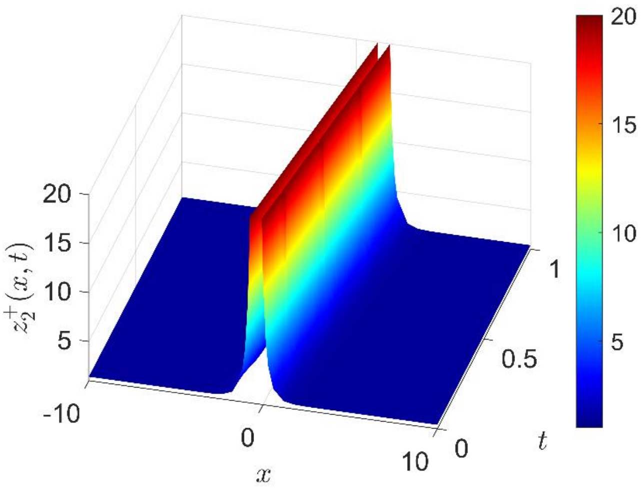
Fig. (2-a) and Fig. (2-b) illustrate the effect of the parameters γ and σ on ${z_1}^{+}(x,t)$ utilizing the parameters $v = 2, \alpha = \frac{1}{10}, \beta = \frac{1}{2}, \mu = \frac{1}{2}\left(-\frac{3}{25}\sqrt{\frac{201}{2}} - \frac{4}{5}\right) c_0 = 1, c_1 = 3, c_2 = \frac{1}{2}, k_1 = 2,$ and $k_2 = 1$. Soliton amplitude increases in absolute value when γ takes both negative and positive increasing values. There is a downward displacement in the wings of the soliton (Fig. (2-a)). While the soliton has a dark appearance for positive values of σ, the soliton turns into a bright soliton character in the case of negative values of σ. In this respect, σ has an important effect that changes the type of soliton. For positive increasing values of σ, the amplitude decreases depending on the increment. While it takes negative increasing values, the amplitude increases.



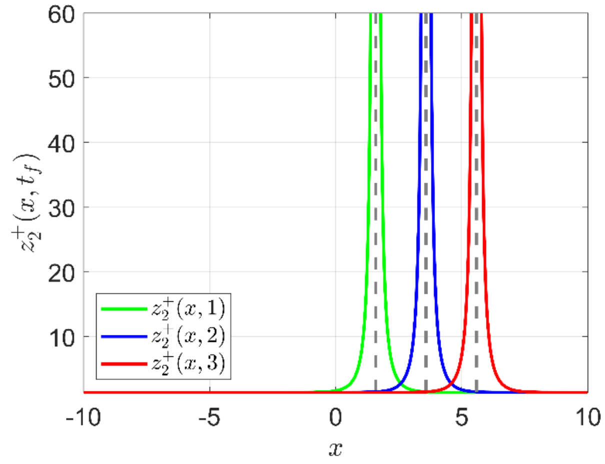
(a) For $\sigma = 1$ (b) For $\gamma = 4$

Figure 2. The effect of the parameters γ and σ on ${z_1}^{+}(x,t)$

In Figure 3, we show the singular solutions ${z_2}^+(x,t)$ of the gDGH equation for $v = 2, \alpha = \frac{1}{10}, \beta = \frac{1}{2}, \gamma = 4, \sigma = 1, \mu = \frac{1}{2}\left(-\frac{3}{25}\sqrt{\frac{201}{2}} - \frac{4}{5}\right), c_0 = 1, c_1 = 3, c_2 = \frac{1}{2}, k_1 = 2,$ and $k_2 = 1$. Fig. (1-a) and Fig. (1-b) represent 3D and 2D demonstrations of the solution, respectively. The wave goes to the right on the horizontal axis while $t_f = 1,2,$ and 3.



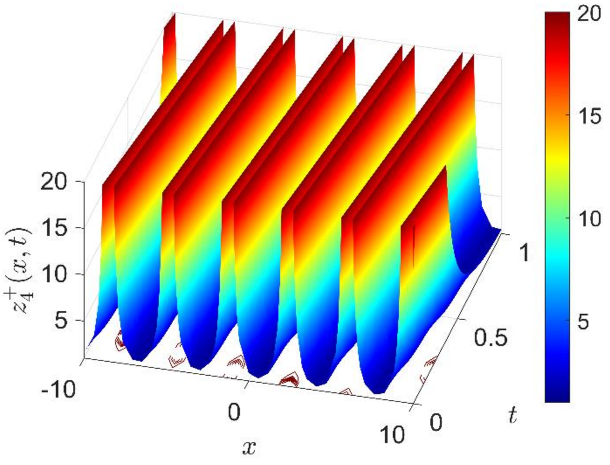
(a)



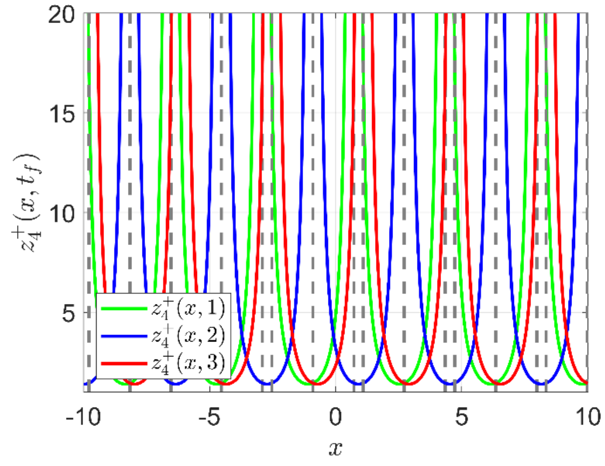
(b)

Figure 3. Three and two-dimensional plots of ${z_2}^+(x,t)$ (singular solution)

Finally, in Figure 4, we plot the solution ${z_2}^+(x,t)$ of the gDGH equation for $v = 2, \alpha = \frac{1}{10}, \beta = \frac{1}{2}, \gamma = 4, \mu = \frac{1}{2}\left(-\frac{4}{5} - \frac{3}{\sqrt{5}}\right), \sigma = 1, c_0 = 1, c_1 = 2, c_2 = 1, k_4 = 2,$ and $k_5 = 1$. It is a periodic solution.



(a)



(b)

Figure 4. Three and two-dimensional plots of ${z_4}^+(x,t)$ (periodic solution)

4. Conclusions and Recommendations

The UREEM was used in this paper to produce many analytical solutions of the gDGH equation with parabolic law nonlinearity, that models shallow water waves. This work effectively provides dark, bright, singular, and periodic solutions. We tested that all extracted solutions satisfied the main equation with the aid of Mathematica, which is a computer algebraic system. Moreover, 2D and 3D schemes are presented for a physical explanation of the obtained solutions. According to the results of the study, for future works, the approach may be utilized to find analytical solutions to nonlinear evolution equations that model real-life problems.

References

Biswas, A., & Kara, A. (2010). 1-Soliton solution and conservation laws of the generalizedDullin–Gottwald–Holm equation. Applied Mathematics and Computation, 217(2),929–932. https://doi.org/10.1016/j.amc.2010.05.085

Braun, M. (1983b). Differential Equations and Their Applications. In Springer eBooks. https://doi.org/10.1007/978-1-4684-0164-6

Cakicioglu, H., Ozisik, M., Secer, A., & Bayram, M. (2023). Optical soliton solutions of Schrödinger–Hirota equation with parabolic law nonlinearity via generalized Kudryashov algorithm. Optical and Quantum Electronics, 55(5). https://doi.org/10.1007/s11082-023-04634-x

Dullin, H. R., Gottwald, G. A., & Holm, D. D. (2001). An Integrable Shallow Water Equation with Linear and Nonlinear Dispersion. Physical Review Letters, 87(19). https://doi.org/10.1103/physrevlett.87.194501

El-Wakil, S., El-Labany, S. K., Zahran, M., & Sabry, R. (2002). Modified extended tanh-function method for solving nonlinear partial differential equations. Physics Letters, 299(2–3), 179–188. https://doi.org/10.1016/s0375-9601(02)00669-2

Farlow, S. J. (2012). Partial Differential Equations for Scientists and Engineers. Dover Publications.

Kudryashov, N. A. (2020). Method for finding highly dispersive optical solitons of nonlinear differential equations. Optik, 206, 163550. https://doi.org/10.1016/j.ijleo.2019.163550

Leta, T. D., & Li, J. (2017). Various Exact Soliton Solutions and Bifurcations of a Generalized Dullin–Gottwald–Holm Equation with a Power Law Nonlinearity. International Journal of Bifurcation and Chaos. https://doi.org/10.1142/s0218127417501292

Levy, R., Shearer, M. (2015). Partial Differential Equations: An Introduction to Theory and Applications. Princeton University Press.

Osman, M. N. M., & Wazwaz, A. (2018b). An efficient algorithm to construct multi-soliton rational solutions of the (2+ 1)-dimensional KdV equation with variable coefficients. Applied Mathematics and Computation, 321, 282–289. https://doi.org/10.1016/j.amc.2017.10.042

Ozisik, M., Secer, A., Bayram, M., & Aydin, H. (2022). An encyclopedia of Kudryashov's integrability approaches applicable to optoelectronic devices. Optik, 265, 169499. https://doi.org/10.1016/j.ijleo.2022.169499

Ozisik, M., Secer, A., & Bayram, M. (2022b). On the examination of optical soliton pulses of Manakov system with auxiliary equation technique. Optik, 268, 169800. https://doi.org/10.1016/j.ijleo.2022.169800

Ozisik, M., Secer, A., & Bayram, M. (2023). On solitary wave solutions for the extended nonlinear Schrödinger equation via the modified F-expansion method. Optical and Quantum Electronics, 55(3). https://doi.org/10.1007/s11082-022-04476-z

Sirendaoreji. (2017). Unified Riccati equation expansion method and its application to two new classes of Benjamin–Bona–Mahony equations. Nonlinear Dynamics, 89(1), 333–344. https://doi.org/10.1007/s11071-017-3457-6

Yang, D., Lou, Q., & Zhang, J. (2022). Bifurcations and exact soliton solutions for generalized Dullin–Gottwald–Holm equation with cubic power law nonlinearity. European Physical Journal Plus, 137(2). https://doi.org/10.1140/epjp/s13360-022-02462-8

Yıldırım, Y. (2021). Optical solitons in birefringent fibers with Biswas–Arshed equation by sine–Gordon equation method. Optik, 227, 165960. https://doi.org/10.1016/j.ijleo.2020.165960

Yin, J. H., Ding, S., Tian, L., & Fan, X. (2013). A New Method for Generating Traveling-Wave Solutions of Coupled Nonlinear Equations. Ukrainian Mathematical Journal, 64(10), 1553–1561. https://doi.org/10.1007/s11253-013-0734-5

Zachmanoglou, E. C., Thoe, D. W. (1986). Introduction to partial differential equations with applications. New York: Dover Publications.

Zayed, E. M., Biswas, A., Asma, M., Ekici, M., Alzahrani, A. K., & Belic, M. R. (2020). Pure-cubic optical soliton perturbation with full nonlinearity by unified Riccati equation expansion. Optik, 223, 165445. https://doi.org/10.1016/j.ijleo.2020.165445

Zhang, Y., & Xia, Y. (2021). Traveling Wave Solutions of Generalized Dullin–Gottwald–Holm Equation with Parabolic Law Nonlinearity. Qualitative Theory of Dynamical Systems, 20(3). https://doi.org/10.1007/s12346-021-00503-8

Zhou, Y., Wang, M., & Chen, X. (2003b). Periodic wave solutions to a coupled KdV equations with variable coefficients. Physics Letters, 308(1), 31–36. https://doi.org/10.1016/s0375-9601(02)01775-9