Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (3) (2024), 608 – 627
DOI : 10.15672/hujms.1270585

Research Article

Refinement of the classical Jensen inequality using finite sequences

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Abstract

This article is dedicated to a refinement of the classical Jensen inequality by virtue of some finite real sequences. Inequalities for various means are obtained from this refinement. Also, from the proposed refinement, the authors acquire some inequalities for Csiszâr Ψ-divergence and for Shannon and Zipf-Mandelbrot entropies. The refinement is further generalized through several finite real sequences.

Mathematics Subject Classification (2020). 26A51, 62B10, 26D15, 26E60

Keywords. Jensen inequality, means, information theory

1. Introduction

Mathematical inequalities particularly for convex functions have a lot of applications in various areas of art, science and technology. Different interesting results regarding mathematical inequalities and their applications in various aspects can be found in [2,4–8,16,18,20,21,23,32,33,35,40–42]. Jensen's inequality may be considered as one of the most dominant inequalities because it gives at once the major part of some well known mathematical inequalities such as Young's, Hölder's, Ky Fan's, Levinson's, and Minkowski's inequalities, etc [14], which can be deduced from this inequality by manipulating different convex functions with some suitable substitutions. Furthermore, this inequality is comprehensively used in distinct areas of science and technology for example statistics [25], qualitative theory of differential and integral equations [24], engineering [9], economics [26], finance [3], information theory and coding [19,36,38] etc. In addition, there are countless papers dealing with generalizations, refinements, counterparts and converse results of Jensen's inequality, (see, for instance [11,12,17,31,34]). Therefore, it deserves to be studied thoroughly and refine it from different point of views.

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Received: 25.03.2023; Accepted: 12.06.2023

Jensen's inequality can be found in the literature of modern applied analysis and states that [22]: if $\mathcal{G} \subset \mathbb{R}$ is an interval, $x_i \in \mathcal{G}$, $\varphi_i > 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \varphi_i = 1$ and $f : \mathcal{G} \to \mathbb{R}$ is a convex function. Then

$$f\left(\sum_{i=1}^{n} \varphi_i x_i\right) \le \sum_{i=1}^{n} \varphi_i f(x_i). \tag{1.1}$$

Particularly, several mathematicians have practiced through different angles to refine Jensen's inequality by determining certain mathematical expressions between the right and left hand sides of this inequality. Motivated by these investigations, this paper deals to refine this inequality by virtue of four finite real sequences. By taking such suitable sequences, inequalities for different means are obtained from this refinement and are presented in Section 2. Section 3 assembles some interesting inequalities for Csiszâr and Rényi divergences, Relative and Shannon entropies, and variational distance etc. An inclusive detail about Zipf's law with inequalities for Zipf-Mandelbrot entropy and its related parametrics have been provided in its subsection. In Section 4, we further generalize the proposed refinement through several finite real sequences. Section 5 is dedicated to concluding remarks of the paper.

2. Main results

For an interval $\mathcal{G} \subset \mathbb{R}$, assume that $f : \mathcal{G} \to \mathbb{R}$ is a convex function. Let $x_i \in \mathcal{G}$ and $\varphi_i, \omega_i, \eta_i, \xi_i, \theta_i \in (0, \infty)$ for $i = 1, 2, \ldots, n$ with the restriction that $\sum_{i=1}^{n} \varphi_i = 1$, $\omega_i + \eta_i = 1$, $\xi_i + \theta_i = 1$ for $i = 1, 2, \ldots, n$. Also, let $\mathcal{I} \subset \{1, 2, \ldots, n\}$ and setting $\bar{\mathcal{I}} := \{1, 2, \ldots, n\}/\mathcal{I}$. Setting the following functional for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_n)$, $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ as strictly positive n-tuples:

$$\begin{aligned}\mathbb{Z}(\mathrm{f}, \boldsymbol{\varphi}, \boldsymbol{\omega}, \boldsymbol{\eta}, \boldsymbol{\xi}, \boldsymbol{\theta}, \boldsymbol{x}; \mathcal{I}) = &\sum_{i \in \mathcal{I}} \varphi_i \omega_i f\left(\frac{\sum_{i \in \mathcal{I}} \varphi_i \omega_i x_i}{\sum_{i \in \mathcal{I}} \varphi_i \omega_i}\right) + \sum_{i \in \mathcal{I}} \varphi_i \eta_i f\left(\frac{\sum_{i \in \mathcal{I}} \varphi_i \eta_i x_i}{\sum_{i \in \mathcal{I}} \varphi_i \eta_i}\right) \\ &+ \sum_{i \in \bar{\mathcal{I}}} \varphi_i \xi_i f\left(\frac{\sum_{i \in \bar{\mathcal{I}}} \varphi_i \xi_i x_i}{\sum_{i \in \bar{\mathcal{I}}} \varphi_i \xi_i}\right) + \sum_{i \in \bar{\mathcal{I}}} \varphi_i \theta_i f\left(\frac{\sum_{i \in \bar{\mathcal{I}}} \varphi_i \theta_i x_i}{\sum_{i \in \bar{\mathcal{I}}} \varphi_i \theta_i}\right).\end{aligned} \tag{2.1}$$

By virtue of the above defined real sequences a refinement has been proposed here in this theorem.

Theorem 2.1. *Let $f : \mathcal{G} \to \mathbb{R}$ be a convex function. Also, let $x_i \in \mathcal{G}$, $\varphi_i, \xi_i, \eta_i, \omega_i, \theta_i \in (0, \infty)$ $(i = 1, 2, \ldots, n)$ such that $\sum_{i=1}^{n} \varphi_i = 1$, $\xi_i + \theta_i = 1$, $\omega_i + \eta_i = 1$ for all $i \in \{1, 2, \ldots, n\}$. Then, provided $\mathcal{I} \subseteq \{1, 2, \ldots, n\}$, the following inequalities hold*

$$f\left(\sum_{i=1}^{n} \varphi_i x_i\right) \le \mathbb{Z}\left(\mathrm{f}, \boldsymbol{\varphi}, \boldsymbol{\omega}, \boldsymbol{\eta}, \boldsymbol{\xi}, \boldsymbol{\theta}, \boldsymbol{x}; \mathcal{I}\right) \le \sum_{i=1}^{n} \varphi_i f(x_i). \tag{2.2}$$

The inequality in (2.2) *reverses for f as a concave function.*

Proof. Since $\sum\limits_{i=1}^{n} \varphi_i x_i = \sum\limits_{i \in \mathcal{I}} \varphi_i x_i + \sum\limits_{i \in \bar{\mathcal{I}}} \varphi_i x_i$ and $\omega_i + \eta_i = 1$, $\xi_i + \theta_i = 1$ for each $i = 1, 2, \ldots, n$, thus f being a convex function, we have

$$\begin{aligned}&\mathbb{Z}(\mathrm{f}, \boldsymbol{\varphi}, \boldsymbol{\omega}, \boldsymbol{\eta}, \boldsymbol{\xi}, \boldsymbol{\theta}, \boldsymbol{x}; \mathcal{I}) \\ &= \sum_{i \in \mathcal{I}} \varphi_i \omega_i f\left(\frac{\sum_{i \in \mathcal{I}} \varphi_i \omega_i x_i}{\sum_{i \in \mathcal{I}} \varphi_i \omega_i}\right) + \sum_{i \in \mathcal{I}} \varphi_i \eta_i f\left(\frac{\sum_{i \in \mathcal{I}} \varphi_i \eta_i x_i}{\sum_{i \in \mathcal{I}} \varphi_i \eta_i}\right) \\ &\quad + \sum_{i \in \bar{\mathcal{I}}} \varphi_i \xi_i f\left(\frac{\sum_{i \in \bar{\mathcal{I}}} \varphi_i \xi_i x_i}{\sum_{i \in \bar{\mathcal{I}}} \varphi_i \xi_i}\right) + \sum_{i \in \bar{\mathcal{I}}} \varphi_i \theta_i f\left(\frac{\sum_{i \in \bar{\mathcal{I}}} \varphi_i \theta_i x_i}{\sum_{i \in \bar{\mathcal{I}}} \varphi_i \theta_i}\right)\end{aligned}$$

$$
\begin{aligned}
&\geq f\Bigg(\sum_{i\in\mathfrak{I}}\varphi_i\omega_i.\frac{\sum_{i\in\mathfrak{I}}\varphi_i\omega_i x_i}{\sum_{i\in\mathfrak{I}}\varphi_i\omega_i}+\sum_{i\in\mathfrak{I}}\varphi_i\eta_i.\frac{\sum_{i\in\mathfrak{I}}\varphi_i\eta_i x_i}{\sum_{i\in\mathfrak{I}}\varphi_i\eta_i}\\
&\quad+\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\xi_i.\frac{\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\xi_i x_i}{\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\xi_i}+\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\theta_i.\frac{\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\theta_i x_i}{\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\theta_i}\Bigg)\\
&=f\Bigg(\sum_{i\in\mathfrak{I}}\varphi_i\omega_i x_i+\sum_{i\in\mathfrak{I}}\varphi_i\eta_i x_i+\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\xi_i x_i+\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\theta_i x_i\Bigg)\\
&=f\Bigg(\sum_{i\in\mathfrak{I}}\varphi_i(\omega_i+\eta_i)x_i+\sum_{i\in\bar{\mathfrak{I}}}\varphi_i(\xi_i+\theta_i)x_i\Bigg)\\
&=f\Bigg(\sum_{i=1}^{n}\varphi_i x_i\Bigg),
\end{aligned}
$$

thus the first inequality in (2.2) directly follows.
Using Jensen's inequality, one can get the following:

$$
\begin{aligned}
\sum_{i=1}^{n}\varphi_i f(x_i)&=\sum_{i\in\mathfrak{I}}\varphi_i f(x_i)+\sum_{i\in\bar{\mathfrak{I}}}\varphi_i f(x_i)\\
&=\sum_{i\in\mathfrak{I}}\varphi_i(\omega_i+\eta_i)f(x_i)+\sum_{i\in\bar{\mathfrak{I}}}\varphi_i(\xi_i+\theta_i)f(x_i)\\
&=\sum_{i\in\mathfrak{I}}\varphi_i\omega_i f(x_i)+\sum_{i\in\mathfrak{I}}\varphi_i\eta_i f(x_i)+\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\xi_i f(x_i)+\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\theta_i f(x_i)\\
&\geq\sum_{i\in\mathfrak{I}}\varphi_i\omega_i f\left(\frac{\sum_{i\in\mathfrak{I}}\varphi_i\omega_i x_i}{\sum_{i\in\mathfrak{I}}\varphi_i\omega_i}\right)+\sum_{i\in\mathfrak{I}}\varphi_i\eta_i f\left(\frac{\sum_{i\in\mathfrak{I}}\varphi_i\eta_i x_i}{\sum_{i\in\mathfrak{I}}\varphi_i\eta_i}\right)\\
&\quad+\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\xi_i f\left(\frac{\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\xi_i x_i}{\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\xi_i}\right)+\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\theta_i f\left(\frac{\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\theta_i x_i}{\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\theta_i}\right)\\
&=\mathbb{Z}(f,\boldsymbol{\varphi},\boldsymbol{\omega},\boldsymbol{\eta},\boldsymbol{\xi},\boldsymbol{\theta},\boldsymbol{x};\mathfrak{I}),
\end{aligned}
$$

and here the second inequality in (2.2) follows. □

Remark 2.2. The Riemann integral version of the above theorem and its related results can be seen in [37].

Remark 2.3. Following is an equivalent form of inequality (2.2).

$$
\sum_{i=1}^{n}\varphi_i f(x_i)\geq\max_{\phi\neq\mathfrak{I}\subset\{1,2,\ldots,n\}}\mathbb{Z}(f,\boldsymbol{\varphi},\boldsymbol{\omega},\boldsymbol{\eta},\boldsymbol{\xi},\boldsymbol{\theta},\boldsymbol{x};\mathfrak{I}),
$$

and

$$
f\left(\sum_{i=1}^{n}\varphi_i x_i\right)\leq\min_{\phi\neq\mathfrak{I}\subset\{1,2,\ldots,n\}}\mathbb{Z}(f,\boldsymbol{\varphi},\boldsymbol{\omega},\boldsymbol{\eta},\boldsymbol{\xi},\boldsymbol{\theta},\boldsymbol{x};\mathfrak{I}).
$$

Corollary 2.4. *For an arbitrary interval $\mathfrak{I}$, assume that $f:\mathfrak{I}\to\mathbb{R}$ is a convex function and let $x_i\in\mathfrak{I}$, $\varphi_i,\xi_i,\eta_i,\ \omega_i,\theta_i\in(0,\infty)$, $i=1,2,\ldots,n$ with the following conditions $\sum_{i=1}^{n}\varphi_i=1$, $\xi_i+\theta_i=1$, $\omega_i+\eta_i=1$ for each $i=1,2,\ldots,n$. Then*

$$
f\left(\sum_{i=1}^{n}\varphi_i x_i\right)\leq\min_{k\in\{1,2,\ldots,n\}}\left\{\left(\sum_{i=1}^{n}\varphi_i\omega_i-\varphi_k\omega_k\right)f\left(\frac{\sum\limits_{i=1}^{n}\varphi_i\omega_i x_i-\varphi_k\omega_k x_k}{\sum\limits_{i=1}^{n}\varphi_i\omega_i-\varphi_k\omega_k}\right)+\varphi_k\xi_k f(x_k)\right.
$$

$$+\left(\sum_{i=1}^{n}\varphi_i\eta_i-\varphi_k\eta_k\right)f\left(\frac{\sum\limits_{i=1}^{n}\varphi_i\eta_i x_i-\varphi_k\eta_k x_k}{\sum\limits_{i=1}^{n}\varphi_i\eta_i-\varphi_k\eta_k}\right)+\varphi_k\theta_k f(x_k)\Bigg\}$$

$$\le\frac{1}{n}\Bigg\{\sum_{k=1}^{n}\left(\sum_{i=1}^{n}\varphi_i\omega_i-\varphi_k\omega_k\right)f\left(\frac{\sum\limits_{i=1}^{n}\varphi_i\omega_i x_i-\varphi_k\omega_k x_k}{\sum\limits_{i=1}^{n}\varphi_i\omega_i-\varphi_k\omega_k}\right)+\sum_{k=1}^{n}\varphi_k\xi_k f(x_k)$$

$$+\sum_{k=1}^{n}\left(\sum_{i=1}^{n}\varphi_i\eta_i-\varphi_k\eta_k\right)f\left(\frac{\sum\limits_{i=1}^{n}\varphi_i\eta_i x_i-\varphi_k\eta_k x_k}{\sum\limits_{i=1}^{n}\varphi_i\eta_i-\varphi_k\eta_k}\right)+\sum_{k=1}^{n}\varphi_k\theta_k f(x_k)\Bigg\}$$

$$\le\max_{k\in\{1,2,\ldots,n\}}\Bigg\{\left(\sum_{i=1}^{n}\varphi_i\omega_i-\varphi_k\omega_k\right)f\left(\frac{\sum\limits_{i=1}^{n}\varphi_i\omega_i x_i-\varphi_k\omega_k x_k}{\sum\limits_{i=1}^{n}\varphi_i\omega_i-\varphi_k\omega_k}\right)+\varphi_k\xi_k f(x_k)$$

$$+\left(\sum_{i=1}^{n}\varphi_i\eta_i-\varphi_k\eta_k\right)f\left(\frac{\sum\limits_{i=1}^{n}\varphi_i\eta_i x_i-\varphi_k\eta_k x_k}{\sum\limits_{i=1}^{n}\varphi_i\eta_i-\varphi_k\eta_k}\right)+\varphi_k\theta_k f(x_k)\Bigg\}$$

$$\le\sum_{i=1}^{n}\varphi_i f(x_i). \tag{2.3}$$

Proof. Taking $\bar{\mathfrak{I}}=\{k\}$, $\mathfrak{I}=\{1,2,\ldots,n\}\backslash\{k\}$, $k\in\{1,2,\ldots,n\}$, we have the following functional

$$\begin{aligned}\mathbb{Z}_k(f,\boldsymbol{\varphi},\boldsymbol{\omega},\boldsymbol{\eta},\boldsymbol{\xi},\boldsymbol{\theta},\boldsymbol{x}) :=&\mathbb{Z}(f,\boldsymbol{\varphi},\boldsymbol{\omega},\boldsymbol{\eta},\boldsymbol{\xi},\boldsymbol{\theta},\boldsymbol{x};\{k\})\\
=&\sum_{i=1,i\neq k}^{n}\varphi_i\omega_i f\left(\frac{\sum\limits_{\substack{i=1\\i\neq k}}^{n}\varphi_i\omega_i x_i}{\sum\limits_{\substack{i=1\\i\neq k}}^{n}\varphi_i\omega_i}\right)+\varphi_k\xi_k f(x_k)\\
&+\sum_{i=1,i\neq k}^{n}\varphi_i\eta_i f\left(\frac{\sum\limits_{\substack{i=1\\i\neq k}}^{n}\varphi_i\eta_i x_i}{\sum\limits_{\substack{i=1\\i\neq k}}^{n}\varphi_i\eta_i}\right)+\varphi_k\theta_k f(x_k)\\
=&\left(\sum_{i=1}^{n}\varphi_i\omega_i-\varphi_k\omega_k\right)f\left(\frac{\sum\limits_{i=1}^{n}\varphi_i\omega_i x_i-\varphi_k\omega_k x_k}{\sum\limits_{i=1}^{n}\varphi_i\omega_i-\varphi_k\omega_k}\right)+\varphi_k\xi_k f(x_k)\\
&+\left(\sum_{i=1}^{n}\varphi_i\eta_i-\varphi_k\eta_k\right)f\left(\frac{\sum\limits_{i=1}^{n}\varphi_i\eta_i x_i-\varphi_k\eta_k x_k}{\sum\limits_{i=1}^{n}\varphi_i\eta_i-\varphi_k\eta_k}\right)+\varphi_k\theta_k f(x_k).\end{aligned}$$

From above Remark 2.3, using the following fact by taking maximum and minimum over k

$$\min_{k\in\{1,2,\ldots,n\}}\alpha_k\le\frac{1}{n}\sum_{k=1}^{n}\alpha_k\le\max_{k\in\{1,2,\ldots,n\}}\alpha_k,$$

we obtain inequality (2.3). □

Definition 2.5. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ and $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)$ be positive n-tuples. Further suppose that $\mathcal{J}$ is a nonempty subset of $\{1, 2, \ldots, n\}$. For $r \in \mathbb{R}$ as the order, the power mean is defined as:

$$\boldsymbol{M}_{[r;\mathcal{J}]}(\boldsymbol{\varphi};\boldsymbol{x}) = \begin{cases} \left(\frac{\sum_{i\in\mathcal{J}}\varphi_i x_i^r}{\sum_{i\in\mathcal{J}}\varphi_i}\right)^{\frac{1}{r}} & , \quad \text{if } r \neq 0, \\ \left(\prod_{i\in\mathcal{J}} x_i^{\varphi_i}\right)^{\frac{1}{\sum_{i\in\mathcal{J}}\varphi_i}} & , \quad \text{if } r = 0, \end{cases}$$

$$\boldsymbol{M}_{[r;\mathcal{J}]}(\boldsymbol{\varphi}.\boldsymbol{\eta};\boldsymbol{x}) = \begin{cases} \left(\frac{\sum_{i\in\mathcal{J}}\varphi_i\eta_i x_i^r}{\sum_{i\in\mathcal{J}}\varphi_i\eta_i}\right)^{\frac{1}{r}} & , \quad \text{if } r \neq 0, \\ \left(\prod_{i\in\mathcal{J}} x_i^{\varphi_i\eta_i}\right)^{\frac{1}{\sum\limits_{i\in\mathcal{J}}\varphi_i\eta_i}} & , \quad \text{if } r = 0, \end{cases}$$

and

$$\boldsymbol{M}_{[r;n]}(\boldsymbol{\varphi};\boldsymbol{x}) = \begin{cases} \left(\frac{\sum_{i=1}^n\varphi_i x_i^r}{\sum_{i=1}^n\varphi_i}\right)^{\frac{1}{r}} & , \quad \text{if } r \neq 0, \\ \left(\prod_{i=1}^n x_i^{\varphi_i}\right) & , \quad \text{if } r = 0. \end{cases}$$

The following corollary provides some inequalities for power mean.

Corollary 2.6. *Let $x_i, \varphi_i, \xi_i, \eta_i, \omega_i, \theta_i$ be strictly positive n-tuples in interval $\mathcal{J}$ when $i = 1, 2, \ldots, n$ with the following restrictions $\sum_{i=1}^n \varphi_i = 1$, $\xi_i + \theta_i = 1$, $\omega_i + \eta_i = 1$ for all $i \in \{1, 2, \ldots, n\}$. Let α, β be some real numbers such that $\beta \geq \alpha$, then*

$$\begin{aligned} \mathbf{M}_{[\alpha;n]}(\boldsymbol{\varphi};\boldsymbol{x}) \leq &\left\{\left(\sum_{i\in\mathcal{J}}\varphi_i\omega_i\right)\mathbf{M}^{\alpha}_{[\beta;\mathcal{J}]}(\boldsymbol{\varphi}.\boldsymbol{\omega};\boldsymbol{x}) + \left(\sum_{i\in\mathcal{J}}\varphi_i\eta_i\right)\mathbf{M}^{\alpha}_{[\beta;\mathcal{J}]}(\boldsymbol{\varphi}.\boldsymbol{\eta};\boldsymbol{x})\right. \\ &\left. + \left(\sum_{i\in\bar{\mathcal{J}}}\varphi_i\xi_i\right)\mathbf{M}^{\alpha}_{[\beta;\bar{\mathcal{J}}]}(\boldsymbol{\varphi}.\boldsymbol{\xi};\boldsymbol{x}) + \left(\sum_{i\in\bar{\mathcal{J}}}\varphi_i\theta_i\right)\mathbf{M}^{\alpha}_{[\beta;\bar{\mathcal{J}}]}(\boldsymbol{\varphi}.\boldsymbol{\theta};\boldsymbol{x})\right\}^{\frac{1}{\alpha}} \\ \leq &\mathbf{M}_{[\beta;n]}(\boldsymbol{\varphi};\boldsymbol{x}), \quad \alpha \neq 0. \end{aligned} \tag{2.4}$$

$$\begin{aligned} \mathbf{M}_{[0;n]}(\boldsymbol{\varphi};\boldsymbol{x}) \leq \exp &\left\{\left(\sum_{i\in\mathcal{J}}\varphi_i\omega_i\right)\log\mathbf{M}_{[\beta;\mathcal{J}]}(\boldsymbol{\varphi}.\boldsymbol{\omega};\boldsymbol{x}) + \left(\sum_{i\in\mathcal{J}}\varphi_i\eta_i\right)\log\mathbf{M}_{[\beta;\mathcal{J}]}(\boldsymbol{\varphi}.\boldsymbol{\eta};\boldsymbol{x})\right. \\ &\left. + \left(\sum_{i\in\bar{\mathcal{J}}}\varphi_i\xi_i\right)\log\mathbf{M}_{[\beta;\bar{\mathcal{J}}]}(\boldsymbol{\varphi}.\boldsymbol{\xi};\boldsymbol{x}) + \left(\sum_{i\in\bar{\mathcal{J}}}\varphi_i\theta_i\right)\log\mathbf{M}_{[\beta;\bar{\mathcal{J}}]}(\boldsymbol{\varphi}.\boldsymbol{\theta};\boldsymbol{x})\right\} \\ \leq &\mathbf{M}_{[\beta;n]}(\boldsymbol{\varphi};\boldsymbol{x}), \quad \alpha = 0. \end{aligned} \tag{2.5}$$

$$\begin{aligned} \mathbf{M}_{[\beta;n]}(\boldsymbol{\varphi};\boldsymbol{x}) \geq &\left\{\left(\sum_{i\in\mathcal{J}}\varphi_i\omega_i\right)\mathbf{M}^{\beta}_{[\alpha;\mathcal{J}]}(\boldsymbol{\varphi}.\boldsymbol{\omega};\boldsymbol{x}) + \left(\sum_{i\in\mathcal{J}}\varphi_i\eta_i\right)\mathbf{M}^{\beta}_{[\alpha;\mathcal{J}]}(\boldsymbol{\varphi}.\boldsymbol{\eta};\boldsymbol{x})\right. \\ &\left. + \left(\sum_{i\in\bar{\mathcal{J}}}\varphi_i\xi_i\right)\mathbf{M}^{\beta}_{[\alpha;\bar{\mathcal{J}}]}(\boldsymbol{\varphi}.\boldsymbol{\xi};\boldsymbol{x}) + \left(\sum_{i\in\bar{\mathcal{J}}}\varphi_i\theta_i\right)\mathbf{M}^{\beta}_{[\alpha;\bar{\mathcal{J}}]}(\boldsymbol{\varphi}.\boldsymbol{\theta};\boldsymbol{x})\right\}^{\frac{1}{\beta}} \\ \geq &\mathbf{M}_{[\beta;n]}(\boldsymbol{\varphi};\boldsymbol{x}), \quad \beta \neq 0. \end{aligned} \tag{2.6}$$

$$\mathbf{M}_{[0;n]}(\boldsymbol{\varphi};\boldsymbol{x}) \geq \exp\left\{\left(\sum_{i\in\mathfrak{I}}\varphi_i\omega_i\right)\log\mathbf{M}_{[\alpha;\mathfrak{I}]}(\boldsymbol{\varphi}.\boldsymbol{\omega};\boldsymbol{x}) + \left(\sum_{i\in\mathfrak{I}}\varphi_i\eta_i\right)\log\mathbf{M}_{[\alpha;\mathfrak{I}]}(\boldsymbol{\varphi}.\boldsymbol{\eta};\boldsymbol{x})\right.$$
$$\left. + \left(\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\xi_i\right)\log\mathbf{M}_{[\alpha;\bar{\mathfrak{I}}]}(\boldsymbol{\varphi}.\boldsymbol{\xi};\boldsymbol{x}) + \left(\sum_{i\in\bar{\mathfrak{I}}}\varphi_i\theta_i\right)\log\mathbf{M}_{[\alpha;\bar{\mathfrak{I}}]}(\boldsymbol{\varphi}.\boldsymbol{\theta};\boldsymbol{x})\right\}$$
$$\geq \mathbf{M}_{[\alpha;n]}(\boldsymbol{\varphi};\boldsymbol{x}), \quad \beta = 0. \tag{2.7}$$

Proof. **A:** First we discuss the convexity of the function $f(z) = z^{\frac{\alpha}{\beta}}$, $z > 0$, $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$ as follows :

Case 1: If $\frac{\alpha}{\beta} \geq 1$ and $\alpha \leq \beta$, then $f(z) = z^{\frac{\alpha}{\beta}}$, $z > 0$ is convex for $\alpha, \beta \in \mathbb{R}^-$. Therefore, utilizing (2.2) for $f(z)$ and $x_i \to {x_i}^{\beta}$, then letting $\frac{1}{\alpha}$ as power , one can obtain (2.4).

Case 2: If $0 < \frac{\alpha}{\beta} < 1$ and $\alpha \leq \beta$, then $f(z) = z^{\frac{\alpha}{\beta}}$, $z > 0$ is concave function for $\alpha, \beta \in \mathbb{R}^+$. Therefore, utilizing (2.2) for $f(z)$ and $x_i \to {x_i}^{\beta}$ and then letting $\frac{1}{\alpha}$ as power, one may also obtain (2.4).

Case 3: If $\frac{\alpha}{\beta} \leq -1$ and $\alpha \leq \beta$, then the function $f(z) = z^{\frac{\alpha}{\beta}}$ for $z > 0$ is convex provided that $\beta \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}^-$. Thus using (2.2) for $f(z)$ and $x_i \to {x_i}^{\beta}$, then letting $\frac{1}{\alpha}$ as power, some one can also obtain (2.4).

For $\alpha = 0$, assuming $\lim\limits_{\alpha\to 0}$ in (2.4), we get (2.5).

B: Here, we discuss the convexity of the function $f(z) = z^{\frac{\beta}{\alpha}}$, $z > 0$, $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$ as follows:

Case 1: If $\frac{\beta}{\alpha} \geq 1$ and $\alpha \leq \beta$, then $f(z) = z^{\frac{\beta}{\alpha}}$, $z > 0$ is convex function for $\alpha, \beta \in \mathbb{R}^+$. Hence, using (2.2) for $f(z)$ and $x_i \to {x_i}^{\alpha}$ and letting $\frac{1}{\beta}$ as power we obtain (2.6).

Case 2: If $0 < \frac{\beta}{\alpha} < 1$ and $\alpha \leq \beta$, then $f(z) = z^{\frac{\beta}{\alpha}}$, $z > 0$ is concave function for $\alpha, \beta \in \mathbb{R}^-$. Hence, using (2.2) for $f(z)$ and $x_i \to {x_i}^{\alpha}$ and letting $\frac{1}{\beta}$ as power we obtain (2.6).

Case 3: Similarly, If $\frac{\beta}{\alpha} \leq -1$ and $\alpha \leq \beta$, then $f(z) = z^{\frac{\beta}{\alpha}}$, $z > 0$ is convex function for $\alpha \in \mathbb{R}^-$, $\beta \in \mathbb{R}^+$. Hence, using (2.2) for $f(z)$ and $x_i \to {x_i}^{\alpha}$ and letting $\frac{1}{\beta}$ as power we obtain (2.6).

For $\beta = 0$, assume that $\lim_{\beta\to 0}$ in (2.6), we get (2.7).

□

Definition 2.7. Let $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ and $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)$ be strictly positive n-tuples. If a function $g : [a, b] \to \mathbb{R}$ is continuous and strictly monotone, and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [a, b]^n$, while $\mathfrak{I}$ is nonempty subset of $\{1, 2, \ldots, n\}$, then the mathematical form of quasi-arithmetic mean is given by

$$\boldsymbol{M}_g^{[\mathfrak{I}]}(\boldsymbol{\varphi};\boldsymbol{x}) = g^{-1}\left(\frac{\sum_{i\in\mathfrak{I}}\varphi_i g(x_i)}{\sum_{i\in\mathfrak{I}}\varphi_i}\right),$$
$$\boldsymbol{M}_g^{[\mathfrak{I}]}(\boldsymbol{\varphi}.\boldsymbol{\eta};\boldsymbol{x}) = g^{-1}\left(\frac{\sum_{i\in\mathfrak{I}}\varphi_i\eta_i g(x_i)}{\sum_{i\in\mathfrak{I}}\varphi_i\eta_i}\right),$$

and

$$\boldsymbol{M}_g^{[n]}(\boldsymbol{\varphi};\boldsymbol{x}) = g^{-1}\left(\frac{\sum_{i=1}^n\varphi_i g(x_i)}{\sum_{i=1}^n\varphi_i}\right). \tag{2.8}$$

The following are some inequalities for quasi-arithmetic mean.

Corollary 2.8. *Let $\Psi \circ g^{-1} : \mathcal{G} \to \mathbb{R}$ be a convex function for g as a strictly monotone and continuous function. Also, let $x_i, \varphi_i, \xi_i, \eta_i, \omega_i, \theta_i$ be strictly positive n-tuples for each $i = 1, 2, \ldots, n$ such that $g(x_i) \in \mathcal{G}$, $\sum_{i=1}^{n} \varphi_i = 1$, $\xi_i + \theta_i = 1$, $\omega_i + \eta_i = 1$ for all $i \in \{1, 2, \ldots, n\}$. Then for $\mathcal{J} \subset \{1, 2, \ldots, n\}$, the following inequalities hold*

$$\Psi\left(\mathbf{M}_g^{[n]}(\boldsymbol{\varphi};\boldsymbol{x})\right) \leq \sum_{i\in\mathcal{J}} \varphi_i\omega_i\Psi\left(\mathbf{M}_g^{[\mathcal{J}]}(\boldsymbol{\varphi}.\boldsymbol{\omega};\boldsymbol{x})\right) + \sum_{i\in\mathcal{J}} \varphi_i\eta_i\Psi\left(\mathbf{M}_g^{[\mathcal{J}]}(\boldsymbol{\varphi}.\boldsymbol{\eta};\boldsymbol{x})\right)$$
$$+ \sum_{i\in\bar{\mathcal{J}}} \varphi_i\xi_i\Psi\left(\mathbf{M}_g^{[\bar{\mathcal{J}}]}(\boldsymbol{\varphi}.\boldsymbol{\xi};\boldsymbol{x})\right) + \sum_{i\in\bar{\mathcal{J}}} \varphi_i\theta_i\Psi\left(\mathbf{M}_g^{[\bar{\mathcal{J}}]}(\boldsymbol{\varphi}.\boldsymbol{\theta};\boldsymbol{x})\right) \leq \sum_{i=1}^{n} \varphi_i\Psi(x_i). \quad (2.9)$$

The inequalities in (2.9) *reverse if the function $\Psi \circ g^{-1}$ is considered as concave.*

Proof. Letting $x_i \to g(x_i)$ and $f \to \Psi \circ g^{-1}$ in (2.2), the required result can be established. □

3. Applications in information theory

Keeping in mind that in the applied and theoretical statistical inference and data processing problems, the information theoretic divergence measures play the role of problem solving oriented tools. The Csiszâr's divergence is a general divergence measure ([1, 10]), which provides various relations and can be used in a binary experiment for the measurement of separation of the distributions understudied. The Csiszâr's Ψ-divergence functional is given by

Definition 3.1 (Csiszâr Divergence)**.** Let $\Psi : [\gamma_1, \gamma_2] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and assume $\boldsymbol{\upsilon} = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ are "positive probability distributions (PPDs)", then the Csiszâr Ψ- divergence functional is given by

$$C_\Psi(\boldsymbol{\upsilon}, \boldsymbol{\sigma}) = \sum_{i=1}^{n} \sigma_i \Psi\left(\frac{\upsilon_i}{\sigma_i}\right),$$

provided that $\frac{\upsilon_i}{\sigma_i} \in [\gamma_1, \gamma_2]$ for $i = 1, 2, \ldots, n$.

satisfying the conditions which explicated undefined expression by Dragomir [13] as follow:

$$\Psi(0) = \lim_{t\to 0^+} \Psi(t), \quad 0\Psi\left(\frac{0}{0}\right) = 0,$$
$$\Psi\left(\frac{\alpha}{0}\right) = \lim_{\varepsilon\to o^+} \varepsilon\Psi\left(\frac{\alpha}{\varepsilon}\right) = \alpha \lim_{t\to\infty} \frac{\Psi(t)}{t}, \quad \alpha > 0.$$

Because of the rapid growing interest and significance of divergences in statistics, information theory and probability theory, the general theory of Ψ-divergences deserves attention. The Csiszâr divergence functional in their natural form can be entertained as a series of some well-known entropies, divergences and distances which are dependent on Jensen's inequality for general and some conditional expectations. These are actually complicated if they are strictly formulated for all recommended functions $\Psi(t)$.

This section gives some important applications for the most familiar among them of our main result.

Theorem 3.2. *Let $\Psi : [\gamma_1, \gamma_2] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $\boldsymbol{\upsilon} = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n) \in \mathbb{R}^n$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^{n} \sigma_i = 1$, $\frac{\upsilon_i}{\sigma_i} \in [\gamma_1, \gamma_2]$ for $i \in \{1, 2, \ldots, n\}$. Also, let $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)$, $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ be some strictly positive tuples with the following conditions $\xi_i + \theta_i = 1$, $\omega_i + \eta_i = 1$ for all $i \in \{1, 2, \ldots, n\}$, then*

$$C_\Psi(\boldsymbol{\upsilon}, \boldsymbol{\sigma}) \geq \sum_{i\in\mathcal{J}} \sigma_i\omega_i\Psi\left(\frac{\sum_{i\in\mathcal{J}} \omega_i\upsilon_i}{\sum_{i\in\mathcal{J}} \sigma_i\omega_i}\right) + \sum_{i\in\mathcal{J}} \sigma_i\eta_i\Psi\left(\frac{\sum_{i\in\mathcal{J}} \eta_i\upsilon_i}{\sum_{i\in\mathcal{J}} \sigma_i\eta_i}\right)$$

$$+ \sum_{i\in\bar{\mathfrak{I}}}\sigma_i\xi_i\Psi\left(\frac{\sum_{i\in\bar{\mathfrak{I}}}\xi_i\upsilon_i}{\sum_{i\in\bar{\mathfrak{I}}}\sigma_i\xi_i}\right)+\sum_{i\in\bar{\mathfrak{I}}}\sigma_i\theta_i\Psi\left(\frac{\sum_{i\in\bar{\mathfrak{I}}}\theta_i\upsilon_i}{\sum_{i\in\bar{\mathfrak{I}}}\sigma_i\theta_i}\right)\geq\Psi\left(\sum_{i=1}^{n}\upsilon_i\right). \quad (3.1)$$

Proof. Taking (2.2) for $\mathcal{G}=[\gamma_1,\gamma_2]$, $f=\Psi$, $x_i=\frac{\upsilon_i}{\sigma_i}$, $\varphi_i=\sigma_i$ for $i\in\{1,2,\ldots,n\}$, we obtain (3.1). □

Corollary 3.3. *Let* $\boldsymbol{\upsilon}=(\upsilon_1,\upsilon_2,\ldots,\upsilon_n)\in\mathbb{R}^n$ *and* $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\ldots,\sigma_n)\in\mathbb{R}^n_+$ *with* $\sum_{i=1}^n\sigma_i=1$, *and let* $\boldsymbol{\eta}=(\eta_1,\eta_2,\ldots,\eta_n)$, $\boldsymbol{\omega}=(\omega_1,\omega_2,\ldots,\omega_n)$, $\boldsymbol{\xi}=(\xi_1,\xi_2,\ldots,\xi_n)$ *and* $\boldsymbol{\theta}=(\theta_1,\theta_2,\ldots,\theta_n)$ *be some positive tuples with the following restrictions* $\xi_i+\theta_i=1$, $\omega_i+\eta_i=1(i=1,2,\ldots,n)$. *Also, let* $\Psi:[\gamma_1,\gamma_2]\to\mathbb{R}$ *be a convex function, then*

$$C_\Psi(\boldsymbol{\upsilon},\boldsymbol{\sigma})\geq\max_{k\in\{1,2,\ldots,n\}}\left\{\left(\sum_{i=1}^{n}\sigma_i\omega_i-\sigma_k\omega_k\right)\Psi\left(\frac{\sum\limits_{i=1}^{n}\omega_i\upsilon_i-\omega_k\upsilon_k}{\sum\limits_{i=1}^{n}\sigma_i\omega_i-\sigma_k\omega_k}\right)+\sigma_k\xi_k\Psi\left(\frac{\upsilon_k}{\sigma_k}\right)\right.$$
$$\left.+\left(\sum_{i=1}^{n}\sigma_i\eta_i-\sigma_k\eta_k\right)\Psi\left(\frac{\sum\limits_{i=1}^{n}\eta_i\upsilon_i-\eta_k\upsilon_k}{\sum\limits_{i=1}^{n}\sigma_i\eta_i-\sigma_k\eta_k}\right)+\sigma_k\theta_k\Psi\left(\frac{\upsilon_k}{\sigma_k}\right)\right\}\geq\Psi\left(\sum_{i=1}^{n}\upsilon_i\right). \quad (3.2)$$

Proof. Taking $\bar{\mathfrak{I}}=\{k\}$, $k\in\{1,2,\ldots,n\}$ in (3.1), we obtain (3.2). □

Definition 3.4 (Shannon entropy)**.** Taking a PPD $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\ldots,\sigma_n)$, the Shannon entropy is defined as:

$$S(\boldsymbol{\sigma})=-\sum_{i=1}^{n}\sigma_i\log\sigma_i.$$

Corollary 3.5. *Assume that* $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\ldots,\sigma_n)$ *is a PPD with* $\frac{1}{\sigma_i}\in[\gamma_1,\gamma_2]\subset\mathbb{R}^+$ *for* $i=1,2,\ldots,n$. *Also, suppose that* $\boldsymbol{\eta}=(\eta_1,\eta_2,\ldots,\eta_n)$, $\boldsymbol{\omega}=(\omega_1,\omega_2,\ldots,\omega_n)$, $\boldsymbol{\xi}=(\xi_1,\xi_2,\ldots,\xi_n)$ *and* $\boldsymbol{\theta}=(\theta_1,\theta_2,\ldots,\theta_n)$ *are some strictly positive tuples with the following restrictions* $\xi_i+\theta_i=1$, $\omega_i+\eta_i=1$ *for all* $i\in\{1,2,\ldots,n\}$, *then*

$$S(\boldsymbol{\sigma})\leq\sum_{i\in\mathfrak{I}}\sigma_i\omega_i\log\left(\frac{\sum_{i\in\mathfrak{I}}\omega_i}{\sum_{i\in\mathfrak{I}}\sigma_i\omega_i}\right)+\sum_{i\in\mathfrak{I}}\sigma_i\eta_i\log\left(\frac{\sum_{i\in\mathfrak{I}}\eta_i}{\sum_{i\in\mathfrak{I}}\sigma_i\eta_i}\right)$$
$$+\sum_{i\in\bar{\mathfrak{I}}}\sigma_i\xi_i\log\left(\frac{\sum_{i\in\bar{\mathfrak{I}}}\xi_i}{\sum_{i\in\bar{\mathfrak{I}}}\sigma_i\xi_i}\right)+\sum_{i\in\bar{\mathfrak{I}}}\sigma_i\theta_i\log\left(\frac{\sum_{i\in\bar{\mathfrak{I}}}\theta_i}{\sum_{i\in\bar{\mathfrak{I}}}\sigma_i\theta_i}\right)\leq\log(n). \quad (3.3)$$

Proof. Taking $\Psi(z)=-\log z$, $z>0$, $\upsilon_i=1$ for $i=1,2,\ldots,n$ in (3.1), result (3.3) is established. □

Corollary 3.6. *Assume that* $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\ldots,\sigma_n)$ *is a PPD. Also, suppose that* $\boldsymbol{\eta}=(\eta_1,\eta_2,\ldots,\eta_n)$, $\boldsymbol{\omega}=(\omega_1,\omega_2,\ldots,\omega_n)$, $\boldsymbol{\xi}=(\xi_1,\xi_2,\ldots,\xi_n)$ *and* $\boldsymbol{\theta}=(\theta_1,\theta_2,\ldots,\theta_n)$ *are positive tuples and* $\xi_i+\theta_i=1$, $\omega_i+\eta_i=1$ *for* $i=1,2,\ldots,n$, *the following inequalities are satisfied*

$$S(\boldsymbol{\sigma})\leq\max_{k\in\{1,2,\ldots,n\}}\left\{\left(\sum_{i=1}^{n}\sigma_i\omega_i-\sigma_k\omega_k\right)\log\left(\frac{\sum\limits_{i=1}^{n}\omega_i-\omega_k}{\sum\limits_{i=1}^{n}\sigma_i\omega_i-\sigma_k\omega_k}\right)-\sigma_k\xi_k\log\sigma_k\right.$$
$$\left.+\left(\sum_{i=1}^{n}\sigma_i\eta_i-\sigma_k\eta_k\right)\log\left(\frac{\sum\limits_{i=1}^{n}\eta_i-\eta_k}{\sum\limits_{i=1}^{n}\sigma_i\eta_i-\sigma_k\eta_k}\right)-\sigma_k\theta_k\log\sigma_k\right\}\leq\log(n). \quad (3.4)$$

Proof. Taking $\bar{\mathfrak{I}}=\{k\}$, $k\in\{1,2,\ldots,n\}$ in (3.3), we get (3.4). □

Definition 3.7 (Kullback-Leibler divergence (Relative entropy)). Assuming $\boldsymbol{\upsilon} = (\upsilon_1, \upsilon_2, \dots, \upsilon_n)$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_n)$ as some PPDs, then the mathematical form of Kullback-Leibler divergence is given by

$$KL(\boldsymbol{\upsilon}, \boldsymbol{\sigma}) = \sum_{i=1}^{n} \upsilon_i \log \frac{\upsilon_i}{\sigma_i}.$$

Corollary 3.8. *Let $\boldsymbol{\upsilon} = (\upsilon_1, \upsilon_2, \dots, \upsilon_n)$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_n)$ be some PPDs with $\frac{\upsilon_i}{\sigma_i} \in [\gamma_1, \gamma_2] \subset \mathbb{R}^+$. Also, assume $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_n)$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_n)$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_n)$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_n)$ as positive tuples but with the conditions that $\xi_i + \theta_i = 1$, $\omega_i + \eta_i = 1$ for all $i \in \{1, 2, \dots, n\}$, then*

$$KL(\boldsymbol{\upsilon}, \boldsymbol{\sigma}) \geq \sum_{i \in \mathfrak{I}} \omega_i \upsilon_i \log \left(\frac{\sum_{i \in \mathfrak{I}} \omega_i \upsilon_i}{\sum_{i \in \mathfrak{I}} \sigma_i \omega_i} \right) + \sum_{i \in \mathfrak{I}} \eta_i \upsilon_i \log \left(\frac{\sum_{i \in \mathfrak{I}} \eta_i \upsilon_i}{\sum_{i \in \mathfrak{I}} \sigma_i \eta_i} \right)$$
$$+ \sum_{i \in \bar{\mathfrak{I}}} \xi_i \upsilon_i \log \left(\frac{\sum_{i \in \bar{\mathfrak{I}}} \xi_i \upsilon_i}{\sum_{i \in \bar{\mathfrak{I}}} \sigma_i \xi_i} \right) + \sum_{i \in \bar{\mathfrak{I}}} \theta_i \upsilon_i \log \left(\frac{\sum_{i \in \bar{\mathfrak{I}}} \theta_i \upsilon_i}{\sum_{i \in \bar{\mathfrak{I}}} \sigma_i \theta_i} \right) \geq 0. \tag{3.5}$$

Proof. Taking $\Psi(z) = z \log(z)$, $z \in [\gamma_1, \gamma_2]$ in (3.1) we obtain (3.5). □

Corollary 3.9. *Letting the assumptions of Corollary 3.8, the following inequalities are satisfied.*

$$KL(\boldsymbol{\upsilon}, \boldsymbol{\sigma}) \geq \max_{k \in \{1,2,\dots,n\}} \left\{ \left(\sum_{i=1}^{n} \omega_i \upsilon_i - \omega_k \upsilon_k \right) \log \left(\frac{\sum\limits_{i=1}^{n} \omega_i \upsilon_i - \omega_k \upsilon_k}{\sum\limits_{i=1}^{n} \sigma_i \omega_i - \sigma_k \omega_k} \right) + \xi_k \upsilon_k \log \left(\frac{\upsilon_k}{\sigma_k} \right) \right.$$
$$\left. + \left(\sum_{i=1}^{n} \eta_i \upsilon_i - \eta_k \upsilon_k \right) \log \left(\frac{\sum\limits_{i=1}^{n} \eta_i \upsilon_i - \eta_k \upsilon_k}{\sum\limits_{i=1}^{n} \sigma_i \eta_i - \sigma_k \eta_k} \right) + \upsilon_k \theta_k \log \left(\frac{\upsilon_k}{\sigma_k} \right) \right\} \geq 0. \tag{3.6}$$

Proof. If we take $\bar{\mathfrak{I}} = \{k\}$, $k \in \{1, 2, \dots, n\}$ in (3.5), then we get (3.6). □

Remark 3.10. It is obvious that

$$\max_{\phi \neq \mathfrak{I} \subset \{1,2,\dots,n\}} \left\{ \sum_{i \in \mathfrak{I}} \omega_i \upsilon_i \log \left(\frac{\sum_{i \in \mathfrak{I}} \omega_i \upsilon_i}{\sum_{i \in \mathfrak{I}} \sigma_i \omega_i} \right) + \sum_{i \in \mathfrak{I}} \eta_i \upsilon_i \log \left(\frac{\sum_{i \in \mathfrak{I}} \eta_i \upsilon_i}{\sum_{i \in \mathfrak{I}} \sigma_i \eta_i} \right) \right.$$
$$\left. + \sum_{i \in \bar{\mathfrak{I}}} \xi_i \upsilon_i \log \left(\frac{\sum_{i \in \bar{\mathfrak{I}}} \xi_i \upsilon_i}{\sum_{i \in \bar{\mathfrak{I}}} \sigma_i \xi_i} \right) + \sum_{i \in \bar{\mathfrak{I}}} \theta_i \upsilon_i \log \left(\frac{\sum_{i \in \bar{\mathfrak{I}}} \theta_i \upsilon_i}{\sum_{i \in \bar{\mathfrak{I}}} \sigma_i \theta_i} \right) \right\}$$
$$\geq \max_{k \in \{1,2,\dots,n\}} \left\{ \left(\sum_{i=1}^{n} \omega_i \upsilon_i - \omega_k \upsilon_k \right) \log \left(\frac{\sum\limits_{i=1}^{n} \omega_i \upsilon_i - \omega_k \upsilon_k}{\sum\limits_{i=1}^{n} \sigma_i \omega_i - \sigma_k \omega_k} \right) + \xi_k \upsilon_k \log \left(\frac{\upsilon_k}{\sigma_k} \right) \right.$$
$$\left. + \left(\sum_{i=1}^{n} \eta_i \upsilon_i - \eta_k \upsilon_k \right) \log \left(\frac{\sum\limits_{i=1}^{n} \eta_i \upsilon_i - \eta_k \upsilon_k}{\sum\limits_{i=1}^{n} \sigma_i \eta_i - \sigma_k \eta_k} \right) + \upsilon_k \theta_k \log \left(\frac{\upsilon_k}{\sigma_k} \right) \right\}.$$

Definition 3.11 (Rényi divergence). Suppose that $\boldsymbol{\upsilon} = (\upsilon_1, \upsilon_2, \dots, \upsilon_n)$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_n)$ are PPDs and $\alpha > 1$, then the mathematical formula of Rényi divergence is given as

$$R_\alpha(\boldsymbol{\upsilon}, \boldsymbol{\sigma}) = \sum_{i=1}^{n} {\upsilon_i}^{\alpha} {\sigma_i}^{1-\alpha}.$$

Corollary 3.12. *Letting the assumptions of Corollary 3.8 and further $\alpha > 1$, the following inequality holds:*

$$R_\alpha(\boldsymbol{\upsilon},\boldsymbol{\sigma}) \geq \left(\sum_{i\in\mathfrak{I}}\sigma_i\omega_i\right)^{1-\alpha}\left(\sum_{i\in\mathfrak{I}}\omega_i\upsilon_i\right)^{\alpha} + \left(\sum_{i\in\mathfrak{I}}\sigma_i\eta_i\right)^{1-\alpha}\left(\sum_{i\in\mathfrak{I}}\eta_i\upsilon_i\right)^{\alpha}$$
$$+ \left(\sum_{i\in\bar{\mathfrak{I}}}\sigma_i\xi_i\right)^{1-\alpha}\left(\sum_{i\in\bar{\mathfrak{I}}}\xi_i\upsilon_i\right)^{\alpha} + \left(\sum_{i\in\bar{\mathfrak{I}}}\sigma_i\theta_i\right)^{1-\alpha}\left(\sum_{i\in\bar{\mathfrak{I}}}\theta_i\upsilon_i\right)^{\alpha} \geq 0. \qquad (3.7)$$

Proof. For the function $\Psi(z) = z^\alpha$, $\alpha > 1$ and $z \in [\gamma_1,\gamma_2]$, we have $\Psi''(z) = \alpha(\alpha-1)z^{\alpha-2} > 0$, which implies that Ψ is convex function. Thus, using (3.1) for $\Psi(z) = z^\alpha$, we obtain (3.7). □

Corollary 3.13. *Letting again the assumptions of Corollary 3.8 and taking $\alpha > 1$, the following inequality also holds:*

$$R_\alpha(\boldsymbol{\upsilon},\boldsymbol{\sigma}) \geq \max_{k\in\{1,2,\dots,n\}}\left\{\left(\sum_{i=1}^{n}\sigma_i\omega_i - \sigma_k\omega_k\right)^{1-\alpha}\left(\sum_{i=1}^{n}\omega_i\upsilon_i - \omega_k\upsilon_k\right)^{\alpha}\right.$$
$$\left. + \left(\sum_{i=1}^{n}\sigma_i\eta_i - \sigma_k\eta_k\right)^{1-\alpha}\left(\sum_{i=1}^{n}\eta_i\upsilon_i - \eta_k\upsilon_k\right)^{\alpha} + {\sigma_k}^{1-\alpha}{\upsilon_k}^{\alpha}(\xi_k+\theta_k)\right\} \geq 0. \qquad (3.8)$$

Proof. If we take $\bar{\mathfrak{I}} = \{k\}$, $k \in \{1,2,\dots,n\}$ in (3.7), then we get (3.8). □

Definition 3.14 (Variational distance)**.** Let $\boldsymbol{\upsilon} = (\upsilon_1,\upsilon_2,\dots,\upsilon_n)$, $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\dots,\sigma_n)$ be PPDs, then the mathematical formula of Variational distance is given as

$$V(\boldsymbol{\upsilon},\boldsymbol{\sigma}) = \sum_{i=1}^{n}|\upsilon_i - \sigma_i|.$$

Corollary 3.15. *Considering the assumptions of Corollary 3.8, the following inequality is satisfied.*

$$V(\boldsymbol{\upsilon},\boldsymbol{\sigma}) \geq \left|\sum_{i\in\mathfrak{I}}\omega_i\upsilon_i - \sum_{i\in\mathfrak{I}}\sigma_i\omega_i\right| + \left|\sum_{i\in\mathfrak{I}}\eta_i\upsilon_i - \sum_{i\in\mathfrak{I}}\sigma_i\eta_i\right|$$
$$+ \left|\sum_{i\in\mathfrak{I}}\xi_i\upsilon_i - \sum_{i\in\mathfrak{I}}\sigma_i\xi_i\right| + \left|\sum_{i\in\mathfrak{I}}\theta_i\upsilon_i - \sum_{i\in\mathfrak{I}}\sigma_i\theta_i\right|. \qquad (3.9)$$

Proof. Using $\Psi(z) = |z-1|$, $z \in [\gamma_1,\gamma_2]$ in (3.1) we obtain (3.9). □

Corollary 3.16. *The following inequality holds by assuming the assumptions of Corollary 3.8:*

$$V(\boldsymbol{\upsilon},\boldsymbol{\sigma}) \geq \max_{k\in\{1,2,\dots,n\}}\left\{\left|\sum_{i=1}^{n}\omega_i\upsilon_i - \sum_{i=1}^{n}\sigma_i\omega_i - \omega_k(\upsilon_k-\sigma_k)\right| + \xi_k\left|\upsilon_k-\sigma_k\right|\right.$$
$$\left. + \left|\sum_{i=1}^{n}\eta_i\upsilon_i - \sum_{i=1}^{n}\sigma_i\eta_i - \eta_k(\upsilon_k-\sigma_k)\right| + \theta_k\left|\upsilon_k-\sigma_k\right|\right\}. \qquad (3.10)$$

Proof. Taking $\bar{\mathfrak{I}} = \{k\}$, $k \in \{1,2,\dots,n\}$ in (3.9), we get (3.10). □

Definition 3.17 (Jeffrey's distance)**.** Suppose that $\boldsymbol{\upsilon} = (\upsilon_1,\upsilon_2,\dots,\upsilon_n)$, $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\dots,\sigma_n)$ are some PPDs, then the mathematical form of Jeffrey's distance is given by

$$J(\boldsymbol{\upsilon},\boldsymbol{\sigma}) = \sum_{i=1}^{n}(\upsilon_i - \sigma_i)\log\frac{\upsilon_i}{\sigma_i}.$$

Corollary 3.18. *The following inequality holds by considering the assumptions of Corollary 3.8:*

$$\begin{aligned}J(\boldsymbol{\upsilon},\boldsymbol{\sigma}) \geq &\left(\sum_{i\in\mathfrak{I}}\omega_i\upsilon_i-\sum_{i\in\mathfrak{I}}\omega_i\sigma_i\right)\log\left(\frac{\sum\limits_{i\in\mathfrak{I}}\omega_i\upsilon_i}{\sum\limits_{i\in\mathfrak{I}}\sigma_i\omega_i}\right)+\left(\sum_{i\in\mathfrak{I}}\eta_i\upsilon_i-\sum_{i\in\mathfrak{I}}\eta_i\sigma_i\right)\log\left(\frac{\sum\limits_{i\in\mathfrak{I}}\eta_i\upsilon_i}{\sum\limits_{i\in\mathfrak{I}}\sigma_i\eta_i}\right)\\&+\left(\sum_{i\in\bar{\mathfrak{I}}}\xi_i\upsilon_i-\sum_{i\in\bar{\mathfrak{I}}}\xi_i\sigma_i\right)\log\left(\frac{\sum\limits_{i\in\bar{\mathfrak{I}}}\xi_i\upsilon_i}{\sum\limits_{i\in\bar{\mathfrak{I}}}\sigma_i\xi_i}\right)+\left(\sum_{i\in\bar{\mathfrak{I}}}\theta_i\upsilon_i-\sum_{i\in\bar{\mathfrak{I}}}\theta_i\sigma_i\right)\log\left(\frac{\sum\limits_{i\in\bar{\mathfrak{I}}}\theta_i\upsilon_i}{\sum\limits_{i\in\bar{\mathfrak{I}}}\sigma_i\theta_i}\right)\\&\geq 0. \qquad (3.11)\end{aligned}$$

Proof. Using the function $\Psi(z)=(z-1)\log z$, $z\in[\gamma_1,\gamma_2]$ in (3.1), we obtain (3.11). □

Corollary 3.19. *The following inequality holds by considering the assumptions of Corollary 3.8:*

$$\begin{aligned}J(\boldsymbol{\upsilon},\boldsymbol{\sigma}) \geq \max_{k\in\{1,2,\ldots,n\}}&\left[\left(\sum_{i=1}^n\omega_i\upsilon_i-\sum_{i=1}^n\sigma_i\omega_i-\omega_k(\upsilon_k-\sigma_k)\right)\log\left(\frac{\sum\limits_{i=1}^n\omega_i\upsilon_i-\omega_k\upsilon_k}{\sum\limits_{i=1}^n\sigma_i\omega_i-\sigma_k\omega_k}\right)\right.\\&+\left(\sum_{i=1}^n\eta_i\upsilon_i-\sum_{i=1}^n\sigma_i\eta_i-\eta_k(\upsilon_k-\sigma_k)\right)\log\left(\frac{\sum\limits_{i=1}^n\eta_i\upsilon_i-\eta_k\upsilon_k}{\sum\limits_{i=1}^n\sigma_i\eta_i-\sigma_k\eta_k}\right)\\&\left.+(\upsilon_k-\sigma_k)(\xi_k+\theta_k)\log\frac{\upsilon_k}{\sigma_k}\right]\geq 0. \qquad (3.12)\end{aligned}$$

Proof. If we take $\bar{\mathfrak{I}}=\{k\}$, $k\in\{1,2,\ldots,n\}$ in (3.11), then we get (3.12). □

Definition 3.20 (Bhattacharyya coefficient)**.** The mathematical formula for the Bhattacharyya coefficient is given for two PPDs $\boldsymbol{\upsilon}=(\upsilon_1,\upsilon_2,\ldots,\upsilon_n)$ and $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\ldots,\sigma_n)$ by

$$B(\boldsymbol{\upsilon},\boldsymbol{\sigma})=\sum_{i=1}^n\sqrt{\upsilon_i\sigma_i}.$$

Corollary 3.21. *The following inequality holds by taking the assumptions of Corollary 3.8:*

$$\begin{aligned}B(\boldsymbol{\upsilon},\boldsymbol{\sigma}) \leq &\sqrt{\sum_{i\in\mathfrak{I}}\omega_i\upsilon_i\sum_{i\in\mathfrak{I}}\omega_i\sigma_i}+\sqrt{\sum_{i\in\mathfrak{I}}\eta_i\upsilon_i\sum_{i\in\mathfrak{I}}\eta_i\sigma_i}\\&+\sqrt{\sum_{i\in\bar{\mathfrak{I}}}\xi_i\upsilon_i\sum_{i\in\bar{\mathfrak{I}}}\xi_i\sigma_i}+\sqrt{\sum_{i\in\bar{\mathfrak{I}}}\theta_i\upsilon_i\sum_{i\in\bar{\mathfrak{I}}}\theta_i\sigma_i}. \qquad (3.13)\end{aligned}$$

Proof. The function $\Psi(z)=-\sqrt{z}$, $z\in[\gamma_1,\gamma_2]$ is convex, because $\Psi''(z)=\frac{1}{4z^{\frac{3}{2}}}>0$. Using $\Psi(Z)$ in (3.1), we obtain (3.13). □

Corollary 3.22. *The following inequality holds by taking the assumptions of Corollary 3.8:*

$$\begin{aligned}B(\boldsymbol{\upsilon},\boldsymbol{\sigma}) \leq \max_{k\in\{1,2,\ldots,n\}}&\left\{\sqrt{\left(\sum_{i=1}^n\omega_i\upsilon_i-\omega_k\upsilon_k\right)\left(\sum_{i=1}^n\omega_i\sigma_i-\omega_k\sigma_k\right)}+\xi_k\sqrt{\sigma_k\upsilon_k}\right.\\&\left.+\sqrt{\left(\sum_{i=1}^n\eta_i\upsilon_i-\eta_k\upsilon_k\right)\left(\sum_{i=1}^n\eta_i\sigma_i-\eta_k\sigma_k\right)}+\theta_k\sqrt{\sigma_k\upsilon_k}\right\}. \qquad (3.14)\end{aligned}$$

Proof. If we take $\bar{\mathfrak{I}} = \{k\}$ for $k \in \{1, 2, \ldots, n\}$ in (3.13), the we obtain (3.14). □

Definition 3.23 (Hellinger distance)**.** The mathematical formula of the Hellinger distance is given for two PPDs $\boldsymbol{\upsilon} = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ by

$$H(\boldsymbol{\upsilon}, \boldsymbol{\sigma}) = \sum_{i=1}^{n} (\sqrt{\upsilon_i} - \sqrt{\sigma_i})^2.$$

Corollary 3.24. *The following inequality holds by taking the assumptions of Corollary 3.8:*

$$\begin{aligned} H(\boldsymbol{\upsilon}, \boldsymbol{\sigma}) \geq & \left(\sqrt{\sum_{i \in \mathfrak{I}} \omega_i \upsilon_i} - \sqrt{\sum_{i \in \mathfrak{I}} \omega_i \sigma_i} \right)^2 + \left(\sqrt{\sum_{i \in \mathfrak{I}} \eta_i \upsilon_i} - \sqrt{\sum_{i \in \mathfrak{I}} \eta_i \sigma_i} \right)^2 \\ + & \left(\sqrt{\sum_{i \in \bar{\mathfrak{I}}} \xi_i \upsilon_i} - \sqrt{\sum_{i \in \bar{\mathfrak{I}}} \xi_i \sigma_i} \right)^2 + \left(\sqrt{\sum_{i \in \bar{\mathfrak{I}}} \theta_i \upsilon_i} - \sqrt{\sum_{i \in \bar{\mathfrak{I}}} \theta_i \sigma_i} \right)^2 \\ \geq & \ 0. \end{aligned} \tag{3.15}$$

Proof. Using the function $\Psi(z) = (\sqrt{z} - 1)^2$, $z \in [\gamma_1, \gamma_2]$ in (3.1), we obtain (3.15). □

Corollary 3.25. *The following inequality holds by letting the assumptions of Corollary 3.8:*

$$\begin{aligned} H(\boldsymbol{\upsilon}, \boldsymbol{\sigma}) \geq \max_{k \in \{1,2,\ldots,n\}} & \left\{ \left(\sqrt{\sum_{i=1}^{n} \omega_i \upsilon_i - \omega_k \upsilon_k} - \sqrt{\sum_{i=1}^{n} \sigma_i \omega_i - \sigma_k \omega_k} \right)^2 + \xi_k \left(\sqrt{\upsilon_k} - \sqrt{\sigma_k} \right)^2 \right. \\ & \left. + \left(\sqrt{\sum_{i=1}^{n} \eta_i \upsilon_i - \eta_k \upsilon_k} - \sqrt{\sum_{i=1}^{n} \sigma_i \eta_i - \sigma_k \eta_k} \right)^2 + \theta_k \left(\sqrt{\upsilon_k} - \sqrt{\sigma_k} \right)^2 \right\} \geq 0. \end{aligned} \tag{3.16}$$

Proof. If we take $\bar{\mathfrak{I}} = \{k\}$, $k \in \{1, 2, \ldots, n\}$ in (3.15), then we get (3.16). □

Definition 3.26 (Triangular discrimination)**.** The mathematical form of the Triangular discrimination is given for two PPDs $\boldsymbol{\upsilon} = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ by

$$D^{\Delta}(\boldsymbol{\upsilon}, \boldsymbol{\sigma}) = \sum_{i=1}^{n} \frac{(\upsilon_i - \sigma_i)^2}{\upsilon_i + \sigma_i}.$$

Corollary 3.27. *The following inequality holds by keeping the assumptions of Corollary 3.8:*

$$\begin{aligned} 0 \leq & \frac{\left(\sum_{i \in \mathfrak{I}} \omega_i \upsilon_i - \sum_{i \in \mathfrak{I}} \omega_i \sigma_i \right)^2}{\sum_{i \in \mathfrak{I}} \omega_i \upsilon_i + \sum_{i \in \mathfrak{I}} \omega_i \sigma_i} + \frac{\left(\sum_{i \in \mathfrak{I}} \eta_i \upsilon_i - \sum_{i \in \mathfrak{I}} \eta_i \sigma_i \right)^2}{\sum_{i \in \mathfrak{I}} \eta_i \upsilon_i + \sum_{i \in \mathfrak{I}} \eta_i \sigma_i} \\ & + \frac{\left(\sum_{i \in \bar{\mathfrak{I}}} \xi_i \upsilon_i - \sum_{i \in \bar{\mathfrak{I}}} \xi_i \sigma_i \right)^2}{\sum_{i \in \bar{\mathfrak{I}}} \xi_i \upsilon_i + \sum_{i \in \bar{\mathfrak{I}}} \xi_i \sigma_i} + \frac{\left(\sum_{i \in \bar{\mathfrak{I}}} \theta_i \upsilon_i - \sum_{i \in \bar{\mathfrak{I}}} \theta_i \sigma_i \right)^2}{\sum_{i \in \bar{\mathfrak{I}}} \theta_i \upsilon_i + \sum_{i \in \bar{\mathfrak{I}}} \theta_i \sigma_i} \\ \leq & D^{\Delta}(\boldsymbol{\upsilon}, \boldsymbol{\sigma}). \end{aligned} \tag{3.17}$$

Proof. If the function $\Psi(z) = \frac{(z-1)^2}{z+1}$, $z \in [\gamma_1, \gamma_2]$, then $\Psi''(z) = \frac{8}{(z+1)^3} \geq 0$, so definitely the function $\Psi(z)$ is convex. Therefore, using the function in (3.1), we obtain (3.17). □

Corollary 3.28. *The following inequality holds under the assumptions of Corollary 3.8:*

$$D^{\Delta}(\boldsymbol{\upsilon},\boldsymbol{\sigma}) \geq \max_{k\in\{1,2,\dots,n\}} \left\{ \frac{\left(\sum_{i=1}^{n}\omega_i\upsilon_i - \sum_{i=1}^{n}\sigma_i\omega_i - \omega_k(\upsilon_k-\sigma_k)\right)^2}{\sum_{i=1}^{n}\omega_i\upsilon_i + \sum_{i=1}^{n}\sigma_i\omega_i - \omega_k(\upsilon_k+\sigma_k)} + \xi_k.\frac{(\upsilon_k-\sigma_k)^2}{\upsilon_k+\sigma_k} \right.$$
$$\left. + \frac{\left(\sum_{i=1}^{n}\eta_i\upsilon_i - \sum_{i=1}^{n}\sigma_i\eta_i - \eta_k(\upsilon_k-\sigma_k)\right)^2}{\sum_{i=1}^{n}\eta_i\upsilon_i + \sum_{i=1}^{n}\sigma_i\eta_i - \eta_k(\upsilon_k+\sigma_k)} + \theta_k.\frac{(\upsilon_k-\sigma_k)^2}{\upsilon_k+\sigma_k} \right\} \geq 0.$$

Proof. Taking $\bar{\mathfrak{I}} = \{k\}$, $k \in \{1,2,\dots,n\}$ in (3.17), we obtain the result of Corollary 3.28. □

3.1. Applications for the Zipf-Mandelbrot entropy

In information science, Zipf's law may be considered as one of the most important and basic laws. Zipf's law says that largest occurrence of the event that is the size of $\imath^{th}$ is inversely proportional to it's rank (i.e $f(i) = 1/i^{\boldsymbol{s}}$, where $f(i)$ represents the number of occurrences of the i^{th} ranked and s takes a positive value close to unit). As by assuming $\boldsymbol{\gamma}$ and $\boldsymbol{\rho}$ as rank and frequency of the word respectively then in linguistics, Zipf obtained by the constant: $\mathbf{C} = \boldsymbol{\rho}.\boldsymbol{\gamma}$ (see [39]).

This law can also be used to obtain web site traffic, solar flare intensity, the size of moon craters, earthquake magnitude, city populations and this has also some useful applications in geology.

In 1966, a well-known mathematician Benoit Mandelbrot gave generalized form of the Zipf law, which is now called as the Zipf-Mandelbrot Law. This law actually provided a generalization regarding low-rank words in corpus [29]: $g(i) = \frac{c}{(i+\varpi)^s}$, where $i < 100$ and if we substitute $\varpi = 0$, will obtain Zipf's law. For some interesting applications of the Zipf-Mandelbrot law, the following references can be found ([15, 27, 28, 30]).
The following is well-known mathematical form of the Zipf-Mandelbrot entropy:

$$Z_{ME}(Q,\varpi,s) = \frac{s}{Q_{n,\varpi,s}}\sum_{i=1}^{n}\frac{\log(i+\varpi)}{(i+\varpi)^s} + \log Q_{n,\varpi,s}, \tag{3.18}$$

where $0 \leq \varpi$, $0 < s$, n is a positive integer, $Q_{n,\varpi,s} = \sum_{i=1}^{n}\frac{1}{(i+\varpi)^s}$ and the probability mass function (Zipf-Mandelbrot law) is defined by: $G(i,n,\varpi,s) = \frac{1/(i+\varpi)^s}{Q_{n,\varpi,s}}$.

Now here the Zipf-Mandelbrot entropy is estimated through some inequalities as follows:

Corollary 3.29. *Let $0 \leq \varpi$, $s, \sigma_i > 0$, $i = 1,2,\dots,n$ with $\sum_{i=1}^{n}\sigma_i = 1$. Further assume that $\boldsymbol{\eta} = (\eta_1,\eta_2,\dots,\eta_n)$, $\boldsymbol{\omega} = (\omega_1,\omega_2,\dots,\omega_n)$, $\boldsymbol{\xi} = (\xi_1,\xi_2,\dots,\xi_n)$ and $\boldsymbol{\theta} = (\theta_1,\theta_2,\dots,\theta_n)$ are some suitable positive tuples with the following conditions $\xi_i+\theta_i = 1$, $\omega_i+\eta_i = 1$ when $i \in \{1,2,\dots,n\}$, then*

$$-Z_{ME}(Q,\varpi,s) - \sum_{i=1}^{n}\frac{\log\sigma_i}{(i+\varpi)^s Q_{n,\varpi,s}} \geq \sum_{i\in\mathfrak{I}}\frac{\omega_i}{(i+\varpi)^s Q_{n,\varpi,s}}\log\left(\frac{\sum_{i\in\mathfrak{I}}\frac{\omega_i}{(i+\varpi)^s Q_{n,\varpi,s}}}{\sum_{i\in\mathfrak{I}}\sigma_i\omega_i}\right)$$
$$+\sum_{i\in\mathfrak{I}}\frac{\eta_i}{(i+\varpi)^s Q_{n,\varpi,s}}\log\left(\frac{\sum_{i\in\mathfrak{I}}\frac{\eta_i}{(i+\varpi)^s Q_{n,\varpi,s}}}{\sum_{i\in\mathfrak{I}}\sigma_i\eta_i}\right) + \sum_{i\in\bar{\mathfrak{I}}}\frac{\xi_i}{(i+\varpi)^s Q_{n,\varpi,s}}\log\left(\frac{\sum_{i\in\bar{\mathfrak{I}}}\frac{\xi_i}{(i+\varpi)^s Q_{n,\varpi,s}}}{\sum_{i\in\bar{\mathfrak{I}}}\sigma_i\xi_i}\right)$$

$$+\sum_{i\in\bar{\mathfrak{I}}}\frac{\theta_i}{(i+\varpi)^sQ_{n,\varpi,s}}\log\left(\frac{\sum_{i\in\bar{\mathfrak{I}}}\frac{\theta_i}{(i+\varpi)^sQ_{n,\varpi,s}}}{\sum_{i\in\bar{\mathfrak{I}}}\sigma_i\theta_i}\right)\geq 0. \tag{3.19}$$

Proof. We have the following identity for $\upsilon_i=\frac{1}{(i+\varpi)^sQ_{n,\varpi,s}}$, $i\in\{1,2,\ldots,n\}$.

$$\begin{aligned}\sum_{i=1}^n\upsilon_i\log\frac{\upsilon_i}{\sigma_i}=&\sum_{i=1}^n\frac{1}{(i+\varpi)^sQ_{n,\varpi,s}}\log\frac{1}{(i+\varpi)^sQ_{n,\varpi,s}}-\sum_{i=1}^n\frac{\log\sigma_i}{(i+\varpi)^sQ_{n,\varpi,s}}\\
=&-\sum_{i=1}^n\frac{1}{(i+\varpi)^sQ_{n,\varpi,s}}\log\left((i+\varpi)^sQ_{n,\varpi,s}\right)-\sum_{i=1}^n\frac{\log\sigma_i}{(i+\varpi)^sQ_{n,\varpi,s}}\\
=&-\sum_{i=1}^n\frac{s\log(i+\varpi)}{(i+\varpi)^sQ_{n,\varpi,s}}-\sum_{i=1}^n\frac{\log Q_{n,\varpi,s}}{(i+\varpi)^sQ_{n,\varpi,s}}-\sum_{i=1}^n\frac{\log\sigma_i}{(i+\varpi)^sQ_{n,\varpi,s}}\\
=&-\frac{s}{Q_{n,\varpi,s}}\sum_{i=1}^n\frac{\log(i+\varpi)}{(i+\varpi)^s}-\frac{\log Q_{n,\varpi,s}}{Q_{n,\varpi,s}}\sum_{i=1}^n\frac{1}{(i+\varpi)^s}-\sum_{i=1}^n\frac{\log\sigma_i}{(i+\varpi)^sQ_{n,\varpi,s}}\\
=&-Z_{ME}(Q,\varpi,s)-\sum_{i=1}^n\frac{\log\sigma_i}{(i+\varpi)^sQ_{n,\varpi,s}},\end{aligned}$$

where $Q_{n,\varpi,s}=\sum_{i=1}^n\frac{1}{(i+\varpi)^s}$, and $\sum_{i=1}^n\frac{1}{(i+\varpi)^sQ_{n,\varpi,s}}=1$. Therefore, utilizing (3.5) for $\upsilon_i=\frac{1}{(i+\varpi)^sQ_{n,\varpi,s}}$, $i=1,2,\ldots,n$, we obtain (3.19). □

Corollary 3.30. *The following inequalities hold by taking the assumptions of Corollary 3.29:*

$$\begin{aligned}&-Z_{ME}(Q,\varpi,s)-\sum_{i=1}^n\frac{\log\sigma_i}{(i+\varpi)^sQ_{n,\varpi,s}}\geq\\
&\max_{k\in\{1,2,\ldots,n\}}\left\{\left(\sum_{i=1}^n\frac{\omega_i}{(i+\varpi)^sQ_{n,\varpi,s}}-\frac{\omega_k}{(k+\varpi)^sQ_{n,\varpi,s}}\right)\log\left(\frac{\sum_{i=1}^n\frac{\omega_i}{(i+\varpi)^sQ_{n,\varpi,s}}-\frac{\omega_k}{(k+\varpi)^sQ_{n,\varpi,s}}}{\sum_{i=1}^n\sigma_i\omega_i-\sigma_k\omega_k}\right)\right.\\
&+\left(\sum_{i=1}^n\frac{\eta_i}{(i+\varpi)^sQ_{n,\varpi,s}}-\frac{\eta_k}{(k+\varpi)^sQ_{n,\varpi,s}}\right)\log\left(\frac{\sum_{i=1}^n\frac{\eta_i}{(i+\varpi)^sQ_{n,\varpi,s}}-\frac{\eta_k}{(k+\varpi)^sQ_{n,\varpi,s}}}{\sum_{i=1}^n\sigma_i\eta_i-\sigma_k\eta_k}\right)\\
&\left.+\frac{(\xi_k+\theta_k)}{(k+\varpi)^sQ_{n,\varpi,s}}\log\left(\frac{1}{\sigma_k.(k+\varpi)^sQ_{n,\varpi,s}}\right)\right\}\geq 0.\end{aligned} \tag{3.20}$$

Proof. Taking $\bar{\mathfrak{I}}=\{k\}$, $k\in\{1,2,\ldots,n\}$ in (3.19), we get (3.20). □

Remark 3.31. By using Remark 3.10, we also have

$$\begin{aligned}&\max_{\phi\neq\mathfrak{I}\subset\{1,2,\ldots,n\}}\left\{\sum_{i\in\mathfrak{I}}\frac{\omega_i}{(i+\varpi)^sQ_{n,\varpi,s}}\log\left(\frac{\sum_{i\in\mathfrak{I}}\frac{\omega_i}{(i+\varpi)^sQ_{n,\varpi,s}}}{\sum_{i\in\mathfrak{I}}\sigma_i\omega_i}\right)\right.\\
&+\sum_{i\in\mathfrak{I}}\frac{\eta_i}{(i+\varpi)^sQ_{n,\varpi,s}}\log\left(\frac{\sum_{i\in\mathfrak{I}}\frac{\eta_i}{(i+\varpi)^sQ_{n,\varpi,s}}}{\sum_{i\in\mathfrak{I}}\sigma_i\eta_i}\right)+\sum_{i\in\bar{\mathfrak{I}}}\frac{\xi_i}{(i+\varpi)^sQ_{n,\varpi,s}}\log\left(\frac{\sum_{i\in\bar{\mathfrak{I}}}\frac{\xi_i}{(i+\varpi)^sQ_{n,\varpi,s}}}{\sum_{i\in\bar{\mathfrak{I}}}\sigma_i\xi_i}\right)\end{aligned}$$

$$+\sum_{i\in\bar{\mathfrak{I}}}\frac{\theta_i}{(i+\varpi)^sQ_{n,\varpi,s}}\log\left(\frac{\sum_{i\in\bar{\mathfrak{I}}}\frac{\theta_i}{(i+\varpi)^sQ_{n,\varpi,s}}}{\sum_{i\in\bar{\mathfrak{I}}}\sigma_i\theta_i}\right)\Bigg\}$$

$$\geq\max_{k\in\{1,2,\dots,n\}}\Bigg\{\left(\sum_{i=1}^n\frac{\omega_i}{(i+\varpi)^sQ_{n,\varpi,s}}-\frac{\omega_k}{(k+\varpi)^sQ_{n,\varpi,s}}\right)$$

$$\times\log\left(\frac{\sum_{i=1}^n\frac{\omega_i}{(i+\varpi)^sQ_{n,\varpi,s}}-\frac{\omega_k}{(k+\varpi)^sQ_{n,\varpi,s}}}{\sum_{i=1}^n\sigma_i\omega_i-\sigma_k\omega_k}\right)$$

$$+\left(\sum_{i=1}^n\frac{\eta_i}{(i+\varpi)^sQ_{n,\varpi,s}}-\frac{\eta_k}{(k+\varpi)^sQ_{n,\varpi,s}}\right)\log\left(\frac{\sum_{i=1}^n\frac{\eta_i}{(i+\varpi)^sQ_{n,\varpi,s}}-\frac{\eta_k}{(k+\varpi)^sQ_{n,\varpi,s}}}{\sum_{i=1}^n\sigma_i\eta_i-\sigma_k\eta_k}\right)$$

$$+\frac{(\xi_k+\theta_k)}{(k+\varpi)^sQ_{n,\varpi,s}}\log\left(\frac{1}{\sigma_k.(k+\varpi)^sQ_{n,\varpi,s}}\right)\Bigg\}. \tag{3.21}$$

The Zipf-Mandelbrot entropy is also estimated through Zipf's law for different parameters as follows:

Corollary 3.32. *Let $\varpi_1,\varpi_2\geq 0$, $s_1,s_2>0$. Also, let $\boldsymbol{\eta}=(\eta_1,\eta_2,\dots,\eta_n)$, $\boldsymbol{\omega}=(\omega_1,\omega_2,\dots,\omega_n)$, $\boldsymbol{\xi}=(\xi_1,\xi_2,\dots,\xi_n)$, $\boldsymbol{\theta}=(\theta_1,\theta_2,\dots,\theta_n)$ be positive tuples with the restrictions that $\xi_i+\theta_i=1$, $\omega_i+\eta_i=1$ for all $i\in\{1,2,\dots,n\}$, then*

$$-Z_{ME}(Q,\varpi_1,s_1)+\sum_{i=1}^n\frac{\log((i+\varpi_2)^{s_2}Q_{n,\varpi_2,s_2})}{(i+\varpi_1)^{s_1}Q_{n,\varpi_1,s_1}}$$

$$\geq\sum_{i\in\mathfrak{I}}\frac{\omega_i}{(i+\varpi_1)^{s_1}Q_{n,\varpi_1,s_1}}\log\left(\frac{\sum_{i\in\mathfrak{I}}\frac{\omega_i}{(i+\varpi_1)^{s_1}Q_{n,\varpi_1,s_1}}}{\sum_{i\in\mathfrak{I}}\frac{\omega_i}{(i+\varpi_2)^{s_2}Q_{n,\varpi_2,s_2}}}\right)$$

$$+\sum_{i\in\mathfrak{I}}\frac{\eta_i}{(i+\varpi_1)^{s_1}Q_{n,\varpi_1,s_1}}\log\left(\frac{\sum_{i\in\mathfrak{I}}\frac{\eta_i}{(i+\varpi_1)^{s_1}Q_{n,\varpi_1,s_1}}}{\sum_{i\in\mathfrak{I}}\frac{\eta_i}{(i+\varpi_2)^{s_2}Q_{n,\varpi_2,s_2}}}\right)$$

$$+\sum_{i\in\bar{\mathfrak{I}}}\frac{\xi_i}{(i+\varpi_1)^{s_1}Q_{n,\varpi_1,s_1}}\log\left(\frac{\sum_{i\in\bar{\mathfrak{I}}}\frac{\xi_i}{(i+\varpi_1)^{s_1}Q_{n,\varpi_1,s_1}}}{\sum_{i\in\bar{\mathfrak{I}}}\frac{\xi_i}{(i+\varpi_2)^{s_2}Q_{n,\varpi_2,s_2}}}\right)$$

$$+\sum_{i\in\bar{\mathfrak{I}}}\frac{\theta_i}{(i+\varpi_1)^{s_1}Q_{n,\varpi_1,s_1}}\log\left(\frac{\sum_{i\in\bar{\mathfrak{I}}}\frac{\theta_i}{(i+\varpi_1)^{s_1}Q_{n,\varpi_1,s_1}}}{\sum_{i\in\bar{\mathfrak{I}}}\frac{\theta_i}{(i+\varpi_2)^{s_2}Q_{n,\varpi_2,s_2}}}\right)$$

$$\geq 0. \tag{3.22}$$

Proof. Suppose that we have $\upsilon_i=\frac{1}{(i+\varpi_1)^{s_1}Q_{n,\varpi_1,s_1}}$ and $\sigma_i=\frac{1}{(i+\varpi_2)^{s_2}Q_{n,\varpi_2,s_2}}$ for $i=1,2,\dots,n$, then analogously in the proof of Corollary 3.29, we get

$$\sum_{i=1}^n\upsilon_i\log\upsilon_i=\sum_{i=1}^n\frac{1}{(i+\varpi_1)^{s_1}Q_{n,\varpi_1,s_1}}\log\frac{1}{(i+\varpi_1)^{s_1}Q_{n,\varpi_1,s_1}}=-Z_{ME}(Q,\varpi_1,s_1),$$

and

$$\sum_{i=1}^{n} \upsilon_i \log \sigma_i = \sum_{i=1}^{n} \frac{1}{(i+\varpi_1)^{s_1} Q_{n,\varpi_1,s_1}} \log \frac{1}{(i+\varpi_2)^{s_2} Q_{n,\varpi_2,s_2}}$$
$$= -\sum_{i=1}^{n} \frac{\log((i+\varpi_2)^{s_2} Q_{n,\varpi_2,s_2})}{(i+\varpi_1)^{s_1} Q_{n,\varpi_1,s_1}},$$

where $\sum_{i=1}^{n} \upsilon_i = \sum_{i=1}^{n} \frac{1}{(i+\varpi_1)^{s_1} Q_{n,\varpi_1,s_1}} = 1$ and $\sum_{i=1}^{n} \sigma_i = \sum_{i=1}^{n} \frac{1}{(i+\varpi_2)^{s_2} Q_{n,\varpi_2,s_2}} = 1$. Therefore, utilizing (3.5) for $\upsilon_i = \frac{1}{(i+\varpi_1)^{s_1} Q_{n,\varpi_1,s_1}}$ and $\sigma_i = \frac{1}{(i+\varpi_2)^{s_2} Q_{n,\varpi_2,s_2}}$, $i = 1, 2, \ldots, n$, we obtain (3.22). □

Corollary 3.33. *The following inequalities hold by letting the assumptions of Corollary 3.32:*

$$-Z_{ME}(Q, \varpi_1, s_1) + \sum_{i=1}^{n} \frac{\log((i+\varpi_2)^{s_2} Q_{n,\varpi_2,s_2})}{(i+\varpi_1)^{s_1} Q_{n,\varpi_1,s_1}}$$
$$\geq \max_{k\in\{1,2,\ldots,n\}} \Bigg\{ \frac{(\xi_k+\theta_k)}{(k+\varpi_1)^{s_1} Q_{n,\varpi_1,s_1}} \log \left(\frac{(k+\varpi_2)^{s_2} Q_{n,\varpi_2,s_2}}{(k+\varpi_1)^{s_1} Q_{n,\varpi_1,s_1}} \right)$$
$$+ \left(\sum_{i=1}^{n} \frac{\omega_i}{(i+\varpi_1)^{s_1} Q_{n,\varpi_1,s_1}} - \frac{\omega_k}{(k+\varpi_1)^{s_1} Q_{n,\varpi_1,s_1}} \right) \tag{3.23}$$
$$ă \times \log \left(\frac{\sum_{i=1}^{n} \frac{\omega_i}{(i+\varpi_1)^{s_1} Q_{n,\varpi_1,s_1}} - \frac{\omega_k}{(k+\varpi_1)^{s_1} Q_{n,\varpi_1,s_1}}}{\sum_{i=1}^{n} \frac{\omega_i}{(i+\varpi_2)^{s_2} Q_{n,\varpi_2,s_2}} - \frac{\omega_k}{(k+\varpi_2)^{s_2} Q_{n,\varpi_2,s_2}}} \right)$$
$$+ \left(\sum_{i=1}^{n} \frac{\eta_i}{(i+\varpi_1)^{s_1} Q_{n,\varpi_1,s_1}} - \frac{\eta_k}{(k+\varpi_1)^{s_1} Q_{n,\varpi_1,s_1}} \right)$$
$$\times \log \left(\frac{\sum_{i=1}^{n} \frac{\eta_i}{(i+\varpi_1)^{s_1} Q_{n,\varpi_1,s_1}} - \frac{\eta_k}{(k+\varpi_1)^{s_1} Q_{n,\varpi_1,s_1}}}{\sum_{i=1}^{n} \frac{\eta_i}{(i+\varpi_2)^{s_2} Q_{n,\varpi_2,s_2}} - \frac{\eta_k}{(k+\varpi_2)^{s_2} Q_{n,\varpi_2,s_2}}} \right) \Bigg\}$$
$$\geq 0. \tag{3.24}$$

Proof. Taking $\bar{\mathfrak{I}} = \{k\}$, $k \in \{1, 2, \ldots, n\}$ in (3.22), we obtain (3.24). □

4. Further Generalization

Theorem 4.1. *Suppose that f is a real valued convex function defined on $\mathcal{G}$. Also, let $s_i \in \mathcal{G}, \mu_i \geq 0, (i = 1, 2, \ldots, n)$ and u_ℓ^r be some positive tuples for $\ell = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, s$ with and $\mathcal{H} := \sum_{i=1}^{n} \mu_i$, $\sum_{\ell=1}^{m} u_\ell^r = 1$, for each r. Suppose that $L_1, L_2, \ldots, L_s$ are some nonempty subsets of $\{1, 2, \ldots, m\}$ with the condition that $L_k \cap L_t = \emptyset$ for different values of k and t while $\cup_{r=1}^{s} L_r = \{1, 2, \ldots, m\}$. Furthermore if $\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_s$ are some nonempty subsets of $\{1, 2, \ldots, n\}$ with the condition that $\mathfrak{J}_k \cap \mathfrak{J}_t = \emptyset$ for different values of k and t while $\cup_{r=1}^{s} \mathfrak{J}_r = \{1, 2, \ldots, n\}$, then the following inequalities hold:*

$$\frac{1}{\mathcal{H}} \sum_{i=1}^{n} \mu_i f(s_i)$$
$$\geq \frac{1}{\mathcal{H}} \sum_{\mathfrak{J}_1} \sum_{\ell\in L_1} u_\ell^1 \mu_i f \left(\frac{\sum_{\mathfrak{J}_1} \sum_{\ell\in L_1} u_\ell^1 \mu_i s_i}{\sum_{\mathfrak{J}_1} \sum_{\ell\in L_1} u_\ell^1 \mu_i} \right) + \cdots + \frac{1}{\mathcal{H}} \sum_{\mathfrak{J}_1} \sum_{\ell\in L_s} u_\ell^1 \mu_i f \left(\frac{\sum_{\mathfrak{J}_1} \sum_{\ell\in L_s} u_\ell^1 \mu_i s_i}{\sum_{\mathfrak{J}_1} \sum_{\ell\in L_s} u_\ell^1 \mu_i} \right)$$

$$+\frac{1}{\mathcal{H}}\sum_{\mathfrak{J}_2}\sum_{\ell\in L_1}u_\ell^2\mu_i f\left(\frac{\sum_{\mathfrak{J}_2}\sum_{\ell\in L_1}u_\ell^2\mu_i s_i}{\sum_{\mathfrak{J}_2}\sum_{\ell\in L_1}u_\ell^2\mu_i}\right)+\cdots+\frac{1}{\mathcal{H}}\sum_{\mathfrak{J}_2}\sum_{\ell\in L_s}u_\ell^2\mu_i f\left(\frac{\sum_{\mathfrak{J}_2}\sum_{\ell\in L_s}u_\ell^2\mu_i s_i}{\sum_{\mathfrak{J}_2}\sum_{\ell\in L_s}u_\ell^2\mu_i}\right)$$

$$+$$

$$\vdots$$

$$+\frac{1}{\mathcal{H}}\sum_{\mathfrak{J}_s}\sum_{\ell\in L_1}u_\ell^s\mu_i f\left(\frac{\sum_{\mathfrak{J}_s}\sum_{\ell\in L_1}u_\ell^s\mu_i s_i}{\sum_{\mathfrak{J}_s}\sum_{\ell\in L_1}u_\ell^s\mu_i}\right)+\cdots+\frac{1}{\mathcal{H}}\sum_{\mathfrak{J}_s}\sum_{\ell\in L_s}u_\ell^s\mu_i f\left(\frac{\sum_{\mathfrak{J}_s}\sum_{\ell\in L_s}u_\ell^s\mu_i s_i}{\sum_{\mathfrak{J}_s}\sum_{\ell\in L_s}u_\ell^s\mu_i}\right)$$

$$\geq f\left(\frac{1}{\mathcal{H}}\sum_{i=1}^{n}\mu_i s_i\right). \tag{4.1}$$

The direction of inequalities reverses in (4.1) for f as a concave function.

Proof. Since it is given that $\sum_{\ell=1}^{m}u_\ell^r=\sum_{\ell\in\cup_{r=1}^{s}L_r}u_\ell^r=1$ for each $r=1,2,\ldots,s$, therefore for the subsets $\mathfrak{J}_r$ of $\{1,2,\ldots,n\}$, one has

$$\begin{aligned}
&\sum_{i=1}^{n}\mu_i f(s_i)\\
=&\sum_{\mathfrak{J}_1}\sum_{\ell\in\cup_{r=1}^{s}L_r}u_\ell^1\mu_i f(s_i)+\sum_{\mathfrak{J}_2}\sum_{\ell\in\cup_{r=1}^{s}L_r}u_\ell^2\mu_i f(s_i)+\ldots+\sum_{\mathfrak{J}_s}\sum_{\ell\in\cup_{r=1}^{s}L_r}u_\ell^s\mu_i f(s_i)\\
=&\sum_{\mathfrak{J}_1}\sum_{\ell\in L_1}u_\ell^1\mu_i f(s_i)+\cdots+\sum_{\mathfrak{J}_1}\sum_{\ell\in L_s}u_\ell^1\mu_i f(s_i)\\
&+\sum_{\mathfrak{J}_2}\sum_{\ell\in L_1}u_\ell^2\mu_i f(s_i)+\cdots+\sum_{\mathfrak{J}_2}\sum_{\ell\in L_s}u_\ell^2\mu_i f(s_i)\\
&+\\
&\vdots\\
&+\sum_{\mathfrak{J}_s}\sum_{\ell\in L_1}u_\ell^s\mu_i f(s_i)+\cdots+\sum_{\mathfrak{J}_s}\sum_{\ell\in L_s}u_\ell^s\mu_i f(s_i).
\end{aligned}\tag{4.2}$$

If we use the integral Jensen inequality in the terms of right hand side of (4.2), then we get the following result

$$\begin{aligned}
&\frac{1}{\mathcal{H}}\sum_{i=1}^{n}\mu_i f(s_i)\\
\geq&\frac{1}{\mathcal{H}}\Bigg(\sum_{\mathfrak{J}_1}\sum_{\ell\in L_1}u_\ell^1\mu_i f\left(\frac{\sum_{\mathfrak{J}_1}\sum_{\ell\in L_1}u_\ell^1\mu_i s_i}{\sum_{\mathfrak{J}_1}\sum_{\ell\in L_1}u_\ell^1\mu_i}\right)+\cdots+\sum_{\mathfrak{J}_1}\sum_{\ell\in L_s}u_\ell^1\mu_i f\left(\frac{\sum_{\mathfrak{J}_1}\sum_{\ell\in L_s}u_\ell^1\mu_i s_i}{\sum_{\mathfrak{J}_1}\sum_{\ell\in L_s}u_\ell^1\mu_i}\right)\\
&+\sum_{\mathfrak{J}_2}\sum_{\ell\in L_1}u_\ell^2\mu_i f\left(\frac{\sum_{\mathfrak{J}_2}\sum_{\ell\in L_1}u_\ell^2\mu_i s_i}{\sum_{\mathfrak{J}_2}\sum_{\ell\in L_1}u_\ell^2\mu_i}\right)+\cdots+\sum_{\mathfrak{J}_2}\sum_{\ell\in L_s}u_\ell^2\mu_i f\left(\frac{\sum_{\mathfrak{J}_2}\sum_{\ell\in L_s}u_\ell^2\mu_i s_i}{\sum_{\mathfrak{J}_2}\sum_{\ell\in L_s}u_\ell^2\mu_i}\right)\\
&+\\
&\vdots\\
&+\sum_{\mathfrak{J}_s}\sum_{\ell\in L_1}u_\ell^s\mu_i f\left(\frac{\sum_{\mathfrak{J}_s}\sum_{\ell\in L_1}u_\ell^s\mu_i s_i}{\sum_{\mathfrak{J}_s}\sum_{\ell\in L_1}u_\ell^s\mu_i}\right)+\cdots+\sum_{\mathfrak{J}_s}\sum_{\ell\in L_s}u_\ell^s\mu_i f\left(\frac{\sum_{\mathfrak{J}_s}\sum_{\ell\in L_s}u_\ell^s\mu_i s_i}{\sum_{\mathfrak{J}_s}\sum_{\ell\in L_s}u_\ell^s\mu_i}\right)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
\geq & f\Bigg(\frac{1}{\mathcal{H}}\sum_{\mathcal{J}_1}\sum_{\ell\in L_1}u_\ell^1\mu_i s_i+\cdots+\frac{1}{\mathcal{H}}\sum_{\mathcal{J}_1}\sum_{\ell\in L_s}u_\ell^1\mu_i s_i+\frac{1}{\mathcal{H}}\sum_{\mathcal{J}_2}\sum_{\ell\in L_1}u_\ell^2\mu_i s_i\\
&+\cdots+\frac{1}{\mathcal{H}}\sum_{\mathcal{J}_2}\sum_{\ell\in L_s}u_\ell^2\mu_i s_i+\cdots+\frac{1}{\mathcal{H}}\sum_{\mathcal{J}_s}\sum_{\ell\in L_1}u_\ell^s\mu_i s_i+\cdots+\frac{1}{\mathcal{H}}\sum_{\mathcal{J}_s}\sum_{\ell\in L_s}u_\ell^s\mu_i s_i\Bigg)\\
= & f\left(\frac{1}{\mathcal{H}}\sum_{i=1}^{n}\mu_i s_i\right),
\end{aligned}
$$

which gives the result (4.1). □

Remark 4.2. Taking $m = s = 2$ in Theorem 4.1, one may obtain Theorem 2.1. Similar applications of Theorem 4.1 can be acquired as acquired for Theorem 2.1 in the previous sections.

5. Conclusion

Being a part of modern applied analysis, Jensen's inequality has been proved to be very useful tool for the solution of different problems in various areas of science, art and technology. From 1906, occasionally a lot of mathematicians tried to refine, generalize, improve or extend this inequality. In this flow, based on some suitable and real sequences we have obtained a new refinement of this inequality. Then various interesting inequalities for different means are also obtained. The proposed refinement also enabled us to acquire inequalities for the class of Csiszâr Ψ- divergence. This refinement is further generalized through several finite real sequences. The idea further motivates the mathematicians to establish such results in the future.

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