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Some Spectrum Properties in C*- Algebras

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Abstract

In this study, we show that a * - homomorphism $\varphi : A \to B$ between unital commutative C*- algebras A and B with $A^{-1} = \varphi^{-1}(B^{-1})$ satisfies the property to preserve spectrum and adjoint mapping $\varphi^* : \Delta(B) \to \Delta(A)$ is surjective, that is, φ^* maps maximal ideal space of B to maximal ideal space of A.

Keywords: C*- algebra, Gelfand transform, maximal ideal, spectrum, complex homomorphism

C*- Cebirlerinde Bazı Spektrum Özellikleri

Özet

Bu çalışmada, birimli değişmeli A ve B C*- cebirleri arasında tanımlı $A^{-1} = \varphi^{-1}(B^{-1})$ şartını sağlayan bir φ * - homomorfizminin spektrumu koruma özelliğini sağladığı ve $\varphi^* : \Delta(B) \to \Delta(A)$ adjoint dönüşümünün örten olduğu yani B cebirinin maksimal idealler uzayını A cebirinin maksimal idealler uzayına dönüştürdüğü gösterildi.

Anahtar Kelimeler: C*- cebiri, Gelfand dönüşümü, maksimal ideal, spektrum, kompleks homomorfizm

1. Introduction

There are many studies on invertible elements of C*- algebras and the property to preserve spectrum of a homomorphism between C*- algebras. The related studies can be found in references as [1-3]. In this paper, the relation between the property preserve spectrum of a homomorphism from one C*- algebra to another, invertible elements of these C*- algebras and the mapping of their maximal ideals is examined.

In this section, basic definitions and properties related to C^*- algebras will be given.

Let A be a complex algebra. An involution on A is a mapping $* : x \to x^*$ from A into A satisfying the following conditions.

i. $(x+y)^* = x^* + y^*$,
ii. $(\lambda x)^* = \bar{\lambda} x^*$,
iii. $(xy)^* = y^* x^*$,
iv. $(x^*)^* = x$

for all $x, y \in A$ and $\lambda \in \mathbb{C}$. Then A is called a * - algebra or an algebra with involution.

If * - algebra A is a Banach algebra and involution on it is isometric; that is, $\|x^*\| = \|x\|$ for all $x \in A$, then A is called a Banach * - algebra.

If * - algebra A is a Banach algebra and its norm satisfies the equation $\|x^* x\| = \|x\|^2$ for all $x \in A$, then A is said to be a C*- algebra. [4]

Let A and B be C*- algebras, $\varphi : A \to B$ be a mapping. If φ satisfies the following conditions for all $x, y \in A$ and $\lambda \in \mathbb{C}$, then this mapping is called a * - homomorphism.

i. $\varphi(x+y) = \varphi(x) + \varphi(y)$,
ii. $\varphi(\lambda x) = \lambda \varphi(x)$,
iii. $\varphi(xy) = \varphi(x)\varphi(y)$,
iv. $\varphi(x^*) = \varphi(x)^*$.

It is said to be a * - isomorphism if a * - homomorphism φ is a bijection. [5]

If A is a unital Banach algebra, then the set $\{\lambda \in \mathbb{C} : (x - \lambda 1_A) \notin A^{-1}\}$ is called spectrum of x in A, denoted by $\sigma_A(x)$, where A^{-1} denotes the set of invertible elements of A. $\sigma_A(x)$ is a nonempty compact subset of $\mathbb{C}$ for every x in A. The resolvent set of x is defined by $\rho_A(x) = \mathbb{C} \setminus \sigma_A(x)$.

The spectral radius of x is characterized by $r_A(x) = \sup\{ |\lambda| : \lambda \in \sigma_A(x)\}$.
If A is a unital commutative Banach algebra, then for every x in A, the limit

$$r_A(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$$

exists and $r_A(x) \le \|x\|$. Also for every $x, y \in A$, $r_A(x+y) \le r_A(x) + r_A(y)$ and $r_A(xy) \le r_A(x) r_A(y)$.

When A is a commutative complex algebra with unit, every proper ideal of A is contained in a maximal ideal of A and every maximal ideal of A is closed. The set of all maximal ideals in A is denoted by $M(A)$.

Let A is a complex algebra and ϕ is a linear functional on A. If $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$, then ϕ is called a complex homomorphism on A. The set of nonzero complex homomorphisms on A is denoted by $\Delta(A)$.

For $x \in A$, $\hat{x} : \Delta(A) \to \mathbb{C}$, Gelfand transform of x, is defined by $\hat{x}(h) = h(x)$ for every h in $\Delta(A)$. The set $\hat{A} = \{\hat{a} : a \in A\}$ is called the set of Gelfand transforms on A. [6]

The ε-open neighbourhood $U_\varepsilon(h_0, a_1, \dots, a_n)$ at any $h_0 \in \Delta(A)$ with respect to the Gelfand topology is given

$$\{h \in \Delta(A) : |\hat{a}_i(h_0) - \hat{a}_i(h)| < \varepsilon\}$$

where $\varepsilon > 0$, $n \in \mathbb{N}$ and $a_1, \dots, a_n$ are arbitrary elements of A. [7]

The following is true when A is a unital commutative Banach algebra.

i. Every maximal ideal of A is the kernel of some $h \in \Delta(A)$.

ii. If $h \in \Delta(A)$, then the kernel of h is a maximal ideal of A.

iii. An element $x \in A$ is invertible in A if and only if $h(x) \neq 0$ for every $h \in \Delta(A)$.

iv. $\lambda \in \sigma(x)$ if and only if $h(x) = \lambda$ for some $h \in \Delta(A)$. [6]

2. Spectrum Properties in C*- Algebras

In this section, it will be obtained that under what conditions equality $\sigma_A(x) = \sigma_B\big(\varphi(x)\big)$ for any $x \in A$, the property to preserve spectrum of $\varphi : A \to B$, will be satisfied when A and B are unital commutative C^*- algebras and φ is a * - homomorphism from A to B.

Proposition 2.1. Let A and B be unital commutative C^*- algebras, φ be a * - homomorphism from A to B and $\varphi(1_A) = 1_B$. Then for every $x \in A$, $\sigma_B(\varphi(x)) \subset \sigma_A(x)$. [5]

Theorem 2.2. Let A and B be unital commutative C^*- algebras, $\varphi : A \to B$ be a * - homomorphism with $\varphi(1_A) = 1_B$. Then $\varphi(x) \in B^{-1}$ for any $x \in A^{-1}$.

Proof. $0 \notin \sigma_A(x)$ for an arbitrary $x \in A^{-1}$. $0 \notin \sigma_B(\varphi(x))$ follows from Proposition 2.1 and this proves $\varphi(x) \in B^{-1}$.

Corollary 2.3. $A^{-1} \subset \varphi^{-1}(B^{-1})$.

Theorem 2.4. Let A and B be unital commutative C^*- algebras, $\varphi : A \to B$ be a * - homomorphism with $\varphi(1_A) = 1_B$. In this case, $A^{-1} = \varphi^{-1}(B^{-1})$ if and only if $0 \notin \sigma_A(x)$ whenever $0 \notin \sigma_B(\varphi(x))$ for any $x \in A$.

Proof. Let $A^{-1} = \varphi^{-1}(B^{-1})$. Suppose that $0 \notin \sigma_B(\varphi(x))$ for an arbitrary $x \in A$. In this case, $\varphi(x) \in B^{-1}$ and hence $x \in A^{-1}$ so that $0 \notin \sigma_A(x)$ for every $x \in A$.
Conversely, assume that $0 \notin \sigma_A(x)$ whenever $0 \notin \sigma_B(\varphi(x))$ for any $x \in A$. Since $0 \notin \sigma_B(\varphi(a))$ for any $a \in \varphi^{-1}(B^{-1})$, $a \in A^{-1}$ by hypothesis and hence $\varphi^{-1}(B^{-1}) \subset A^{-1}$. According to Corollary 2.3, $A^{-1} = \varphi^{-1}(B^{-1})$.

Corollary 2.5. Let A and B be unital commutative C^*- algebras, $\varphi : A \to B$ be a * - homomorphism with $\varphi(1_A) = 1_B$. Then $\sigma_A(x) = \sigma_B(\varphi(x))$ for every $x \in A$ if and only if $A^{-1} = \varphi^{-1}(B^{-1})$.

Proof. First, suppose that $\sigma_A(x) = \sigma_B(\varphi(x))$ for every $x \in A$. In that case, one says $0 \notin \sigma_B(\varphi(a))$ for any $a \in \varphi^{-1}(B^{-1})$. Hence $\sigma_A(a) = \sigma_B(\varphi(a))$ implies $0 \notin \sigma_A(a)$, that is, $a \in A^{-1}$. Then $\varphi^{-1}(B^{-1}) \subset A^{-1}$. Again, using Corollary 2.3, it follows that $A^{-1} = \varphi^{-1}(B^{-1})$.
Conversely, let $A^{-1} = \varphi^{-1}(B^{-1})$. Given any $\lambda \in \mathbb{C} - \sigma_B(\varphi(x))$, $\varphi(x - \lambda 1_A) \in B^{-1}$ for

any $x \in A$, that is, $x - \lambda 1_A \in \varphi^{-1}(B^{-1})$ for any $x \in A$ and hence it is clear that $\lambda \notin \sigma_A(x)$, since $x - \lambda 1_A \in A^{-1}$ by hypothesis. Thus, we have seen that $\sigma_A(x) \subset \sigma_B(\varphi(x))$ for every $x \in A$ and we obtain $\sigma_A(x) = \sigma_B(\varphi(x))$ for every $x \in A$ by Proposition 2.1.

Corollary 2.6. If $A^{-1} = \varphi^{-1}(B^{-1})$, then $r_A(x) = r_B(\varphi(x))$ for every $x \in A$.

3. Mapping of Maximal Ideals in C*- Algebras

Let φ be a * - homomorphism between unital commutative C*- algebras A and B and also A^* and B^* be algebraic duals of A and B, respectively. Surjectivity of $\varphi^* : \Delta(B) \to \Delta(A)$ which is obtained from $\varphi^* : B^* \to A^*$ means that φ^* maps $M(B)$ to $M(A)$. In this section, it will be obtained that under what conditions this property will be satisfied.

Theorem 3.1. Let A and B be unital commutative C*- algebras, $\varphi : A \to B$ be a * - homomorphism. Then $\varphi^* f$ is also a * - homomorphism for every $f \in \Delta(B)$.

Proof. For every $f \in \Delta(B)$ and $x, y \in A$,

$$\begin{aligned}(\varphi^* f)(xy) &= f\big(\varphi(xy)\big)\\ &= f\big(\varphi(x)\big) f\big(\varphi(y)\big)\\ &= (\varphi^* f)(x)(\varphi^* f)(y)\end{aligned}$$

and hence $\varphi^* f \in \Delta(A)$. Also, since

$$\begin{aligned}(\varphi^* f)(x^*) &= f\big(\varphi(x^*)\big)\\ &= \underline{f(\varphi(x)^*)}\\ &= \overline{f\big(\varphi(x)\big)}\\ &= \overline{(\varphi^* f)(x)}\end{aligned}$$

for every $f \in \Delta(B)$ and $x \in A$, it is clear that $\varphi^* f$ is a * - homomorphism.

Corollary 3.2. Let A and B be unital commutative C*- algebras, $\varphi : A \to B$ be a * - homomorphism. Then $\varphi^* \Delta(B) \subset \Delta(A)$.

Theorem 3.3. Let A and B be unital commutative C*- algebras, $\varphi : A \to B$ be a * - homomorphism with $\varphi(1_A) = 1_B$. In that case, $A^{-1} = \varphi^{-1}(B^{-1})$ if and only if $\varphi^* \Delta(B) = \Delta(A)$.

Proof. Let $A^{-1} = \varphi^{-1}(B^{-1})$. Then for every $g \in \Delta(A)$, there exists $I \in M(A)$ such that $Kerg = I$. If we denote by J_0 the smallest ideal of B containing $\varphi(I)$, then $J_0 = B$ or $J_0 \neq B$.
If $J_0 \neq B$, then there exists $J \in M(B)$ such that $J_0 \subset J$ and also $f \in \Delta(B)$ such that $Kerf = J$. Since $I \in M(A)$ and $A/_I \cong \mathbb{C}$, there exists $\lambda \in \mathbb{C}$ and $t \in I$ such that $a = \lambda.1 + t$ for any $a \in A$. Therefore,

$$(\varphi^* f)(a) = (\varphi^* f)(\lambda.1 + t) = \lambda + f\big(\varphi(t)\big).$$

Again for $t \in I$, $\varphi(t) \in Kerf$ and hence $(\varphi^* f)(a) = \lambda$. Thus, we can write $a = (\varphi^* f)(a).1 + t$.
Using the fact that $t \in I = Kerg$, $g(a) = (\varphi^* f)(a)$. Then it is easily seen that $g = \varphi^* f \in \varphi^* \Delta(B)$ and obtained that $\Delta(A) \subset \varphi^* \Delta(B)$.
If J_0 were all of B, then there would be $b_1, b_2, \dots, b_n \in B$ and $a_1, a_2, \dots, a_n \in I$ such that

$$\sum_{i=1}^{n} b_i . \varphi(a_i) = 1.$$

Since $b_i . \|a_i\| \in B$ and $\frac{a_i}{\|a_i\|} \in I$, we can assume that $\|a_i\| = 1$ for each $i = 1,2, \dots, n$.
Let

$$\max_{1 \le i \le n} \|b_i\| = M$$

and a neighbourhood U at $g \in \Delta(A)$ with respect to the Gelfand topology for $0 < \varepsilon < 1$ be

$$\left\{h \in \Delta(A) : |\hat{a}_i(h) - \hat{a}_i(g)| < \frac{\varepsilon}{M.n}, 1 \le i \le n\right\}.$$

Then, since $a_i \in I = Kerg$ for each $i = 1,2, \dots, n$,

$$U = \left\{ h \in \Delta(A) : |\hat{a}_i(h)| < \frac{\varepsilon}{M.n}, 1 \le i \le n \right\}$$

As A is regular, there is a $m \in A$ such that

$$\widehat{m}(h) = \begin{cases} 1 & , h = g \\ 0 & , h \in \Delta(A) - U \\ \le 1 & , \text{otherwise} \end{cases}.$$

Thus for any $k \in \Delta(A)$,

$$\begin{aligned}\left|(a_i . m)^\wedge(k)\right| &= |\hat{a}_i(k) . \widehat{m}(k)|\\ &= |\hat{a}_i(k)| . |\widehat{m}(k)|\end{aligned}$$

$$< \frac{\varepsilon}{M.n}$$

and hence

$$\sup\{ \left|(a_i.m)^{\wedge}(k)\right| : k \in \Delta(A)\} < \frac{\varepsilon}{M.n}.$$

Also

$$\begin{aligned} r(a_i m) &= \sup\{ |k(a_i.m)| : k \in \Delta(A)\} \\ &= \sup\{ \left|(a_i.m)^{\wedge}(k)\right| : k \in \Delta(A)\} \end{aligned}$$

implies

$$r(a_i m) < \frac{\varepsilon}{M.n}.$$

On the other hand, if we remember

$$\sum_{i=1}^{n} b_i.\varphi(a_i) = 1,$$

then it is clear that

$$\begin{aligned} \varphi(m) &= \varphi(m).\sum_{i=1}^{n} b_i.\varphi(a_i) \\ &= \sum_{i=1}^{n} b_i.\varphi(a_i)\,.\varphi(m) \\ &= \sum_{i=1}^{n} b_i.\varphi(a_i m). \end{aligned}$$

Then,

$$\begin{aligned} r(\varphi(m)) &= r\left(\sum_{i=1}^{n} b_i.\varphi(a_i m)\right) \\ &\le \sum_{i=1}^{n} r(b_i) r(\varphi(a_i m)) \\ &\le \sum_{i=1}^{n} \|b_i\|.r(\varphi(a_i m)) \\ &\le M.\sum_{i=1}^{n} \frac{\varepsilon}{M.n} \\ &= \varepsilon \end{aligned}$$

and hence $r(\varphi(m)) < \varepsilon < 1$. Moreover, since

$$\begin{aligned} r(m) &= \sup\{ |k(m)| : k \in \Delta(A)\} \\ &= \sup\{ |\widehat{m}(k)| : k \in \Delta(A)\} \\ &= 1, \end{aligned}$$

$r(\varphi(m)) = 1$ by hypothesis, which is a contradiction. This contradiction shows that $J_0 \neq B$. Consequently, $\varphi^*\Delta(B) = \Delta(A)$ by Corollary 3.2.

Conversely, let $\varphi^*\Delta(B) = \Delta(A)$. For any point $x \in \varphi^{-1}(B^{-1})$ and $h \in \Delta(A)$, $\varphi(x) \in B^{-1}$ and there exists $u \in \Delta(B)$ such that $\varphi^* u = h$. Thus $u(\varphi(x)) \neq 0$, so that $\varphi^* u(x) = h(x) \neq 0$ and hence $x \in A^{-1}$. Thus it is obtained that $\varphi^{-1}(B^{-1}) \subset A^{-1}$ and $\varphi^{-1}(B^{-1}) = A^{-1}$ by Corollary 2.3.

Corollary 3.4. $A^{-1} = \varphi^{-1}(B^{-1})$ if and only if $\varphi^* M(B) = M(A)$.

4. References

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