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Kronecker Solutions for the Matrix Differantial Equations

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Abstract

In this paper, the linear matrix differantial equations which is a special case of matrix differantial equations has been formulated by the consepts of of the matrix differantial equations and Kronecker products and investigated by the Kronecker products. The formulation of the matrix differantial equation obtained by use of the linear matrix equations and Kronecker products have been applied to the matrix differantial equations and some important results have been found. It is shown that in solutions of the equation and its reduced case have emerged the importance of generalized inverse matrix and matrix functions.

Keywords: Matrix functions, Kronecker products, Matrix differantial equations.

Matris Diferansiyel Denklemler için Kronecker Çözümler

Özet

Bu çalışmada, matris diferansiyel denklemlerinin özel bir hali olan doğrusal matris diferansiyel denklemleri Kronecker çarpım ve matris diferansiyel denklemleri kavramalarıyla formüle edilmiş ve Kronecker çarpımlarla incelenmiştir. Kronecker çarpım ve doğrusal matris denklemler kullanılarak elde edilen doğrusal matris diferansiyel denklem formulasyonu matris diferansiyel denklemlere uygulanmış ve bazı önemli sonuçlar bulunmuştur. Denklemin ve onun indirgenmiş durumu genelleştirilmiş ters matris ve matris fonksiyonunun önemini ortay çıkarmıştır.

Anahtar Kelimeler: Matris Fonksiyonlar, Kronecker Çarpım, Matris Diferensiyel Denklemler.

1. Introduction

In this paper only algebra was taken into consideration. Which have been motivated largely by the fact that they enables us to treat linear matrix differantial equations and the linear matrix differantial equations as if they were vector equations of the Kronecker products of matrices.

The use of Kronecker products arises in a variety of other mathematical applications as well. A review of Kronecker product applications in linear system theory has been presented in [1,5,6,7].The Kronecker product has had a long history. Until 19th century, it has not been used sufficiently in any area of applied mathematics.

Statisticians now have at their disposal a large body of results concerning the Kronecker product and its uses in linear matrix calculus and linear matrix differantial calculus [7,9].

This paper focuses on a newly proposed generalization of Kronecker product [2], and outlines its utility with some algebraic properties for all solutions of the differantial matrix equations and its special cases. This paper investigates some algebraic results concerning the solutions of the linear matrix differantial equation

$$\frac{dx}{dt} = AXB + CXD, \quad X(t_0) = X_0 \tag{1.1}$$

Utulizing the spectral decomposotion and generalized inverses of a matrix and using the Kronecker product. A basic method is to express (1.1) in an equivalent vector form as follows

$$\frac{dx}{dt} = (B^T \otimes A + D^T \otimes C)x, \quad x(t_0) = x_0 \tag{1.2}$$

Which is a linear differantial equation with singular columns of x, $\otimes$ denotes Kronecker product and B^T is thetranspose of B. I shall use algebraic results derived to analyse the solutions of (1.1) and (1.2). I shall relationship amongst results and I shall argue that only one of them is viable.

2. Kronecker Products

Let A=$[a_{ij}]$ be an mxn matrix and B be pxq matrix. The mpxnq Kronecker product of A and B, A⊗B is defined as [2,3,4,9]

$$A \otimes B=\left[a_{ij}B\right]. \tag{2.1}$$

The mnxmn Kronecker sum of A and B, A⊕B, is defined as

$$A \oplus B = A \otimes I_n + I_m \otimes B, \tag{2.2}$$

where A and B are mxm and nxn matrices.

Theorem 2.1.

Let f be an analytic function and A be an nxn matrix. Then

$$f(A \otimes I_m) = f(A) \otimes I_m , \tag{2.3}$$

Theorem 2.2.

Let A be an mxm and B be an nxn matrix. Then

$$\exp(A \oplus B) = \exp(A) \otimes \exp(B) \tag{2.4}$$

where ⊕ and ⊗ are Kronecker sum and Kronevker Product, respectively.

Theorem 2.3.

Let A be an pxq and B be sxt, and D a qxs matrix. Then

$$vec(ADB) = (B^T \otimes A) . d \tag{2.5}$$

where vec(ADB) is ptx1 vector formed from the columns of ABD and d is qsx1 vector formed from the columns of D.

Now consider the derivative of a matrix A=$[a_{rv}]$ with respect to a scaler b to be

$$\frac{\partial A}{\partial B} = \left[\frac{\partial a_{rv}}{\partial b}\right] \tag{2.6}$$

$\frac{\partial A}{\partial B}$ is taken to be a partitioned matrix whose **ik**th partitio

$$\frac{\partial A}{\partial B} = \left[\frac{\partial A}{\partial b_{ik}}\right], \tag{2.7}$$

where B=$[b_{ik}]$ is a rectangular matrix.

3. Linear Matrix Equations

The linear matrix equation for the unknown matrix X such that

$$A X B = C, \tag{3.1}$$

where A is an mxn, B is a pxq, C is an mxq and X is an nxp matrix. We can view this as a linear equation in the form

$$(B^T \otimes A) x = C, \tag{3.2}$$

where

$$X^T = [X_{11}, X_{21}, \ldots, X_{n1}, \ldots, X_{1p}, X_{2p}, \ldots, X_{np}] \tag{3.3}$$

$$C^T = [C_{11}, C_{21}, \ldots, C_{m1}, \ldots, C_{1q}, C_{2q}, \ldots, C_{mq}] \tag{3.4}$$

and B^T is the transpose of B[8].

Theorem 3.1.

A necessary and sufficient condition for the equation AXB = C to have a solution is that

$$AA^+CB^+B = C \tag{3.5}$$

In which case the general solution is

$$X = A^+CB^+ + Y - A^+AYBB^+ \tag{3.6}$$

where Y is an arbitrary matrix.

4. Matrix Differential Equations

We shall be concerned with systems of first order linear differential equations of the form

$$\frac{dx}{dt} = A\,x + b\,, \quad , \quad x(t_0) = x_0 \tag{4.1}$$

where A is an nxn constant matrix and x(t) and b(t) are vector valued functions of the real variable t, and b(t) is continious in some interval containing t_0 . If b(t) = 0, Equation (4.1) becomes of the form

$$\frac{dx}{dt} = \mathrm{A}\, x\,, \quad x(t_0) = x_0 \qquad (4.2)$$

which called as homogeneous initial value problem[3,4,5].

Theorem 4.1.

The solution of (4.2) is

$$X = e^{A(t-t_0)} x_0 \qquad (4.3)$$

Theorem 4.2.

The solution of (4.1) is

$$X = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} X = e^{A(t-u)} \mathrm{b(u)du} \qquad (4.4)$$

We now consider the matrix differential equation

$$\frac{dx}{dt} = AXB + CXD\,, \quad x(t_0) = x_0 \qquad (4.5)$$

(4.5) can be expressed in usual form of linear systems of differential equations as follows

$$\frac{dx}{dt} = (B^T \otimes A + D^T \otimes C)x, \quad x(t_0) = x_0 \qquad (4.6)$$

where if $X = [x_{ij}(t)]$ then X is the mnx1 vector

$$X^T = [X_{11}, \dots, X_{m1}, \dots, X_{1n}, \dots, X_{mn}] \qquad (4.7)$$

formed from the columns of X.

A special case of (4.5), where B and C are identity matrices, has been extensively studied in literatüre. In this case we obtain the following differential equation

$$\frac{dx}{dt} = AX + XB\,, \quad x(t_0) = x_0 \qquad (4.8)$$

which can be expressed as

$$\frac{dx}{dt} = (B^T \oplus A)\mathrm{X}\,, \quad x(t_0) = x_0 \qquad (4.9)$$

(4.6) is a special case of the homogeneous initial value problem. The general solution of (4.6) is

$$X = \exp(Mt)\,.\,x_0 \qquad (4.10)$$

where

M = $B^T \otimes A + D^T \otimes C$ and x_0 = X(0).

Corollary 4.1.

The linear matrix differential equations

$$\frac{dx}{dt} = AXB\,, \quad x(t_0) = x_0 \qquad (4.11)$$

is equivalent to

$$\frac{dx}{dt} = (B^T \otimes A)\mathrm{X}\,, \quad x(t_0) = x_0 \qquad (4.12)$$

The general solutions of (4.11) and (4.12) recpectively, are

$$X = e^{A} x_0\, e^{Bt} \qquad (4.13)$$

and

$$X = e^{(B^T \otimes A)\mathrm{t}} x_0, \quad X(0) = x_0 \qquad (4.14)$$

5. Conclusion

Let us note that the some special types of linear matrix equations which have important linear equations theory and a class of matrix differential equations can be investigated by use of Kronecker product and Kronecker sum of matrices.

A basic method is to express (1.1) in an equivalent vector form (1.2) which is a linear differential equation with singular constant coefficient and use algebraic results on the Krocner products to analyse the solutions of (1.1) and (1.2). It is then shown that the solutions of (1.1) and (1.2) have common algebraic characterizations.

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