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A new filter in residuated lattices

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Abstract

In this paper, we introduce the notion of node elements and nodal filters of residuated lattices. We define nodal filter of a residuated lattice as a node of the set of all filters in residuated lattices and we study their properties. Next, we investigate the relationships with the other types of filters and special sets in residuated lattice. Finally we obtain a characterization of the nodal filters in terms of congruences.

Keywords: residuated lattice, node, (nodal, implicative, positive implicative, normal) filter.

1. Introduction and preliminaries

The concept of a commutative residuated lattice was firstly introduced by M. Ward and R. P. Dilworth as generalization of ideal lattices of rings [9]. The properties of a commutative residuated lattice were presented in [8]. Also, the lattice of filters of a commutative residuated lattice was investigated [7].

J. G. Varlet introduced the notion of node and nodal filter in semilattices [8]. J. Duda defined congruence generated filters and obtained some interesting results [3].

In this paper, we define the notion of node in a residuated lattice and we state relationship between this notion and dense elements, regular elements and nilpotent elements. By use of the node elements we introduce the nodal filters and define connected elements. Next, we show that the nodal filters satisfy the intersection property. By means of the notions of the upper bounds and lower bounds of the filters in residuated lattice A, the normal filters are defined. Also, we investigate the relationships between this notion and other filters in residuated lattices.

We recollect some definitions and result which will be used in the every time they are used.

Definition 1.1. [9] *A residuated lattice is an algebra* A = $(A,\wedge,\vee,*,\longrightarrow,0,1)$ *with four binary operations and two constant 0,1 such that:*
$(A,\wedge,\vee,0,1)$ *is a bounded lattice with 0 as smallest and 1 as greatest element,*

● $*$ *is commutative and associative, with 1 as neutral element, and*

● $x*y\leq z$ *if and only if* $x\leq y\longrightarrow z$*, for all x,y and z in A (residuation principle).*

For any element x of a residuated lattice and positive integer n, we denote: $x^{-}=x\longrightarrow 0$ and $x^n=\underbrace{x*\ldots*x}_{n\ times}$.

The following lemma also holds in any residuated lattice.

Lemma 1.2. [7] *In each residuated lattice A, the following relations hold for all x,y,z∈A,*

(1) $x\leq y\longrightarrow(x*y)$.

(2) $x\leq y$ *if and only if* $x\longrightarrow y=1$.

(3) *If* $x\leq y$*, then* $y\longrightarrow z\leq x\longrightarrow z$ *and* $z\longrightarrow x\leq z\longrightarrow y$.

(4) $y\leq(y\longrightarrow x)\longrightarrow x$.

(5) $y\longrightarrow x\leq(z\longrightarrow y)\longrightarrow(z\longrightarrow x)$.

(6) $x*y\leq x,y$*, hence* $x*y\leq x\wedge y$ *and* $x*0=0$.

(7) $x\leq y$ *implies* $x*z\leq y*z$.

(8) $1\longrightarrow x=x$, $x\longrightarrow x=1$, $x\leq y\longrightarrow x$, $x\longrightarrow 1=1$.

Definition 1.3. [5]

● *An element x of A is said to be regular if and only if* $x^{--}=x$.

● *An element x of A is said to be dense if and only if* $x^{-}=0$.

● *An element x of A is said to be nilpotent if and only if there exists* $n\in N$ *such that* $x^n=0$.

Definition 1.4. [7] *A filter of a residuated lattice A to be a non-void subset F of A such that:*

(i) *If* $x,y\in F$*, then* $x*y\in F$,

(ii) *If $x \in F$ and $x \leq y$, then $y \in F$.*

For all $x,y \in A$, we write $x \sim_F y$ if and only if $x \longrightarrow y$ and $y \longrightarrow x$ are both in F.

Definition 1.5. [10] *Let F be a filter of residuated lattice A. Then*

- *F is proper if $F \neq A$. A proper filter F is prime if for all $x,y \in A$, $x \vee y \in F$ implies $x \in F$ or $y \in F$.*
- *A proper filter F of A is said to be a maximal filter if and only if it is not included in any other proper filter of A. A proper filter F of A is maximal if and only if for all $x \in A$, the following equivalence holds: $x \notin F$ if and only if there exists $n \in \mathbb{N} = N \setminus \{0\}$ such that $(x^n)^- \in F$.*
- F is called a Boolean filter if $x \vee x^- \in F$, for all $x \in A$.

Definition 1.6. [10] *Let A be a residuated lattice and F be a filter of A. We define*

- *The set of double complemented elements by $D(F) = \{x \in A \mid x^{--} \in F\}$.*
- *If F is a proper filter of A. The intersection of all maximal filters of A that contain F is called the radical of F and is denoted by Rad(F). Then $Rad(F) = \{x \in A \mid (x^n)^- \longrightarrow x \in F$, for all $n \in \mathbb{N}\}$.*

Definition 1.7. [7, 10] *A non-empty subset F of A is called:*

- *A positive implicative filter of A if $1 \in F$ and $x \longrightarrow (y \longrightarrow z) \in F$ and $x \longrightarrow y \in F$ imply that $x \longrightarrow z \in F$.*
- *An implicative filter if $1 \in F$ and $x \longrightarrow ((y \longrightarrow z) \longrightarrow y) \in F$ and $x \in F$ imply $y \in F$.*
- *An obstinate filter if $0 \notin F$ and $x,y \notin F$ imply $x \longrightarrow y \in F$ and $y \longrightarrow x \in F$.*

Definition 1.8. [7] *The principal filter generated by element a of A i.e. the set $\{x \in A$: $x \geq a^n$, for some $n \geq 1\}$ will be denoted by $[a)$.*

2. Nodal filters

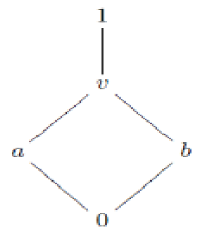
From now $(A,\wedge,\vee,*,\longrightarrow,0,1)$ or simply A is a residuated lattice.

Definition 2.1. *A node of A is an element which is comparable with every element of A. It is clear that 0,1 are node in every residuated lattice.*

Remark 2.2. *Let $a \in A$. a is node if and only if for every $x \in A$ either $a \longrightarrow x = 1$ or $x \longrightarrow a = 1$.*

Let A be a linear residuated lattice. It is clear that every element a of A is a node of A.

Example 2.3. *Let $A = \{0,a,b,v,1\}$. Define $*,\longrightarrow$ as follow:*



$*$	0	a	b	v	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
v	0	a	b	v	v
1	0	a	b	v	1

$\longrightarrow$	0	a	b	v	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
v	0	a	b	1	1
1	0	a	b	v	1

Then A is a residuated lattice. Clearly v is a node and b is not a node.

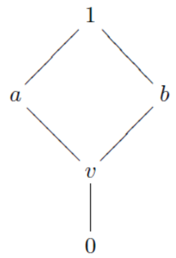
Example 2.4. *Let $A = \{0,a,b,1\}$, where $0 \leq a \leq b \leq 1$. Define $*$ and $\longrightarrow$ as follow:*

$*$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\longrightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(A,\wedge,\vee,*,\longrightarrow,0,1)$ is a residuated lattice. It is clear that b is a node of A but b is not a dense element, regular element and nilpotent element of A.

Example 2.5. *Let $A = \{0,a,b,v,1\}$. Define $*$ and $\longrightarrow$ as follow:*

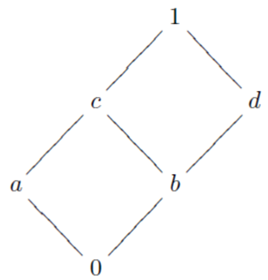


→	0	v	a	b	1
0	1	1	1	1	1
v	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	v	a	b	1

*	0	v	a	b	1
0	0	0	0	0	0
v	0	v	v	v	v
a	0	v	a	v	a
b	0	v	v	b	b
1	0	v	a	b	1

Then $(A,\wedge,\vee, *, \longrightarrow,0,1)$ *is a residuated lattice. Clearly a is a dense element but is not a node.*

Example 2.6. *Let* $A = \{0,a,b,c,d,1\}$. *Define* $*$ *and* $\longrightarrow$ *as follow:*

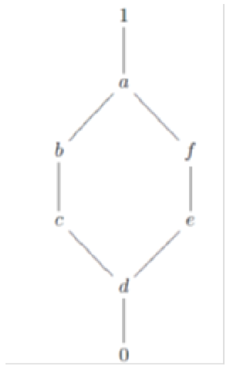


*	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

→	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

Then $(A,\wedge,\vee,*,\longrightarrow,0,1)$ *is a residuated lattice. It is clear that b is not a node but is a nilpotent element.*

Example 2.7. *Let* $A = \{0,a,b,c,d,e,f,1\}$. *Define* $*$ *and* $\longrightarrow$*as follow:*



*	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	c	c	0	d	d	a
b	0	c	c	c	0	0	d	b
c	0	c	c	c	0	0	0	c
d	0	0	0	0	0	0	0	d
e	0	d	0	0	0	d	d	e
f	0	d	d	0	0	d	d	f
1	0	a	b	c	d	e	f	1

→	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	a	a	f	f	f	1
b	e	1	1	a	f	f	f	1
c	f	1	1	1	f	f	f	1
d	a	1	1	1	1	1	1	1
e	b	1	a	a	a	1	1	1
f	c	1	a	a	a	a	1	1
1	0	a	b	c	d	e	f	1

Then $(A,\wedge,\vee,*,\longrightarrow,0,1)$ *is a residuated lattice. f is a nilpotent element and b is a regular element but both of them are not node of A.*

Definition 2.8. *An element* $a \in A$ *is called distributive if and only if* $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$, *for all x, y in A.*

Lemma 2.9. *Let* $a \in A$ *be a node of A. Then a is a distributive element of A.*

Proof. Let a be a node of A and $x,y \in A$. We consider the following cases:

Case 1: $a \geq x,y$. This implies $a \geq x \wedge y$ and thus $a \vee (x \wedge y) = a$on the other hand, we have $(a \vee x) \wedge (a \vee y) = a$.

Case 2: $a \le x,y$. Then we get that $a \le x \wedge y$ and thus, $a \vee (x \wedge y) = x \wedge y$ further, we have $(a \vee x) \wedge (a \vee y) = x \wedge y$.
Case 3: $x \le a \le y$. then $a \vee (x \wedge y) = a$ and $(a \vee x) \wedge (a \vee y) = a$.
Case 4: $x \ge a \ge y$ is similar to case 3.

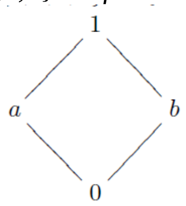
In the following example we show that there exists a distributive element of A that is not a node of A.

Example 2.10. *Consider Example 2.3. It is clear that a is a distributive element. We have $a \not\le b$, $b \not\le a$. So a is not a node.*
The set F(A) of all filters of a residuated lattice A is a residuated lattice in which for any $F,G \in$ F(A), $F * G = F \cap G$ and $F \longrightarrow G$ is the filter generated by $F \cup G$.

Lemma 2.11. *If F is a filter of A and* F(A) *is a set of all filters of A, then the following conditions are equivalent:*
(1) *Let G be a filter of A. Then we have either $G \subseteq F$ or $F \subseteq G$,*
(2) *Let $x \in F$, $y \notin F$. Then $x \gneqq y$,*
(3) *F is a node of* F(A).
Proof. (2)⇒ (1) Suppose that there exists a filter G incomparable with F.
Then there are element x and y such that $x \in F \backslash G$, $y \in G \backslash F$ and $x \ngeq y$.
(1)⇒ (3) Immediate by the definition of a node.
(3)⇒ (2) If F is a node of F(A), then for every $x \in F$ and every $y \notin F$ we have $[y) \nsubseteq F$. So $[x) \subset F \subset [y)$ and $y \lneqq x$.

Definition 2.12. *A filter satisfying one of the condition* (1) − (3) *will be called a nodal filter.*
$F = \{1\}$ is a nodal filter of A.

Example 2.13. (a) *In Example 2.5,* F(A) = $\{\{a,1\},\{b,1\},\{v,a,b,1\},\{1\}\}$. *Clearly, $F = \{v,a,b,1\}$ is a nodal filter of A.*
(b) *Let* $A = \{0,a,b,1\}$. *Define $*$ and $\longrightarrow$ as follow:*



$*$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\longrightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then $(A, \wedge, \vee, *, \longrightarrow, 0, 1)$ is a residuated lattice. Clearly, $F = \{b,1\}$ and $G = \{a,1\}$ are not nodal filter of A.
(c) *In Example 2.7, $F = \{a,1\}$ is a nodal filter of A but $Rad(F) = \{a,b,c,1\}$ is not a nodal filter of A.*

Definition 2.14. *A filter F of A is said to be distributive if F is distributive as an element of* F(A).

Example 2.15. *In Example 2.5, $F = \{v,a,b,1\}$ is distributive element of* F(A).

Corollary 2.16. *Every nodal filter of A is distributive element of* F(A).
In the following example we show that $[x)$ is a nodal filter of A but x is not a node element of A.

Example 2.17. *Consider Example 2.7. Clearly, e is not a node but $F = [e) = A$ is a nodal filter of A.*
Two nodal filters are always comparable but the next example shows that the converse is not true.

Example 2.18. *In Example 2.5, $G = \{b,1\}$, $F = \{v,a,b,1\}$ are comparable but G is not nodal filter.*
The class of all nodal filters forms a totally ordered structure.

Proposition 2.19. *Let F be a nodal filter of A. Then F satisfies the intersection property.*

Lemma 2.20. *The set $N(A)$ of all nodal filters of A, ordered by inclusion, is a chain that greatest element is A. Then $N(A)$ has the least element* $\{1\}$.

Clearly, in chains all filters are nodal.
The following example shows that the extension property does not hold for nodal filters.

Example 2.21. *Consider Example 2.3. Let F = {v,1} and G = {a,v,1}. Then F, G are filters of A and F ⊆G. Clearly, F is a nodal filter of A but G is not a nodal filter of A.*
In the following examples we consider the relation between a nodal filter and the other filters of *A*.

Example 2.22. *Consider Example 2.3. It is clear that F = {v,1} is a nodal filter of A but F is not prime, Boolean and obstinate filter of A. Also, G = {a,v,1} is prime, Boolean and obstinate filter of A but G is not a nodal filter of A.*

Example 2.23. *Let A = {0,a,b,1}. Define ∗ ,⟶ as follow:*

∗	0	*a*	*b*	1
0	0	0	0	0
a	0	*a*	0	*a*
b	0	0	*b*	*b*
1	0	*a*	*b*	1

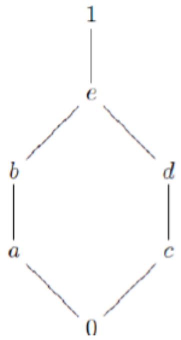
⟶	0	*a*	*b*	1
0	1	1	1	1
a	*b*	1	*b*	1
b	*a*	*a*	1	1
1	0	*a*	*b*	1

Then (A,∧,∨,∗,⟶,0,1) is a residuated lattice. F = {1} is a nodal filter of A but F is not a positive implicative filter of A.
Definition 2.24. *We say that two element x and y of A are connected (in symbol (x,y)∈R) if there is no nodal filter which separates them, let us notice that (x,y) ∉R implies either x⪇y or y⪇x.*

Example 2.25.*Let A = {0,a,b,c,d,e,1}. Define ∗and ⟶as follow:*

∗	0	*a*	*b*	*c*	*d*	*e*	1
0	0	0	0	0	0	0	0
a	0	*a*	*a*	0	0	*a*	*a*
b	0	*a*	*a*	0	0	*a*	*b*
c	0	0	0	*c*	*c*	*c*	*c*
d	0	0	0	*c*	*c*	*c*	*d*
e	0	*a*	*a*	*c*	*c*	*e*	*e*
1	0	*a*	*b*	*c*	*d*	*e*	1

→	0	*a*	*b*	*c*	*d*	*e*	1
0	1	1	1	1	1	1	1
a	*d*	1	1	*d*	*d*	1	1
b	*d*	*e*	1	*d*	*d*	1	1
c	*b*	*b*	*b*	1	1	1	1
d	*b*	*b*	*b*	*e*	1	1	1
e	0	*b*	*b*	*d*	*d*	1	1
1	0	*a*	*b*	*c*	*d*	*e*	1



Then (A,∧,∨,∗,⟶,0,1) is a residuated lattice and F = {e,1} is a nodal filter of A. Clearly, (a,e) ∉R and a,b are connected (in symbol (a,b) ∈R).

Proposition 2.26. *The relation R is reflexive and transitive.*
Proof. By Definition 2.24, we have (*x*,*x*) ∈*R* so *R* is reflexive. Let (*x*,*y*) ∈*R*, (*y*,*z*) ∈*R* and (*x*,*z*) ∉*R*. Then there is a nodal filter *F* such that *x* ∈*F* and *z* ∉*F*. So *y* ∈*F* and (*y*,*z*) ∉ *R*. That is a contradiction. Thus *R* is transitive.

Theorem 2.27. *The relation R enjoys the following properties:*
(1) *Any R-class contains at most one node.*
(2) *An R-class is totally ordered if and only if it is a singleton.*
(3) *A/R is a chain dually isomorphic to N(A).*
Proof. (1) Let *a* and *b* be connected nodes of *A*. We have either a⪈b or b⪇ a. In the first case, *a* and *b* are separated by the nodal filter [*b*).

(2) Let [*a*)*R* (the *R*-class of *a*) be totally ordered and (*b*,*a*) ∈*R*, *a* ≠*b*. Any *x* ∈*A* is comparable with *a* and *b*. Hence both *a* and *b* are nodes, which is a contradiction.

(3) We define the mapping $\boldsymbol{\alpha}$: A/R⟶N(A) by $\boldsymbol{\alpha}$(c)=F_c where F_c is a nodal filter generated by the *R*-class *c*. In fact, F_c = {*x* ∈*A* :*x*≥ *y*, *y*∈*c*}. Obviously *α* is bijective and *c*⊆*c'* in*A/R*if and only if$F_{c'}$⊆F_c in *N*(*A*).

Example 2.28. *Consider Example 2.4. Clearly, N(A) = {{1},{b,1},{a,b,1},A} and R = {(1,1),(0,0),(a,a),(b,b)}. We have A/R=*

{[0],[*a*],[*b*],[1]}. *So* A/R *is a chain dually isomorphic to N*(*A*).

In the following example we show that *R* is not antisymmetric

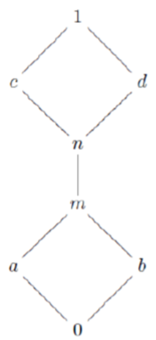
Example 2.29. *Consider Example 2.29. It is clear that* F(*A*) = {{1},{*e*,1},{*a*,*b*,*e*,1},{*c*,*d*,*e*,1},*A*}.*So* {1}, {*e*,1} *are nodal filter and* (*a*,*c*)∈*R*, (*c*,*a*)∈*R but a≠c. Thus R is not antisymmetric.*

Theorem 2.30. *Let F be a filter of A. If for all a,b∈A, a ⟶b ≠* 1, *a ⟶b ∈F imply a,b∈F, then F is nodal filter.*

Proof. Let there exist *a∉F* and *b∈F* such that a≰ b. Then *a,b∈A*, *a ⟶b ≠* 1 and *a⟶b∈F* so *a∈F*. That is a contradiction.

In the following examples we show that the converse of above theorem is not true.

Example 2.31. *Let A* = {0,*a*,*b*,*m*,*n*,*c*,*d*,1}. *Define ∗ and ⟶ as follow:*



∗	0	a	b	m	n	c	d	1
0	0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a	a
b	0	0	b	b	b	b	b	b
m	0	a	b	m	m	m	m	m
n	0	a	b	m	n	n	n	n
c	0	a	b	m	n	c	n	c
d	0	a	b	m	n	n	d	d
1	0	a	b	m	n	c	d	1

→	0	a	b	m	n	c	d	1
0	1	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1	1
b	a	a	1	1	1	1	1	1
m	0	a	b	1	1	1	1	1
n	0	a	b	m	1	1	1	1
c	0	a	b	m	d	1	d	1
d	0	a	b	m	c	c	1	1
1	0	a	b	m	n	c	d	1

Then (*A*,∧,∨,∗,⟶,0,1) *is a residuated lattice and F* = {*d*,1} *is a nodal filter of A. Also, for n,c∈A we have c ⟶n ≠*1,*c ⟶n ∈F but c,n∉F.*

Definition 2.32. *If F^+ is upper bounds of F and F− is lower bounds of F, then we define*

$$(F')^u = \{a \in A: a \in F^+(F^-(F))\}$$

Now we direct our attention to the Mac Neille completion of *A*. Or, more precisely, to the dual of the lattice.

It means that to every subset *F* of *A*, we associate it's "Dedekind cut" $(F')^u$, i.e. all upper bounds to the set of lower bounds of *F*. We call a filter *F* of *A* normal if *F* = $(F')^u$obviously any principal filter is normal.

Filter *F* = {1} always is normal filter of A.

Example 2.33.

(a) *Consider Example 2.29. If F* = {*e*,1}, *then F*− = {0, *a*, *b*, *c*, *d*,*e*}, F^+(*F*−) = {*e*,1}. *So* $(F')^u$= {*e*,1}, *then F is a normal filter of A.*

(b) Consider Example 2.35. If G = {*n*,*d*,1}, *then G*− = {0,*a*,*b*,*m*,*n*}, G^+(*G*−) = {*n*,*c*,*d*,1}. *So* $(G')^u$= {*n*,*c*,*d*,1}, *then G is not a normal filter of A.*

(c) *Consider Example 2.29. Thus F* = {*e*,1} *is a normal filter of A but is not a maximal filter of A. In Example 2.7, G* = {*e*,*f*,*a*,1} *is a maximal filter of A but is not a normal filter of A.*

Theorem 2.34. *For a non-principal nodal filter F the following conditions are equivalent:*

(1) F is a normal filter of A,
(2) A/F is not a principal ideal,
(3) *Inf F do not exist.*

Proof: (1) ⇒ (2)Let for all non-principal nodal filter *F* of *A* holds $A/F = F$. If $A/F = (a]$, then $(F')^u$= [*a*) ⊃*F*. So *F* is not a normal filter of *A*.

(2)⇒ (3) If *inf F*= *a*, then *a ∉F*, *a* is the greatest element of A\F. Thus A\F is a principal ideal of *A*.

(3)⇒ (1) If *F* is not a normal filter of *A*, then $(F')^u$= [*a*) ⊃*F* and there is an element *a∉F* which is an upper bound of A\F. This element obviously constitutes the greatest element of A\F. Also, it is the infimum of *F*.

For *A* = [0,1], we define *x ∗y* = *x.y* (natural product). Then *A* is a residuated lattice. Clearly, *A* is a chain. We take *F* = (0,1] so *F* is a filter of *A*. Since *F*−((0,1]) = {0} and F^+(*F*−((0,1])) = F^+({0}) = *A≠F*, *F* is not a normal filter of *A*.

Theorem 2.35. *Let F be a nodal filter of A . Then so is $(F')^u$.*

Proof. Since the case F principal is trivial, by virtue of the preceding theorem we may restrict ourselves to the consideration of a non-principal nodal filter F for which *inf* F exists. Let $a = inf\ F$. Thus $a \notin F$ for any $x \in A \backslash F$, we have $x \leq a$, a is a node, where $[a) = (F')^u$is nodal.

3. Congruences generated by filter

To end with we shall characterize nodal filter in terms of congruences. But here also we need some preliminaries. Here filter End(A) will mean the endomorphism of $A = (A,\wedge,\vee,*,\longrightarrow,0,1)$. To every endomorphism α of A is associated a congruence θ_α defined by:

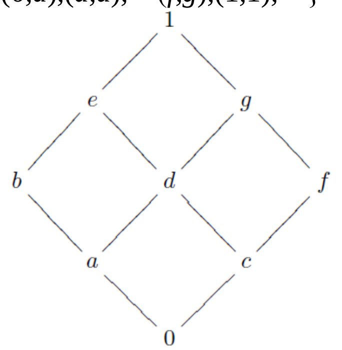
$(x,y) \in \theta_\alpha$ if and only if $\alpha(x) = \alpha(y)$.

An endomorphism α of an algebra $A = \langle A : F \rangle$ is said to be a left vector endomorphism if there exists a subalgebra $B = \langle B : F \rangle$ of A satisfying the following two conditions:

(1) $\bigcup\{[x]\theta_x : [x]\theta_\alpha \cap B \neq \emptyset, x \in A\} = A$, i.e. the union of the θ_α classes which meet B is A.

(2) $\theta_\alpha|B = \omega_B$, where ω_B is the equality on B, B is transversal for $A/\theta_\alpha \in A$.

Example 3.1. *Let $A = \{0,a,b,c,d,e,f,g,1\}$ be a residuated lattice. We define endomorphism α $:A \longrightarrow A$ such that $\alpha(x) = x^2$, for all $x \in A$. We have $\alpha(0) = \alpha(a) = \alpha(c) = \alpha(d) = 0$, $\alpha(b) = \alpha(e) = b$, $\alpha(f) = \alpha(g) = f$ and $\alpha(1) = 1$. So $[0] = \{0,a,c,d\}$, $[b] = \{b,e\}$, $[f] = \{f,g\}$, $[1] = \{1\}$ and θ_α $=\{(0,0),(a,a),(c,c),(b,b),\cdots$ $,(0,a),(0,c),(0,d),(a,d),\cdots(f,g),(1,1),\cdots\}$*



It is clear that B = {g,f,1} is a subalgebra of A satisfying condition (1) ,(2).

For every congruence θ of A, [1]θ is a filter of A. We shall denote it by F_θ. Moreover $(x,y) \in \theta$ if and only if $x * d = y * d$ and $x \longrightarrow d = y \longrightarrow d$ for a suitable $d \in F_\theta$.

*	0	*a*	*b*	*c*	*d*	*e*	*f*	*g*	1
0	0	0	0	0	0	0	0	0	0
a	0	0	*a*	0	0	*a*	0	0	*a*
b	0	*a*	*b*	0	*a*	*b*	0	*a*	*b*
c	0	0	0	0	0	0	*c*	*c*	*c*
d	0	0	*a*	0	0	*a*	*c*	*c*	*d*
e	0	*a*	*b*	0	*a*	*b*	*c*	*d*	*e*
f	0	0	0	*c*	*c*	*c*	*f*	*f*	*f*
g	0	0	*a*	*c*	*c*	*d*	*f*	*f*	*g*
1	0	*a*	*b*	*c*	*d*	*e*	*f*	*g*	1

⟶	0	*a*	*b*	*c*	*d*	*e*	*f*	*g*	1
0	1	1	1	1	1	1	1	1	1
a	*g*	1	1	*g*	1	1	*g*	1	1
b	*b*	*g*	1	*f*	*g*	1	*f*	*g*	1
c	*e*	*e*	*e*	1	1	1	1	1	1
d	*d*	*e*	*e*	*g*	1	1	*g*	1	1
e	*c*	*d*	*e*	*f*	*g*	1	*f*	*g*	1
f	*b*	*b*	*b*	*e*	*e*	*e*	1	1	1
g	*a*	*b*	*b*	*d*	*e*	*e*	*g*	1	1
1	0	*a*	*b*	*c*	*d*	*e*	*f*	*g*	1

In other words, the correspondence between filters and congruences is one to one.

Example 3.2. *In Example 3.1, it is clear that $F = \{f,g,1\}$ is a filter of A.*

Note that $\leq$ is congruence of A. For $g \in F$ we have $(c,d) \in \theta$ i.e. $c \geq d$ if and only if $c * g = d * g$ and $c \longrightarrow g = d \longrightarrow g$.

Theorem 3.3. *Let θ be a congruence on A. Then the following conditions are equivalent:*

(1) [1]θ is a nodal filter of A,

(2) θ is a node of con(A), the congruence lattice of A.

Proof. The mapping $\theta \longrightarrow F_\theta$ of con(A) on to F(A) is an isomorphism and F_θ is a nodal filter if and only if it is a node of F(A).

Theorem 3.4. [1]θα is a nodal filter of A.

Proof. For every endomorphism α of A that α ≠1 we have [1]θα = {1}. So it is nodal filter of A. If α = 1, then [1]θα = A that is nodal filter of A.

Example 3.5. Consider Example 3.1. Clearly, we have [1]θα = {1}. So [1]θα is a nodal filter of A.

4. Conclusion and future research

For analyzing the residuated lattices we continue our studies in residuated lattices and generalized some notions in this structure and get some new results.

In this paper, we have introduced the notions of special elements as node and nodal filters, normal filters of residuated lattices. We have established properties of nodal filters of residuated lattices. We proved the relationships between nodal filters and other types of filters of residuated lattices. Also, by the notions of the upper bounds and lower bounds of the filter F of A we introduced the concept of$(F')^u$ and normal filters of A. Next, we prove that if F is a nodal filter of A, then so is $(F')^u$.

The investigation of other such generalizations can be an interesting object for future work.

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