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On the classification and reduced equations of the second order linear partial differential equations in two independent variables

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Abstract

In this study, we give first the classification and reduced equations of the second degree equations of the form $ax^2+2bxy+cy^2+2ex+2fy+g=0$. We investigate the classification and reduced equations of the second order linear PDEs in two independent variables x and y of the form $Au_{xx}+2Bu_{xy}+Cu_{yy}+2Eu_x+2Fu_y+Gu=0$ using the results we obtained on the classification and reduced equations of the second degree equations. Finally, solving reduced equations of the second order linear PDEs we obtain the solutions of the PDEs. We show that both equations have common algebraic characterizations.

Keywords: Differential equations, partial differential equations, second order linear partial differential equations, second degree equations.

İki Bağımsız Değişkenli İkinci Basamaktan Doğrusal Kısmi Diferansiyel Denklemlerin Sınıflandırılması ve İndirgenmiş Denklemeleri Üzerine

Özet

Bu çalışmada, ilk olarak $ax^2+2bxy+cy^2+2ex+2fy+g=0$. biçiminde ikinci dereceden denklemin sınıflandırılması ve indirgenmiş denklemleri vereceğiz. İkinci dereceden denklemlerin sınıflandırılması ve indirgenmiş denklemleri üzerinde elde edilen sonuçları kullanarak x ve y bağımsız değişkenli $Au_{xx}+2Bu_{xy}+Cu_{yy}+2Eu_x+2Fu_y+Gu=0$ biçimindeki ikinci basamaktan kısmi diferansiyel denkleminin sınıflandırılması ve indirgenmiş denklemlerini inceleyeceğiz. Son olarak ikinci basamaktan doğrusal kısmi diferansiyel denklemlerin indirgenmiş denklemlerin çözümünü kullanarak doğrusal kısmi diferansiyel denklemin çözümlerini elde edeceğiz.Her iki denkleminde ortak cebirsel özelliklere sahip olduğunu göstereceğiz.

Anahtar kelimeler: Diferansiyel Denklemler, Kısmi diferansiyel denklemler, Ikinci basamaktan kısmi diferansiyel denklemler, Ikinci derceden denklemler.

1.Introduction

Differential equations involving two or more independent variables are referred to as partial differential equations. The second order linear partial differential equation in two independent variables is a special case of the partial differential equations (PDEs).

Many engineering problems such as wave propagation, heat conduction, elasticity, vibrations, electrostatics, electrodynamics, etc are formulated as the second order linear partial differential equations in two independent variables x and y of the form [1, 2, 3, 9]

$$L(u) = Au_{xx} + 2Bu_{xy} + Cu_{yy} + 2Eu_x + 2Fu_y + Gu = 0 \quad (1)$$

where $A, B, ..., G$ are constants and $A^2 + B^2 + C^2 \neq 0$.

Partial differential equations are normally classified according to their mathematical form. However, in some case they might be classified according to the particular physical problem being modeled [6, 7, 10].

Let

$$\frac{\partial}{\partial x} \equiv D_x \text{ and } \frac{\partial}{\partial y} \equiv D_y .$$

Then we have

$$L \equiv AD_x^2 + 2BD_xD_y + CD_y^2 + 2ED_x + 2FD_y + G, \quad \textbf{(2)}$$

where L is the linear partial differential operator with constant coefficients $A, B, ..., G$.

In this study, by comparison with the second degree equations of the conic [4, 5, 8]

$$ax^2 + 2bxy + cy^2 + 2ex + 2fy + g = 0, \quad (3)$$

where the coefficients $a, b, ..., g$ represent numbers and $a^2 + b^2 + c^2 \neq 0$, we shall investigate the classification and reduced equations of (1). Using this technique that we consider it can be reduced by variable substitutions to simpler equations depending on whether (1) is elliptic, hyperbolic or parabolic. Thus we show that (1) and (3) have common algebraic characterizations.

2. Classification and reduced equations of the conics

The conics have second degree equations of the form (3). We can express the Eq (3) as follows:

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a & b & e \\ b & c & f \\ e & f & g \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0 ,$$

where

$$M = \begin{bmatrix} a & b & e \\ b & c & f \\ e & f & g \end{bmatrix}$$

is the coefficients matrix of (3).

Solving for y in Eq (3), we obtain

$$y = -\frac{1}{c}(bx + f) \mp \frac{1}{c}\sqrt{-g^* x^2 + 2e^* x - a^*} ,$$

where $a^*, b^*, ..., g^*$ are the cofactors of the coefficients $a, b, ..., g$, respectively.

Let $|M|$ be the determinant of the matrix M. If $|M| = 0$, the conic is degenerate. Therefore we can establish the following theorems which are not difficult to prove.

Theorem 1

(i) If $|M| \neq 0$ and $g^* > 0$, the conic is an ellipse,

(ii) If $|M| \neq 0$ and $g^* < 0$, the conic is a hyperbole,

(iii) If $|M| \neq 0$ and $g^* = 0$, the conic is a parabola.

Theorem 2

(*i*) If $|M| = 0$ and $g^* < 0$, it is two intersecting lines,

(*ii*) If $|M| = 0$, $g^* = 0$ and so $e^* = 0$, it is two parallel straight lines,

(*iii*) If $|M| = 0$ and $g^* > 0$, it is two imaginary intersecting lines.

Let e^*, f^* and g^* are the cofactors of the coefficients *e*, *f* and *g*, respectively. Using the e^*, f^* and g^* we compute the center of the conic as $(\frac{e^*}{g^*}, \frac{f^*}{g^*})$, where $g^* \neq 0$.

We would like to change the coordinate system in order to have the curve at a convenient and familiar location. The process of making this change is called the translation of axes. We now consider the old and new coordinates by the translation equations

$$x = \frac{e^*}{g^*} + \bar{x} \quad \text{and} \quad y = \frac{f^*}{g^*} + \bar{y} . \tag{4}$$

Substituting (4) into (3), we get

$$a\bar{x}^2 + 2b\bar{x}\,\bar{y} + c\bar{y}^2 + \frac{|M|}{g^*} = 0 . \tag{5}$$

Now we also consider the rotation of axes. To do this we take the equations

$$\bar{x} = XCos\theta - YSin\theta$$

$$\bar{y} = XSin\theta + YCos\theta , \tag{6}$$

where

$$Tan2\theta = \frac{2b}{a-c} .$$

If we substitute (6) into (5), we have

$$a_1 X^2 + c_1 Y^2 + \frac{|M|}{g^*} = 0 , \tag{7}$$

where $g^* \neq 0$ and

$$a_1 = aCos^2\theta + bSin2\theta + cSin^2\theta$$

and

$$c_1 = aSin^2\theta - bSin2\theta + cCos^2\theta . \tag{8}$$

From (8) we find out that

$$a_1 + c_1 = a + c \quad \text{and} \quad a_1 c_1 = ac - b^2 .$$

Note that a_1 and c_1 are the real roots of characteristic polynomial of the matrix

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} .$$

We now turn to the (7) which is called as the reduced equation of the conic defined in (3). If $a_1 c_1 > 0$ then the locus is an ellipse and $a_1 c_1 < 0$ then the locus is a hyperbole.

When the conic is parabola, we first find the rotated equation of (3). To do this we take the equations

$$x = \xi Cos\theta - \upsilon Sin\theta \quad \text{and} \quad y = \xi Sin\theta + \upsilon Cos\theta$$

and we substitute in (3). Then we have

$$\upsilon = -\frac{1}{2f_2}(a_2\xi^2 + 2e_2\xi + g_2) \quad \text{or}$$

$$\xi = -\frac{1}{2e_2}(c_2\upsilon^2 + 2f_2\upsilon + g_2) , \tag{9}$$

where

$$e_2 = eCos\theta + fSin\theta , \qquad f_2 = -eSin\theta + fCos\theta$$

and $g_2 = g$.

The extremum points of the conics defined in (9) are

$$(-\frac{e_2}{a_2}, -\frac{c_2^*}{2a_2 f_2}) \quad \text{and} \quad (-\frac{a_2^*}{2c_2 e_2}, -\frac{f_2}{c_2}) ,$$

respectively. The translation of axes,

$$\xi = \mathrm{X} - \frac{e_2}{a_2} , \quad \upsilon = \mathrm{Y} - \frac{c_2^*}{2a_2 f_2}$$

or

$$\xi = \mathrm{X} - \frac{a_2^*}{2c_2 e_2} , \quad \upsilon = \mathrm{Y} - \frac{f_2}{c_2}$$

leads to the equations

$$\mathrm{X}^2 = -\frac{2f_2}{a_2}\mathrm{Y} \quad \text{or} \quad \mathrm{Y}^2 = -\frac{2e_2}{c_2}\mathrm{X} , \tag{10}$$

respectively. Eq (10) is called as the reduced equations of the parabola.

3. The second order linear partial differential equations in two independent variables

We consider the linear operator L defined in (2). This operator can be expressed as follows:

$$L \equiv \begin{bmatrix} D_x & D_y & 1 \end{bmatrix} \begin{bmatrix} A & B & E \\ B & C & F \\ E & F & G \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ 1 \end{bmatrix},$$

Where

$$N = \begin{bmatrix} A & B & E \\ B & C & F \\ E & F & G \end{bmatrix}$$

is the coefficients matrix of L.

Let $|N|$ be determinant of the matrix N and G^* be cofactor of G, where G^* is the discriminant of the L and

$$G^* = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2.$$

Here we establish the following theorems which are proved easily using the results obtained on the conics.

Theorem 3

(i) If $|N| \neq 0$ and $G^* > 0$, Eq (1) is an elliptic differential equation,

(ii) If $|N| \neq 0$ and $G^* < 0$, Eq (1) is a hyperbolic differential equation,

(iii) If $|N| \neq 0$ and $G^* = 0$, Eq (1) is a parabolic differential equation.

Theorem 4

(i) If $|N| = 0$ and $G^* < 0$, the characteristics of (1) are two distinct families of real straight lines,

(ii) If $|N| = 0$ and $G^* = 0$, the characteristics of (1) are two parallel lines,

(iii) If $|N| = 0$ and $G^* > 0$, the characteristics of (1) are two distinct families of imaginary straight lines.

Often a PDE can be reduced to a simpler form with a known solution by a suitable change of variables. In this study, we first consider the following translation equations to obtain the reduced equation of (1)

$$D_x \equiv \frac{E^*}{G^*} + D_{\bar{x}} \text{ and } D_y \equiv \frac{F^*}{G^*} + D_{\bar{y}}, \qquad (11)$$

where E^*, F^* and G^* are cofactors of E, F and G, respectively.

If we substitute (11) into (2), we have

$$L \equiv AD_{\bar{x}}^2 + 2BD_{\bar{x}}D_{\bar{y}} + CD_{\bar{y}}^2 + \frac{|N|}{G^*}, \qquad (12)$$

where $G^* \neq 0$.

When we also substitute following equations

$$D_{\bar{x}} \equiv D_X Cos\theta - D_Y Sin\theta$$

and

$$D_{\bar{y}} \equiv D_X Sin\theta + D_Y Cos\theta$$

into (12), then we obtain

$$L \equiv A_1 D_X^2 + C_1 D_Y^2 + \frac{|N|}{G^*}, \qquad (13)$$

where

$$A_1 = ACos^2\theta + BSin2\theta + CSin^2\theta,$$
$$C_1 = ASin^2\theta - BSin2\theta + CCos^2\theta$$

and

$$A_1 + C_1 = A + C \text{ and } A_1 C_1 = AC - B^2.$$

Note that A_1 and C_1 are eigenvalues of the matrix

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

From (13) we obtain the reduced equation of (1) as

$$L(u) = A_1 u_{XX} + C_1 u_{YY} + \frac{|N|}{G^*} u = 0 . \quad (14)$$

If $A_1C_1 > 0$, it is an elliptic differential equation, and $A_1C_1 < 0$, it is a hyperbolic differential equation.

When $G^* = 0$, we first find the rotated form of L defined in (2). For this we take the equations

$$D_x \equiv D_\xi Cos\theta - D_\upsilon Sin\theta$$

and

$$D_y \equiv D_\xi Sin\theta + D_\upsilon Cos\theta$$

and we substitute in (2). Then we have

$$D_\upsilon \equiv -\frac{1}{2F_2}(A_2 D_\xi^2 + 2E_2 D_\xi + G_2) \quad (15)$$

or

$$D_\xi \equiv -\frac{1}{2E_2}(C_2 D_\upsilon^2 + 2F_2 D_\upsilon + G_2) , \quad (16)$$

where

$E_2 = ECos\theta + FSin\theta$, $F_2 = -ESin\theta + FCos\theta$ and $G_2 = G$.

Using the (15) and (16) we write the following equations, which are called normal equations of the linear partial parabolic differential equations,

$$A_2 u_{\xi\xi} + 2E_2 u_\xi + 2F_2 u_\upsilon + G_2 u = 0$$

or

$$C_2 u_{\upsilon\upsilon} + 2E_2 u_\xi + 2F_2 u_\upsilon + G_2 u = 0 .$$

Now we also consider the translation equations

$$D_\xi \equiv -\frac{E_2}{A_2} + D_X \text{ and } D_\upsilon \equiv -\frac{C_2^*}{2A_2F_2} + D_Y$$

or

$$D_\xi \equiv -\frac{A_2^*}{2C_2E_2} + D_X \text{ and } D_\upsilon \equiv -\frac{F_2}{C_2} + D_Y$$

and substitute into (15) and (16). Then we obtain

$$D_X^2 \equiv -\frac{2F_2}{A_2} D_Y \text{ or } D_Y^2 \equiv -\frac{2E_2}{C_2} D_X ,$$

respectively. Thus we get following equations, respectively,

$$u_{XX} = -\frac{2F_2}{A_2} u_Y \text{ or } u_{YY} = -\frac{2E_2}{C_2} u_X . \quad (17)$$

These equations are reduced equations of the linear partial parabolic differential equations.

We now consider the separating of variables, which is an analytical method, to solve PDEs. Concerning solutions of equations defined in (14) and (17), we first seek solutions of the form

$$u(X,Y) = \mathbf{X}(X)\mathbf{Y}(Y) , \quad (18)$$

where $\mathbf{X}$ is a function of X only and $\mathbf{Y}$ is a function of Y only, and

$$\frac{\partial u}{\partial X} = \mathbf{X}'\mathbf{Y} , \quad \frac{\partial u}{\partial Y} = \mathbf{X}\mathbf{Y}' .$$

We now discuss the solution of the equation

$$u_{XX} = -\frac{2F_2}{A_2} u_Y \quad (19)$$

defined in (17). Assuming a solution of the form (18) substituting in (19) we find

$$\frac{\mathbf{X}''}{\mathbf{X}} = -\frac{2F_2}{A_2} \frac{\mathbf{Y}'}{\mathbf{Y}} .$$

Hence putting

$$\frac{\mathbf{X}''}{\mathbf{X}} = -\omega^2 \text{ and } -\frac{2F_2}{A_2} \frac{\mathbf{Y}'}{\mathbf{Y}} = -\omega^2 ,$$

where ω is any real number, we have

$$\mathbf{X} = k_1 Cos\omega X + k_2 Sin\omega X , \quad \mathbf{Y} = k_3 e^{\frac{A_2}{2F_2}\omega^2 Y} , \quad (20)$$

where k_1, k_2 and k_3 are arbitrary constants. With (20), (18) now becomes

$$u(X,Y) = (\ell_1 Cos\omega X + \ell_2 Sin\omega X) e^{\frac{A_2}{2F_2}\omega^2 Y} ,$$

where ℓ_1 and ℓ_2 are new arbitrary constants.

Now we also consider the solution of the equation (14). We now assume a solution of (14) of the form (18). In this way (14) becomes

$$A_1 \frac{\mathrm{X}''}{\mathrm{X}} = -\frac{C_1 \mathrm{Y}'' + \frac{|N|}{G^*}\mathrm{Y}}{\mathrm{Y}} .$$

This equation is satisfied if we now write

$$A_1 \mathrm{X}'' + \omega^2 \mathrm{X} = 0$$

and

$$C_1 \mathrm{Y}'' + \left(\frac{|N|}{G^*} - \omega^2\right)\mathrm{Y} = 0 ,$$

where ω is any real number. The solutions of these equations are

$$\mathrm{X} = k_1 Cos \frac{\omega}{\sqrt{A_1}} X + k_2 Sin \frac{\omega}{\sqrt{A_1}} X$$

and

$$\mathbf{Y} = k_3 Cos \sqrt{\frac{(|N|/G^*) - \omega^2}{C_1}}\, Y + k_4 Sin \sqrt{\frac{(|N|/G^*) - \omega^2}{C_1}}\, Y ,$$

respectively, where k_1, k_2, k_3 and k_4 are arbitrary constants.

Then we have following corollaries.

Corollary 1 Let $A_1 > 0$, $C_1 > 0$ and $\frac{|N|}{G^*} - \omega^2 > 0$ or $A_1 > 0$, $C_1 < 0$ and $\frac{|N|}{G^*} - \omega^2 < 0$.

Then the solution of (14) is

$$u(\mathrm{X},\mathrm{Y}) = \left(k_1 Cos \frac{\omega}{\sqrt{A_1}} X + k_2 Sin \frac{\omega}{\sqrt{A_1}} X\right) \left(k_3 Cos \sqrt{\frac{(|N|/G^*) - \omega^2}{C_1}}\, Y + k_4 Sin \sqrt{\frac{(|N|/G^*) - \omega^2}{C_1}}\, Y\right).$$

Corollary 2 Let $A_1 > 0$, $C_1 > 0$ and $\frac{|N|}{G^*} - \omega^2 < 0$ or $A_1 > 0$, $C_1 < 0$ and $\frac{|N|}{G^*} - \omega^2 > 0$. Then the solution of (14) is

$$u(\mathrm{X},\mathrm{Y}) = \left(k_1 Cos \frac{\omega}{\sqrt{A_1}} X + k_2 Sin \frac{\omega}{\sqrt{A_1}} X\right) \left(k_3 Cosh \sqrt{\frac{(|N|/G^*) - \omega^2}{C_1}}\, Y + k_4 Sinh \sqrt{\frac{(|N|/G^*) - \omega^2}{C_1}}\, Y\right).$$

Corollary 3 Let $A_1 < 0$, $C_1 < 0$ and $\frac{|N|}{G^*} - \omega^2 < 0$ or $A_1 < 0$, $C_1 > 0$ and $\frac{|N|}{G^*} - \omega^2 > 0$. Then the solution of (14) is

$$u(X,Y) = \left(k_1 e^{\frac{\omega}{\sqrt{-A_1}} X} + k_2 e^{-\frac{\omega}{\sqrt{-A_1}} X}\right) \left(k_3 Cos \sqrt{\frac{(|N|/G^*) - \omega^2}{C_1}}\, Y + k_4 Sin \sqrt{\frac{(|N|/G^*) - \omega^2}{C_1}}\, Y\right).$$

4. Conclusion

In this study, we show that the classification and reduced equations of the second order linear partial differential equations in two independent variables can be investigated using the results we obtained on the classification and reduced equations of the second degree equations, and the reduced equations of PDEs can be easily solved by transforming to a ordinary linear differential equation.

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