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Research Article

A LAGRANGE NEURAL NETWORK FOR NETWORK TRAFFIC ASSIGNMENT OPTIMIZATION

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Abstract: *A solution to the challenge of assigning static network traffic is being sought through this research. To address this issue, a model that uses the Lagrange function has been developed. The Lagrange neural network is a dynamic system that is created by calculating the function's gradient in relation to its variables. This system comprises differential equations that are adapted based on the initial conditions. The theoretical properties of this network are delved into in the paper, including its overall and local stability, and it is proven that the system exhibits global stability. Equilibrium points achieved by numerically solving the network meet the Karush-Kuhn-Tucker requirements, as demonstrated by using the Lyapunov method. It is guaranteed that they will converge towards the user equilibrium of the original problem. Finally, the effectiveness of the presented approach is showcased by providing a numerical example. The results obtained using this approach outperform those obtained using the Frank-Wolfe method.*

Keywords*: Dynamical systems, Graph theory, Lagrange neural network, Network traffic assignment*

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1. Introduction

In the field of transportation engineering and urban planning, traffic assignment models are essential tools. These models offer valuable insights into how vehicular traffic moves through a network of roads and infrastructure. By helping planners and decision-makers optimize traffic management, reduce congestion, and upgrade transportation systems, they play a crucial role in improving overall transportation efficiency.

However, traffic assignment has been a challenging problem for transportation engineers for quite some time. Initially, mathematical models were created to understand how traffic moves through cities, assuming that drivers always choose the shortest route. However, this assumption was found to be inaccurate. Drivers may select longer routes if they believe it will save time or help them avoid certain roads or areas. As a result, more sophisticated traffic assignment models have been developed that consider driver preferences. These models can predict traffic flow, enabling planners to make informed decisions to optimize traffic flow.

Wardrop's research [1] has fundamentally established that in the realm of transportation, travelers do not possess the ability to reduce their travel time merely by opting for an alternative route. This fundamental principle, commonly referred to as User Equilibrium (UE), has emerged as a cornerstone in the domain of traffic assignment. Subsequently, this concept has undergone extensive development and refinement by numerous scholars, rendering it the prevailing standard for the evaluation of traffic assignment algorithms in contemporary practice. Addressing this intricate challenge necessitates a diverse array of models and algorithms, spanning from rudimentary heuristics to sophisticated optimization techniques.

One advanced approach in the field of transportation engineering is the Frank-Wolfe algorithm, which was originally introduced to address the challenging problem of traffic assignment [2]. Beckmann and his colleagues [5] made significant strides in this domain by transforming the UE condition into a nonlinear convex programming problem. This transformation paved the way for the development of efficient algorithms tailored to tackle the UE condition. In a similar vein, Daganzo and Sheffi [4] pioneered a distinctive traffic assignment methodology grounded in the concept of traffic equilibrium, while Sheffi and Powell [3] provided a comprehensive analysis of this issue along with potential remedies. In addition to these fundamental contributions, various alternative methodologies have emerged for predicting User Equilibrium. These include sequential averages ([3], [5]), simulated annealing ([6]), and the application of game theory ([25]). Notably, Javani and Babazadeh [26] introduced an algorithm that identifies descent directions through iterative solutions of a series of quadratic programming subproblems within the framework of truncated quadratic programming. This method, formally known as OD-based Frank-Wolfe truncated quadratic programming, systematically addresses each quadratic programming subproblem associated with Origin-Destination (OD) pairs. These subproblems are efficiently solved using the Frank-Wolfe optimization technique, taking into account only the active paths. The authors introduced a path-based algorithm designed for static traffic assignment problems. This algorithm leverages the Wolfe reduced gradient method, column generation, and speed-up techniques to optimize traffic flow allocation efficiently. It demonstrates superior convergence performance when compared to reference algorithms, as evidenced by testing on scenarios presented in [27].

For an exhaustive compilation of optimization models related to the traffic assignment problem, along with a thorough exposition of the solution methodologies applied to these models, readers are encouraged to consult [22] and [23]. Additionally, for a current overview of research on UE, we recommend referring to [28].

In the context of solving traffic assignment optimization problems subject to specific constraints, the Lagrange function emerges as an invaluable tool for achieving optimality. This specialized function amalgamates the objective function with constraints to maximize the solution while satisfying both the objective and the constraints, constituting what is known as the Lagrange dual problem [22]. By leveraging the gradient of the Lagrange function, a system of differential equations can be derived. Given that the primary purpose of the Lagrange function in network traffic assignment problems is to minimize system variables, the derivatives of these variables with respect to time must exhibit a decreasing trend in the negative direction. Simultaneously, in pursuit of maximizing the Lagrange variables embedded within the Lagrange function, their derivatives must manifest an ascending trajectory in the positive direction. This intricate interplay yields a system of equations that bears a resemblance to an artificial neural network, thus being denoted as a Lagrange programming neural network [17]. This neural network variant finds application in diverse mathematical optimization contexts.

To construct a Lagrange programming neural network (LPNN), a conventional neural network architecture is augmented with a specialized layer responsible for computing the Lagrange multipliers associated with constraints. The mathematical underpinnings of LPNN and its training process through the backpropagation algorithm are expounded upon in this paper. Furthermore, a mathematical rendition of the Lagrange neural network grounded in the duality theorem is presented in [18-20], where the authors furnish empirical evidence substantiating the efficacy of these methodologies in addressing a spectrum of optimization challenges ([19], [20]). In [24], the authors introduce an artificial neural network method tailored for traffic assignment, particularly aimed at optimizing the duration of green signals at traffic intersections, culminating in observed enhancements in traffic flow.

To the best of our knowledge, the utilization of Lagrange neural networks to optimize network traffic assignment problems has not been explored in existing literature. Thus this study centers on addressing a network traffic assignment problem through the application of a neural network approach. Furthermore, it transitions the initially static network traffic assignment problem into a dynamic one, allowing an examination of how traffic flows on connections evolve over time. To establish this dynamic framework, a differential equation system is formulated using the Lagrange function, interpreting the given network traffic assignment problem as a neural network. The stability of this system and its convergence towards user equilibrium are demonstrated with the aid of a Lyapunov function. Lastly, a numerical example is presented to elucidate the solution process.

Here is the structure of the paper: The second section presents the optimization problem concerning network traffic assignment UE, and the Karush-Kuhn-Tucker criteria. Section 3 introduces the Lagrange function and outlines the creation of a Lagrange neural network. A numerical example is given in Section 4, and concluding remarks are provided in Section 5. This text contains instructions for preparing manuscripts for the journal.

2. Mathematical Formulation

2.1. Constrained Nonlinear Optimization for The Network Traffic Assignment Problem

The theory of UE is of utmost importance in transportation engineering. It provides valuable insights into traffic behavior within a network. According to this theory, drivers opt for the most efficient route to reach their destination based on travel time. As more drivers make informed decisions, the network reaches a state of equilibrium where no one can improve their travel time by switching routes. This equilibrium flow is a reliable mechanism for approximating traffic patterns in any given network.

The primary aim is to achieve a balanced distribution of traffic across all edges, denoted as $e \in E$, in a connected directed graph $G = (V, E)$. This distribution is dictated by the traffic demand F^w for each origin-destination (OD) pair $w \in W,$ with the ultimate objective of attaining UE. Under UE conditions, travel times on all utilized routes are equal and do not surpass those of any unused routes.

This model defines the symbol G as a network that comprises a collection of vertices denoted as V and a group of edges identified as E that connect these vertices. The set W includes all origin-destination pairs that exist within the network. Each pair of origin-destination, denoted as $w \in W$ the set R^w comprises all possible routes that connect the origin and destination of that specific pair. The flow of traffic through each edge $e \in E$ is depicted by the vector $x = (\ldots, x_e, \ldots)$ x_e^w, representing the traffic flow through the edge e for the OD pair w. The capacity of each edge $e \in E$ is illustrated by the vector $c = (\ldots, c_e, \ldots)$, whi $t_e(x_e)$ ch is a function that confidently models the travel time (delay) of the flow x_e through a congested edge e.

The problem of traffic assignment is tackled by formulating it as a nonlinear optimization problem that aims to achieve UE ([21]). The problem can be mathematically described as follows:

$$\begin{aligned}
&\min_{x(f)} \sum_{e\in E} \int_0^{x_e(f)} t_e(u)du \\
&\text{subject to} \\
&F^w - \sum_{r\in R^w} f_r^w = 0, \quad \forall w \in W \\
&f_r^w \geq 0, \forall r \in R^w, w \in W
\end{aligned} \tag{1}$$

The problem is accompanied by predefined constraints, where $x_e\left(f_r^w\right) = \sum_{w\in W} \sum_{r\in R^w} f_r^w \delta_{e,r}^w \quad \forall e \in E$.

The relevant variables are:

* f represents the traffic flow on all edges

* $x_e(f)$ denotes the flow-dependent travel time on the edge e

* $t_e(u)$ represents the travel time function on the edge e

* F^w stands for the demand for OD pair w

* R^w is the set of routes for the OD pair w

* f_r^w denotes the flow on route r for the OD pair w

* $\delta_{e,r}^w$ is the Kronecker delta, which is 1 if the edge e is on the route r and 0 otherwise.

The main objective of the problem (1) is to minimize the total travel time (or total travel cost) for all traffic flows. This can be achieved by implementing the first set of constraints, which ensures that the demand for each OD pair is met by the traffic flows on the network. Additionally, the second set of constraints guarantees that the traffic flow on each route is always non-negative. Lastly, the definitional constraints establish a link between the traffic flow on each edge and the traffic flow on each route.

2.2. Karush Kuhn Tucker (KKT) Conditions For The Nonlinear Network Traffic Assignment Problem

It is crucial to abide by the constraints of problem (1) when dealing with traffic assignment problems. To achieve optimization, it is imperative to satisfy the Karush-Kuhn-Tucker (KKT) conditions. These conditions provide a framework for the requirements to attain a feasible solution. If all KKT conditions are met and the optimization problem is convex, then the solution is guaranteed to be optimal. To obtain the Lagrange function for the traffic assignment problem (1), Lagrange multipliers must be added for each constraint of the problem (1) and included in the objective function.

Thus the Lagrangian function, which solves the problem, is expressed below:

$$\mathrm{L}(x(f),\lambda) = \sum_{e\in E} \int_0^{x_e(f)} t_e(u)du + \sum_{w\in W} \lambda_r^w \left(F^w - \sum_{r\in R^w} f_r^w \right) \tag{2}$$

It consists of flow variables $x_e(f)$ and Lagrange multipliers λ_r^w.

The first term in function (2) represents the total travel time on all edges concerning the flow variable f, which is the same as the objective function of problem (1).

The second term includes the Lagrange multiplier associated with demand constraints, ensuring that the total traffic demand between each OD pair is satisfied.

The third term includes the Lagrange multiplier associated with flow constraints, ensuring that the flow on each route is non-negative. The Lagrange multiplier λ_r^w represents both the flow conservation constraint and the non-negativity constraint.

In order to optimize problem (1), we can re-write the function (2) by substituting all $e \in E$ with $x_e = \sum_{w \in W} \sum_{r \in R^w} f_r^w \delta_{e,r}^w$. This leads to the following result:

$$\mathrm{L}(f, \lambda) = \sum_{e \in E} \int_0^{\sum_{w \in W} \sum_{r \in R^w} f_r^w \delta_{e,r}^w} t_e(u) du + \sum_{w \in W} \lambda_r^w \left(F^w - \sum_{r \in R^w} f_r^w \right) \tag{3}$$

We can take the derivative of function (3) with respect to f_r^w using the chain rule, which yields:

$$\frac{\partial \mathrm{L}}{\partial f_r^w} = t_e(x_e) \delta_{e,r}^w - \lambda_r^w. \tag{4}$$

In order to ensure the success of the traffic assignment problem, it is essential that the KKT conditions are satisfied. These conditions encompass the following key points:

1. Primal feasibility: The decision variables must adhere to the constraints of the problem:

$$F^w - \sum_{r \in R^w} f_r^w \geq 0, \quad \forall w \in W, \quad f_r^w \geq 0, \quad \forall r \in R^w, w \in W \tag{5}$$

2. Dual feasibility: Lagrange multipliers must be non-negative:

$$\lambda_r^w \geq 0, \quad \forall r \in R^w, w \in W \tag{6}$$

3. Complementary slackness: The product of decision variables and their corresponding Lagrange multipliers must be zero:

$$\lambda_r^w \left(F^w - \sum_{r \in R^w} f_r^w \right) = 0, \quad \forall w \in W \tag{7}$$

4. Gradient of the Lagrangian (Stationary): The gradient of the Lagrangian function with respect to the decision variable f_r^w must be zero:

$$\frac{\partial \mathrm{L}}{\partial f_r^w} = t_e(x_e) \delta_{e,r}^w - \lambda_r^w = 0 \tag{8}$$

3. Lagrange Neural Network for Traffic Assignment as Nonlinear Optimization

This section will showcase the power of KKT conditions (4)- (8) in transforming the traffic assignment optimization problem (1) into a neural network using the Lagrange function (2). Our approach involves calculating the gradient of the Lagrange function for each variable and solving a set of differential equations using numerical methods. We are confident that we can prove that the equilibrium point of the obtained dynamic system provides the UE for the optimization problem (1).

Now, let's consider the primal variables as f_r^w, and λ_r^w as the Lagrange multipliers. We can express the gradient of the Lagrangian function (2) with respect to each variable, as follows:

$$\nabla_{f_r^w} \mathrm{L}(f, \lambda, \mu) = \frac{\partial \mathrm{L}}{\partial f_r^w} = t_e(x_e) \delta_{e,r}^w - \lambda_r^w \tag{9}$$

$$\nabla_{\lambda_r^w} \mathrm{L}(f, \lambda, \mu) = F^w - \sum_{r \in R^w} f_r^w \tag{10}$$

where ∇ denotes the gradient operator.

At t t he time, $f_r^w(t)$ represents the flow value for the route r and OD pair w. The dynamical system can be expressed using a gradient algorithm.

$$\frac{df_r^w}{dt} = -\frac{\partial \mathrm{L}}{\partial f_r^w} = -\left(t_e(x_e) \delta_{e,r}^w - \lambda_r^w \right) \tag{11}$$

$$\frac{d\lambda_w}{dt} = F^w - \sum_{r\in R^w} f_r^w \tag{12}$$

In order to maintain non-negative variable conditions within the system, we define the flow variable as $\max\{(f,\lambda),0\}$. The first equation (11) outlines the update rule for flow f_r^w, which is obtained by taking the negative derivative of the function (2). The second equation (12) presents the updated rules for the Lagrange multiplier λ_r^w. These dynamic equations are repeatedly applied to adjust the values of f_r^w λ_r^w and until they reach convergence.

Theorem 1: Suppose that (f^*,λ^*) is a stationary point of function (2), and that the Lagrangian function is positive definite and f^* is a regular point of the constrained optimization problem (1). Then, (f^*,λ^*) is an asymptotically stable point of neural network (11)-(12).

Proof: To prove that (f^*,λ^*) is an asymptotically stable point of neural network (11)- (12). We need to demonstrate that the system's dynamics converge to this point $t\to\infty$.

First, we observe that f^* is a stationary point of function (2), which implies (from KKT conditions) that $\nabla_f \mathrm{L}(f^*,\lambda^*)=0$. By substituting the pre-definitional constraint $x_e=\sum_{w\in W}\sum_{r\in R^w} f_r^w\delta_{e,r}^w$, and Lagrange multiplier into equation (8), we obtain the following equation:

$$\begin{aligned}\frac{\partial \mathrm{L}}{\partial f_r^w} &= t_e\left(\sum_{w\in W}\sum_{r\in R^w} f_r^w\delta_{e,r}^w\right)\frac{\partial\left(\sum_{w\in W}\sum_{r\in R^w} f_r^w\delta_{e,r}^w\right)}{\partial f_r^w}-\lambda_r^w \\ &= t_e(\sum_{w\in W}\sum_{r\in R^w} f_r^w\delta_{e,r}^w)\delta_{e,r}^w-\lambda_r^w\end{aligned}$$

By using the equation mentioned earlier, we can find the UE for the optimization problem (1). From the KKT conditions (5)- (8), we obtain $t_e(\sum_{w\in W}\sum_{r\in R^w} f_r^w\delta_{e,r}^w)\delta_{e,r}^w-\lambda_r^w=0 \Rightarrow \lambda_r^w = t_e(\sum_{w\in W}\sum_{r\in R^w} f_r^w\delta_{e,r}^w)\delta_{e,r}^w$. Here, since $f_r^w\ge 0,\ \lambda_r^w\ge 0$ and then, complementary slackness condition from (7) must be either $\lambda_r^w=0$ or $\left(F^w-\sum_{r\in R^w} f_r^w\right)=0$. In this way, the UE of the problem (1) is obtained as:

$$UE=\begin{cases} t_e(\sum_{w\in W}\sum_{r\in R^w} f_r^w\delta_{e,r}^w)\delta_{e,r}^w, & \lambda_r^w=0 \\ t_e(\sum_{w\in W}\sum_{r\in R^w} f_r^w\delta_{e,r}^w)\delta_{e,r}^w-\lambda_r^w, & \lambda_r^w>0\end{cases}$$

If the above KKT conditions for the traffic assignment problem are met by (f^*,λ^*), then the equation provides a valid solution. Moving on to the neural network's dynamics, we can use the update rules to rephrase equations (11) and (12) for f_r^w λ_r^w and. Linearizing the equation system around a point (f^*,λ^*) involves computing the Jacobian matrix.

$$J = \begin{bmatrix} -\frac{\partial}{\partial f_r^w}\left(-\left(t_e(\sum_{w\in W}\sum_{r\in R^w} f_r^w\delta_{e,r}^w)\delta_{e,r}^w - \lambda_r^w\right)\right) & 0 \\ \frac{\partial}{\partial f_r^w}\left(F^w - \sum_{r\in R^w} f_r^w\right) & 0 \end{bmatrix}_{(f^*,\lambda^*)}$$

Since f^* is a regular point of the constrained optimization problem (1), it can be deduced that the Hessian matrix of the Lagrangian is positive definite at f^*. Therefore, it follows that the Jacobian matrix J is negative definite at (f^*,λ^*). By applying the Hartman-Grobman theorem, it can be determined that the system dynamics in the vicinity of (f^*,λ^*) are locally equivalent to those of a linearized system. Given that J is negative definite, the linearized system is asymptotically stable at (f^*,λ^*). Thus, it can be concluded that the original nonlinear system dynamics are also asymptotically stable at (f^*,λ^*). The proof is now complete.

3.1. Stability of The Dynamical System

Lyapunov's method is a powerful tool that allows us to analyze the stability of dynamical systems. However, in order to use this method effectively, it is essential to find an appropriate Lyapunov function, which can be a challenging task. To implement Lyapunov's method and show that it is a reliable tool, we need to find a Lyapunov function $V(f,\lambda)$ that is positive definite and has a negative definite derivative along the trajectories of the dynamical system.

One possible Lyapunov function for neural network (11)- (12) is:

$$V(f,\lambda) = \frac{1}{2}\sum_{w\in W}\sum_{r\in R}\left|\left(\frac{\partial \mathrm{L}(f,\lambda)}{\partial f_r^w}\right)\right|^2 + \frac{1}{2}\left|\left(\sum_{w\in W}\left(\sum_{r\in R^w}(F^w - f_r^w)\right)\right)\right|^2 \tag{13}$$

To demonstrate that $V(f,\lambda)$ is indeed a Lyapunov function, we must verify that it satisfies the two necessary conditions:

1. $V(f,\lambda) \ge 0$ for all (f,λ), and $V(f,\lambda) = 0$ only at (f,λ).
2. $\frac{dV}{dt} \le 0$ for all f λ. and.

It is evident that the first condition is met as both terms $V(f,\lambda)$ are non-negative. Furthermore, $V(f,\lambda) = 0$ only when both terms are zero. The first term equals zero if and only if $\frac{\partial L(f,\lambda)}{\partial f_r^w} = 0$ for all $r \in R$ $w \in W$ and, which indicates that (f,λ) is a stationary point of the system. The second term equals zero if and only if $\sum_{w\in W}\left(\sum_{r\in R^w}(F^w - f_r^w)\right) = 0$, which implies that the flow conservation constraint is met. To establish the second condition, we must differentiate $V(f,\lambda)$ with respect to time and demonstrate that it is negative or zero. We obtain:

$$\frac{dV}{dt} = \sum_{w\in W}\sum_{r\in R}\left(\frac{\partial V}{\partial f_r^w}\frac{df_r^w}{dt}\right) + \frac{\partial V}{\partial \lambda}\frac{d\lambda}{dt} = -\sum_{w\in W}\sum_{r\in R}\left|\frac{\partial L(f,\lambda)}{\partial f_r^w}\right|^2 + \left|\sum_{w\in W}\left(\sum_{r\in R^w}(F^w - f_r^w)\right)\right|^2 \le 0$$

The last inequality follows from the Cauchy-Schwarz inequality, which states that $|\langle u,v\rangle|^2 \leq |u|^2|v|^2$ any vector u v. $V(f,\lambda)$, is a Lyapunov function for the neural network, which implies that the stationary point (f,λ) is stable.

Through the usage of the Cauchy-Schwarz inequality, which guarantees that $|\langle u,v\rangle|^2 \leq |u|^2|v|^2$ for all vectors u v, we can conclusively prove the validity of the aforementioned inequality. This, in turn, confirms that $V(f,\lambda)$ is a Lyapunov function for the neural network, providing affirmation of the stability of the stationary point (f,λ). Therefore, we have successfully demonstrated the effectiveness of Lyapunov's method in evaluating the stability of the given dynamical system. Furthermore, we have established a Lyapunov function that solidifies the stability of the system at the origin.

To solve the above dynamical system using Euler's method, we need to discretize the system in time. Let Δt be the time step. Then, we can approximate the derivatives by finite differences:

$$\frac{df_r^w}{dt} \approx \frac{f_{r,n+1}^w - f_{r,n}^w}{\Delta t}, \qquad \frac{d\lambda_r^w}{dt} \approx \frac{\lambda_{r,n+1}^w - \lambda_{r,n}^w}{\Delta t} \tag{14}$$

where $f_{r,n}^w$, and $\lambda_{r,n}^w$, denote the values of f_r^w $\lambda_{r,n}^w$ and at time step n. Substituting these approximations into the dynamical system, we get:

$$\frac{f_{r,n+1}^w - f_{r,n}^w}{\Delta t} = -\frac{\partial \mathrm{L}}{\partial f_r^w} = -\left[t_e \left(\sum_{w\in W} \sum_{r\in R^w} f_{r,n}^w \delta_{e,r}^w \right) \delta_{e,r}^w - \lambda_{r,n}^w \right] \tag{15}$$

$$\frac{\lambda_{r,n+1}^w - \lambda_{r,n}^w}{\Delta t} = F^w - \sum_{r\in R^w} f_{r,n}^w \tag{16}$$

Solving for $f_{r,n+1}^w$ and $\lambda_{r,n+1}^w$, we get:

$$f_{r,n+1}^w = f_{r,n}^w - \Delta t \left[t_e \left(\sum_{w\in W} \sum_{r\in R^w} f_{r,n}^w \delta_{e,r}^w \right) \delta_{e,r}^w + \lambda_{r,n}^w \right] \tag{17}$$

$$\lambda_{r,n+1}^w = \lambda_{r,n}^w + \Delta t (F^w - \sum_{r\in R^w} f_{r,n}^w) \tag{18}$$

In order to ensure that all variables are non-negative, the system is reformulated as follows:

$$f_{r,n+1}^w = \max \left\{ f_{r,n}^w - \Delta t \left[t_e \left(\sum_{w\in W} \sum_{r\in R^w} f_{r,n}^w \delta_{e,r}^w \right) \delta_{e,r}^w + \lambda_{r,n}^w \right], 0 \right\} \quad 19)$$

$$\lambda_{r,n+1}^w = \max \left\{ \lambda_{r,n}^w + \Delta t (F^w - \sum_{r\in R^w} f_{r,n}^w), 0 \right\} \tag{20}$$

These equations give us a way to update values of f_r^w and λ_r^w at each time step $n+1$ based on their values at the previous time step n. To start the iteration, we need to initialize these variables at the time step $n=0$. Let $f_{r,0}^w$ $\lambda_{r,0}^w$ and denote their initial values. Then, we can use the following algorithm to solve the dynamical system using Euler's method:

1. Set $n=0$ and initialize $f_{r,n}^w = f_{r,0}^w$ $\lambda_{r,n}^w = \lambda_{r,0}^w$, for all $r \in R^w$ and $w \in W$.
2. Set the time step Δt.
3. For $n = 0,1,2,\ldots,$ do the following:
   a. Compute $f_{r,n+1}^w$ and $\lambda_{r,n+1}^w$ use the updated equations above.

b. Set $f_{r,n}^{w} = f_{r,n+1}^{w}$ and $\lambda_{r,n}^{w} = \lambda_{r,n+1}^{w}$ for all $r \in R^{w}$ $w \in W$ and.

c. Increment n and repeat step 3.

This algorithm will iteratively update the values of f_r^w and λ_r^w at each time step until convergence. The convergence criteria can vary depending on the specific problem and implementation, but a common one is to stop the iteration when the change in the objective function value or the norm of the gradient falls below a certain threshold.

4. A Numerical Example

In this section, we provide a numerical example to illustrate the computation of traffic flow within a network, based on demand and travel time functions. Consider the network depicted in Figure 1. The travel time for each edge is defined as a function of the flow on that edge, denoted as x_e given by $t(x_e) = 1 + 0.15x_e$.

We have two OD pairs, namely (1,5) and (2,5), with demand values of $F^{1,5} = 20$ $F^{2,5} = 30$ and respectively. To determine the traffic flow, we must account for all potential routes between these OD pairs. For OD pair (1,5), the set of all feasible routes is defined as $R^{1,5} = [(1,3,5),(1,4,5)]$, and for OD pair (2,5), it is defined as $R^{2,5} = [(2,3,5),(2,4,5)]$. Each route is represented as a sequence of nodes.

Subsequently, we can calculate the objective function for the traffic assignment problem as follows:

$$\begin{aligned}\sum_{e\in E}\int_0^{x_e(f)} t_e(u)du &= \int_0^{x_{(1,3)}(f)} t_{(1,3)}(u)du + \int_0^{x_{(3,5)}(f)} t_{(3,5)}(u)du + \int_0^{x_{(1,4)}(f)} t_{(1,4)}(u)du + \int_0^{x_{(4,5)}(f)} t_{(4,5)}(u)du \\ &+ \int_0^{x_{(2,3)}(f)} t_{(2,3)}(u)du + \int_0^{x_{(2,4)}(f)} t_{(2,4)}(u)du + \int_0^{x_{(3,5)}(f)} t_{(3,5)}(u)du + \int_0^{x_{(4,5)}(f)} t_{(4,5)}(u)du \\ &= \int_0^{x_{(1,3)}(f)} (1+0.15u)du + \int_0^{x_{(3,5)}(f)} (1+0.15u)du + \int_0^{x_{(1,4)}(f)} (1+0.15u)du + \int_0^{x_{(4,5)}(f)} (1+0.15u)du \\ &+ \int_0^{x_{(2,3)}(f)} (1+0.15u)du + \int_0^{x_{(2,4)}(f)} (1+0.15u)du + \int_0^{x_{(3,5)}(f)} (1+0.15u)du + \int_0^{x_{(4,5)}(f)} (1+0.15u)du\end{aligned} \quad (21)$$

where $x_e(f)$ is the flow on edge e, and $t_e(u) = 1 + 0.15u$ is the travel time function for edge e.

For each node w in the network, the total flow into the node must equal the total flow out of the node. In other words, the demand for each OD pair must be met by the traffic flows on the network. The flow conservation constraint is given by:

$$\text{For OD pair } (1,5):\ f_{(1,3,5)}^{(1,5)} + f_{(1,4,5)}^{(1,5)} = 20;\ \text{For OD pair } (2,5):\ f_{(2,3,5)}^{(2,5)} + f_{(2,4,5)}^{(2,5)} = 30$$

The traffic flow on each route must be non-negative. The non-negative flow constraint is given by:

$$f_{(1,3,5)}^{(1,5)} \geq 0,\ f_{(1,4,5)}^{(1,5)} \geq 0,\ f_{(2,3,5)}^{(2,5)} \geq 0,\ f_{(2,4,5)}^{(2,5)} \geq 0$$

To calculate these constraints, we need to determine which routes use each edge. Based on the definitions given in the problem statement, we can obtain that:

- Edge (1,3) is used by route (1,3,5) in $R^{(1,5)}$; $x_{(1,3)} = f_{(1,3,5)}^{(1,5)}$
- Edge (1,4) is used by route (1,4,5) in $R^{(1,5)}$; $x_{(1,4)} = f_{(1,4,5)}^{(1,5)}$.
- Edge (2,3) is used by route (2,3,5) in $R^{(2,5)}$; $x_{(2,3)} = f_{(2,3,5)}^{(2,5)}$.
- Edge (2,4) is used by route (2,4,5) in $R^{(2,5)}$; $x_{(2,4)} = f_{(2,4,5)}^{(2,5)}$.

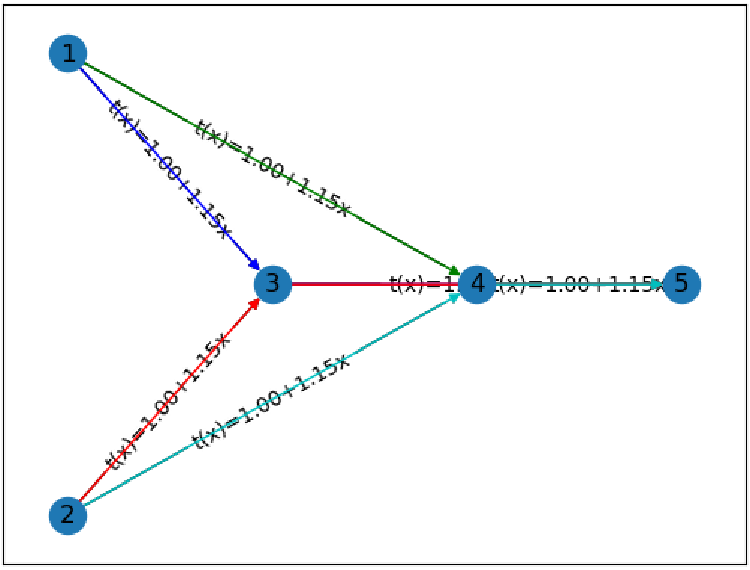

Figure 1. Network for the given problem with routes

Using these values, an optimization problem equivalent to problem (1) can be obtained as follows:

$$\begin{aligned}
\min_{f} \quad & \left(f^{(1,5)}_{(1,3,5)} + f^{(1,5)}_{(1,4,5)} + f^{(2,5)}_{(2,3,5)} + f^{(2,5)}_{(2,4,5)} + 0.075\left((f^{(1,5)}_{(1,3,5)})^2 + (f^{(1,5)}_{(1,4,5)})^2 + (f^{(2,5)}_{(2,3,5)})^2 + (f^{(2,5)}_{(2,4,5)})^2 \right) \right) \\
\text{s.t.} \quad & f^{(1,5)}_{(1,3,5)} + f^{(1,5)}_{(1,4,5)} = 20 \\
& f^{(2,5)}_{(2,3,5)} + f^{(2,5)}_{(2,4,5)} = 30 \\
& f^{(1,5)}_{(1,3,5)} \geq 0,\ f^{(1,5)}_{(1,4,5)} \geq 0,\ f^{(2,5)}_{(2,3,5)} \geq 0,\ f^{(2,5)}_{(2,4,5)} \geq 0
\end{aligned} \tag{22}$$

For this problem, the Lagrange function equivalent to function (2) is obtained as follows:

$$\begin{aligned}
L(f, \lambda_1, \lambda_2) = {} & f^{(1,5)}_{(1,3,5)} + f^{(1,5)}_{(1,4,5)} + f^{(2,5)}_{(2,3,5)} + f^{(2,5)}_{(2,4,5)} + 0.075\left((f^{(1,5)}_{(1,3,5)})^2 + (f^{(1,5)}_{(1,4,5)})^2 + (f^{(2,5)}_{(2,3,5)})^2 + (f^{(2,5)}_{(2,4,5)})^2 \right) \\
& + \lambda_1 (20 - f^{(1,5)}_{(1,3,5)} - f^{(1,5)}_{(1,4,5)}) + \lambda_2 (30 - f^{(2,5)}_{(2,3,5)} - f^{(2,5)}_{(2,4,5)})
\end{aligned} \tag{23}$$

where λ_1 λ_2 and are the Lagrange multipliers.

After calculating the derivative of the Lagrange function with respect to each variable f and λ, we have a dynamical system that includes a system of differential equations that corresponds to equations (11)-(12).

The derivative for function (23) with respect to f and λ are as follows:

$$\begin{aligned}
& \frac{\partial L}{\partial f^{(1,5)}_{(1,3,5)}} = 1 + 0.15 f^{(1,5)}_{(1,3,5)} - \lambda_1 = 0 \qquad && \frac{\partial L}{\partial f^{(1,5)}_{(1,4,5)}} = 1 + 0.15 f^{(1,5)}_{(1,4,5)} - \lambda_1 = 0 \\
& \frac{\partial L}{\partial f^{(2,5)}_{(2,3,5)}} = 1 + 0.15 f^{(2,5)}_{(2,3,5)} - \lambda_2 = 0 \qquad && \frac{\partial L}{\partial f^{(2,5)}_{(2,4,5)}} = 1 + 0.15 f^{(2,5)}_{(2,4,5)} - \lambda_2 = 0 \\
& \frac{\partial L}{\partial \lambda_1} = 20 - f^{(1,5)}_{(1,3,5)} - f^{(1,5)}_{(1,4,5)} = 0 \qquad && \frac{\partial L}{\partial \lambda_2} = 30 - f^{(2,5)}_{(2,3,5)} - f^{(2,5)}_{(2,4,5)} = 0
\end{aligned} \tag{24}$$

Then, the dynamical system is obtained as:

$$\frac{\partial f_{(1,3,5)}^{(1,5)}}{\partial t} = -\left(1+0.15 f_{(1,3,5)}^{(1,5)} - \lambda_1\right) \qquad \frac{\partial f_{(1,4,5)}^{(1,5)}}{\partial t} = -\left(1+0.15 f_{(1,4,5)}^{(1,5)} - \lambda_1\right)$$
$$\frac{\partial f_{(2,3,5)}^{(2,5)}}{\partial t} = -\left(1+0.15 f_{(2,3,5)}^{(2,5)} - \lambda_2\right) \qquad \frac{\partial f_{(2,4,5)}^{(2,5)}}{\partial t} = -\left(1+0.15 f_{(2,4,5)}^{(2,5)} - \lambda_2\right) \tag{25}$$
$$\frac{\partial \lambda_1}{\partial t} = 20 - f_{(1,3,5)}^{(1,5)} - f_{(1,4,5)}^{(1,5)} \qquad \frac{\partial \lambda_2}{\partial t} = 30 - f_{(2,3,5)}^{(2,5)} - f_{(2,4,5)}^{(2,5)}$$

where all variables are accepted as: $\max\left\{\left(f,\lambda\right),0\right\}$.

In order to apply Euler's method (equations (14) to (20)) in the Python 3.0 program, we need to establish the initial conditions and select an appropriate step size.
Let us suppose the initial time points to be denoted as follows:

$$f_{(1,3,5)}^{(1,5)} = 0,\ f_{(1,4,5)}^{(1,5)} = 0,\ f_{(2,3,5)}^{(2,5)} = 0,\ f_{(2,4,5)}^{(2,5)} = 0, \lambda_1 = 0, \lambda_2 = 0$$

A step size of 0.01 may be selected for this computational process. Subsequently, Euler's method can be applied iteratively to update values and Lagrange multipliers. The updated rules are delineated as follows:

$$f_{(1,3,5)}^{(1,5)}(t+\Delta t) = \max\left\{ f_{(1,3,5)}^{(1,5)}(t) - \Delta t \cdot \left(1+0.15 f_{(1,3,5)}^{(1,5)}(t) - \lambda_1\right), 0\right\}$$
$$f_{(1,4,5)}^{(1,5)}(t+\Delta t) = \max\left\{ f_{(1,4,5)}^{(1,5)}(t) + \Delta t \cdot \left(1+0.15 f_{(1,4,5)}^{(1,5)}(t) - \lambda_1\right), 0\right\}$$
$$f_{(2,3,5)}^{(2,5)}(t+\Delta t) = \max\left\{ f_{(2,3,5)}^{(2,5)}(t) + \Delta t \cdot \left(1+0.15 f_{(2,3,5)}^{(2,5)}(t) - \lambda_2\right), 0\right\}$$
$$f_{(2,4,5)}^{(2,5)}(t+\Delta t) = \max\left\{ f_{(2,4,5)}^{(2,5)}(t) - \Delta t \cdot \left(1+0.15 f_{(2,4,5)}^{(2,5)}(t) - \lambda_2\right), 0\right\}$$
$$\lambda_1(t+\Delta t) = \max\left\{\lambda_1(t) + \Delta t \cdot \left(20 - f_{(1,3,5)}^{(1,5)}(t) - f_{(1,4,5)}^{(1,5)}(t)\right), 0\right\}$$
$$\lambda_2(t+\Delta t) = \max\ \left\{\lambda_2(t) + \Delta t \cdot \left(30 - f_{(2,3,5)}^{(2,5)}(t) - f_{(2,4,5)}^{(2,5)}(t)\right), 0\right\}$$

Thus the results obtained after 1000 iterations are as follows: $f_{(1,3,5)}^{(1,5)} = 9.34$, $f_{(1,4,5)}^{(1,5)} = 9.34$, $f_{(2,3,5)}^{(2,5)} = 14.84$, $f_{(2,4,5)}^{(2,5)} = 14.84$, $\lambda_1 = 0.88$, $\lambda_2 = 0.82$.

Accordingly, the time-dependent changes of flows on each route and Lagrange multipliers along the route are presented in Figure II.
In order to calculate the minimum total travel time, we can substitute values of $f_{(1,3,5)}^{(1,5)}$, $f_{(1,4,5)}^{(1,5)}$, $f_{(2,3,5)}^{(2,5)}$, $f_{(2,4,5)}^{(2,5)}$ and into an objective function (22) and then minimize it. That is, by substituting these values into an objective function (22), we have 94.48.

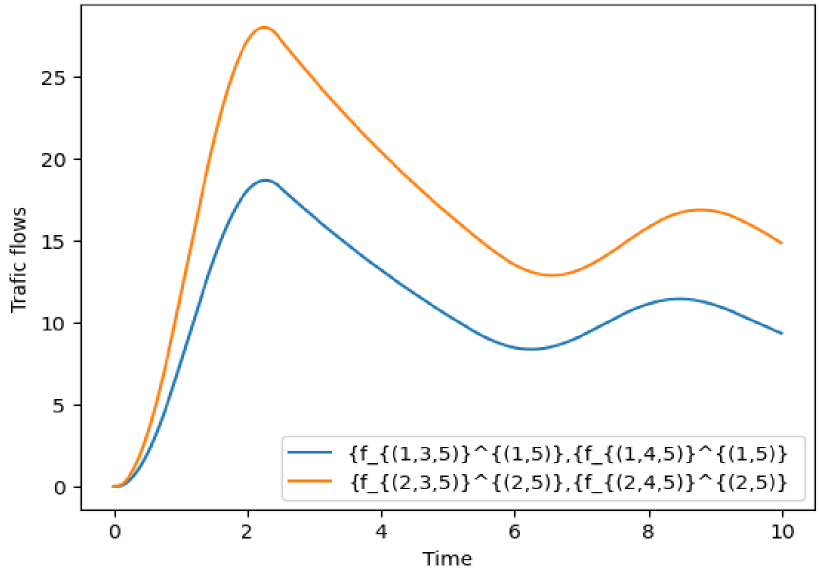

Figure 2. The traffic flow over time

In light of Theorem 1, it has been established that the outcomes derived are representative of the asymptotically stable points within the neural network framework (Equations (11) and (12)). Moreover, these outcomes not only meet the stringent KKT conditions but also fulfill the requirements for the optimization problem outlined in problem (22). Consequently, they constitute the UE of said optimization problem (22).

It is pertinent to note that the utilization of the Frank-Wolfe algorithm yields a solution with a numerical value of 98.75, while the application of the Lagrange neural network model employing the Euler approach results in a solution value of 94.48.

The fundamental goal of this research is to discover the best route preferences for each user in the network. As a result, when the acquired results are compared, it is discovered that the approach provided in this article produces more effective results for the optimization model of classical traffic assignment problems and converges towards UE.

5. Conclusions

This article explored a model for optimizing traffic assignment within a network. It began by establishing the Lagrange function to comprehend the optimal conditions of the static traffic assignment model. The Lagrange function assisted in illustrating the existence of points adhering to the Karush-Kuhn-Tucker conditions. Subsequently, it elucidated how the Lagrange function could be transformed into a dynamic system employing the gradient method. It was mathematically proven that the points derived from this system converged towards the UE points, and further details were provided regarding the asymptotic and Lyapunov stability of the resulting system. A neural network model was also applied to a numerical example, and the Euler method was employed to solve the dynamic system. Finally, the temporal evolution of traffic in the network routes was visually represented using a graphical illustration. Furthermore, the results obtained through the Frank Wolfe algorithm and the methodology presented in this study were compared, with our findings confirming the superior efficacy of the approach proposed herein.

It is important to note that this solution method particularly entails a higher computational workload, as well as a greater demand for storage space, especially in the case of large and intricate networks. However, despite all these associated costs, upon system implementation, it is possible to achieve superior and more reliable results compared to existing methods.

In the future, further research in this field is likely to hold significant potential for advancing our understanding of network traffic allocation and optimization, as well as for developing practical applications. Moreover, this solution process can be seamlessly integrated into network traffic assignment problems, including those that involve analyzing traffic congestion within the network.

Ethical statement:

The author declares that this document does not require ethics committee approval or any special permission. The study does not cause any harm to the environment and does not involve the use of animal or human subjects.

Conflict of interest:

The author claims that he has no conflicts of interest.

Authors' Contributions:

The author made a substantial contribution to the paper's authoring. The final manuscript was read and approved by the author.

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