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Research Article

# Torsion pairs and related modules over trivial ring extensions

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**Abstract**

Let $R \ltimes M$ be a trivial extension of a ring $R$ by an $R$-$R$-bimodule $M$. We first study how to construct torsion pairs over $R \ltimes M$ from torsion pairs over $R$. Some characterizations of finitely generated (presented) modules, flat modules and coherent rings relative to a torsion pair over $R \ltimes M$ are obtained. Then we discuss the transfers of torsion pairs over $R \ltimes M$ to $R$. Finally, some applications are given in Morita context rings.

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**Keywords.** torsion pair, trivial ring extension, Morita context ring

## 1. Introduction

The notion of a torsion pair (torsion theory) was introduced by Dickson in 1966 [2] and is a fundamental topic in ring theory and the representation theory of algebras [18]. Ding and Chen investigated relative flatness of modules and coherence of rings with respect to torsion pairs [3]. Ma and Huang investigated torsion pairs in recollements of abelian categories [10]. Fan and Yao studied the properties of torsion pairs in a triangulated category [5]. Recently, Peng, Ma and Huang described torsion pairs over triangular matrix artin algebras [16].

On the other hand, the notion of a trivial extension of a ring by a bimodule is an important extension of a ring and has played a crucial role in ring theory. Let $R$ be an associative ring and $M$ an $R$-$R$-bimodule, the Cartesian product $R \times M$, with the natural addition and multiplication, given by $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$, becomes a ring. This ring is called the *trivial extension* of the ring $R$ by the bimodule $M$ [6, 17], and denoted by $R \ltimes M$. When $R$ is a commutative ring, this construction is also called *idealization* [14]. The class of trivial ring extensions covers Morita context rings with zero bimodule homomorphisms, particularly, covers formal triangular matrix rings. Many scholars have focused on trivial ring extensions. For example, Palmér and Roos gave some explicit formulae for the global homological dimensions of trivial ring extensions [15]. Fossum, Griffith and Reiten studied the categorical aspect and homological properties of trivial ring extensions [6]. Dumitrescu, Mahdou and Zahir investigated the radical factorization for trivial ring extensions [4].

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The main objective of the present paper is to extend some known results about torsion pairs to a more general setting. We deal with the descent and ascent of torsion pairs between the category of left $R$-modules $R$-Mod and the category of left $R \ltimes M$-modules $R \ltimes M$-Mod.

In Section 2, we recall some basic concepts and facts about trivial ring extensions.

In Section 3, we study how to construct (hereditary) torsion pairs over $R \ltimes M$ from (hereditary) torsion pairs over $R$. It is proven that $(\mathfrak{C}_1, \mathfrak{C}_2)$ is a hereditary torsion pair in $R$-Mod if and only if $(\mathfrak{A}^{\mathfrak{C}_1}, \mathfrak{K}^{\mathfrak{C}_2})$ is a hereditary torsion pair in $R \ltimes M$-Mod (see Theorem 3.3). It is also shown that torsion pairs over $R \ltimes M$ can be created by tilting and cotilting $R$-modules (see Corollary 3.5). In addition, we characterize finitely generated (presented) modules, flat modules and coherent rings relative to a torsion pair over $R \ltimes M$ (see Theorem 3.9).

In Section 4, we investigate how (hereditary) torsion pairs over $R \ltimes M$ induce (hereditary) torsion pairs over $R$. It is proven that, if $(\mathfrak{D}_1, \mathfrak{D}_2)$ is a hereditary torsion pair in $R \ltimes M$-Mod, then $(\mathfrak{Z}^{\mathfrak{D}_1}, \mathfrak{H}^{\mathfrak{D}_2})$ is a hereditary torsion pair in $R$-Mod (see Theorem 4.3).

Section 5 is devoted to torsion pairs over Morita context rings with zero bimodule homomorphisms, which are special examples of trivial ring extensions. We describe explicitly (hereditary) torsion pairs over Morita context rings with zero bimodule homomorphisms (see Theorems 5.1 and 5.2).

## 2. Preliminaries and notations

Throughout this paper, all rings are nonzero associative rings with identity and all modules are unitary. For a ring $R$, $R$-Mod stands for the category of left $R$-modules. $pd(X)$ and $id(X)$ denote the projective and injective dimensions of a module $X$ respectively. All classes of modules are assumed to be closed under isomorphisms and contain 0.

Let $R \ltimes M$ be the trivial extension of a ring $R$ by an $R$-$R$-bimodule $M$. Recall from [6] that the category $R \ltimes M$-Mod is isomorphic to the category $\Xi$ whose objects are couples $(X, f)$ with $X \in R$-Mod and $f \in \mathrm{Hom}_R(M \otimes_R X, X)$ such that the composition $M \otimes_R M \otimes_R X \xrightarrow{M \otimes_R f} M \otimes_R X \xrightarrow{f} X$ is 0 and a morphism $\gamma : (X, f) \to (Y, g)$ is a morphism $\gamma : X \to Y$ in $R$-Mod such that the following diagram commutes.

$$\begin{tikzcd} M \otimes_R X \arrow[r, "M \otimes_R \gamma"] \arrow[d, "f"'] & M \otimes_R Y \arrow[d, "g"] \\ X \arrow[r, "\gamma"] & Y \end{tikzcd}$$

A sequence in $\Xi$ is exact if and only if the sequence of codomains in $R$-Mod is exact.

By the adjointness isomorphism, the category $R \ltimes M$-Mod is also isomorphic to the category $\Upsilon$ whose objects are couples $[X, \alpha]$ with $X \in R$-Mod and $\alpha \in \mathrm{Hom}_R(X, \mathrm{Hom}_R(M, X))$ such that the composition $X \xrightarrow{\alpha} \mathrm{Hom}_R(M, X) \xrightarrow{\mathrm{Hom}_R(M,\alpha)} \mathrm{Hom}_R(M, \mathrm{Hom}_R(M, X))$ is 0 and a morphism $\varphi : [X, \alpha] \to [Y, \beta]$ is a morphism $\varphi : X \to Y$ in $R$-Mod such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ \mathrm{Hom}_R(M, X) \arrow[r, "\mathrm{Hom}_R(M,\varphi)"] & \mathrm{Hom}_R(M, Y) \end{tikzcd}$$

A sequence in $\Upsilon$ is exact if and only if the sequence of domains in $R$-Mod is exact.

There are some important functors as follows.

The functor $\mathbf{T} : R\text{-Mod} \to \Xi$ is given, for every object $X \in R\text{-Mod}$, by $\mathbf{T}(X) = (X \oplus (M \otimes_R X), \mu)$ with $\mu = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : (M \otimes_R X) \oplus (M \otimes_R M \otimes_R X) \to X \oplus (M \otimes_R X)$ and for morphisms by $\mathbf{T}(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & M \otimes_R \alpha \end{pmatrix}$.

The functor $\mathbf{U} : \Xi \to R$-Mod is given, for every object $(X, f) \in \Xi$, by $\mathbf{U}(X, f) = X$ and for morphisms by $\mathbf{U}(\alpha) = \alpha$.

The functor $\mathbf{Z} : R\text{-Mod} \to \Xi$ is given, for every object $X \in R$-Mod, by $\mathbf{Z}(X) = (X, 0)$ and for morphisms by $\mathbf{Z}(\alpha) = \alpha$.

The functor $\mathbf{C} : \Xi \to R$-Mod is given, for every object $(X, f) \in \Xi$, by $\mathbf{C}(X, f) = \operatorname{coker}(f)$ and for morphisms by $\mathbf{C}(\alpha) =$ the induced morphism.

The functor $\mathbf{H} : R\text{-Mod} \to \Upsilon$ is given, for every object $X \in R$-Mod, by $\mathbf{H}(X) = [\operatorname{Hom}_R(M, X) \oplus X, \vartheta]$ with $\vartheta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : \operatorname{Hom}_R(M, X) \oplus X \to \operatorname{Hom}_R(M, \operatorname{Hom}_R(M, X)) \oplus \operatorname{Hom}_R(M, X)$ and for morphisms by $\mathbf{H}(\beta) = \begin{pmatrix} \operatorname{Hom}_R(M, \beta) & 0 \\ 0 & \beta \end{pmatrix}$.

The functor $\mathbf{U} : \Upsilon \to R$-Mod is given, for every object $[X, g] \in \Upsilon$, by $\mathbf{U}[X, g] = X$ and for morphisms by $\mathbf{U}(\alpha) = \alpha$.

The functor $\mathbf{Z} : R\text{-Mod} \to \Upsilon$ is given, for every object $X \in R$-Mod, by $\mathbf{Z}(X) = [X, 0]$ and for morphisms by $\mathbf{Z}(\alpha) = \alpha$.

The functor $\mathbf{K} : \Upsilon \to R$-Mod is given, for every object $[X, g] \in \Upsilon$, by $\mathbf{K}[X, g] = \ker(g)$ and for morphisms by $\mathbf{K}(\alpha) =$ the induced morphism.

We note that the functors $\mathbf{T}$ and $\mathbf{C}$ are right exact, $\mathbf{H}$ and $\mathbf{K}$ are left exact, $\mathbf{U}$ and $\mathbf{Z}$ are exact. There exist important pairs of adjoint functors $(\mathbf{T}, \mathbf{U})$, $(\mathbf{C}, \mathbf{Z})$, $(\mathbf{Z}, \mathbf{K})$ and $(\mathbf{U}, \mathbf{H})$ for which $\mathbf{CT} = id_{R-\text{Mod}}$, $\mathbf{UZ} = id_{R-\text{Mod}}$ and $\mathbf{KH} = id_{R-\text{Mod}}$.

$$R\text{–Mod} \underset{\mathbf{U}}{\overset{\mathbf{T}}{\rightleftarrows}} \Xi \underset{\mathbf{Z}}{\overset{\mathbf{C}}{\rightleftarrows}} R\text{–Mod}, \qquad R\text{–Mod} \underset{\mathbf{K}}{\overset{\mathbf{Z}}{\rightleftarrows}} \Upsilon \underset{\mathbf{H}}{\overset{\mathbf{U}}{\rightleftarrows}} R\text{–Mod},$$

In the rest of the paper, we always identify $R \ltimes M$-Mod with $\Xi$ and $\Upsilon$.

## 3. Transfers of torsion pairs over $R$ to $R \ltimes M$

Let $R \ltimes M$ be a trivial extension of a ring $R$ by an $R$-$R$-bimodule $M$ and $\mathcal{C}$ a class of left $R$-modules. We write

$\mathcal{C}^{\perp_0} = \{L \in R\text{-Mod: } \operatorname{Hom}_R(C, L) = 0 \text{ for all } C \in \mathcal{C}\}$,
${}^{\perp_0}\mathcal{C} = \{L \in R\text{-Mod: } \operatorname{Hom}_R(L, C) = 0 \text{ for all } C \in \mathcal{C}\}$,
$\mathbf{T}(\mathcal{C}) = \{\mathbf{T}(C) \in R \ltimes M\text{-Mod: } C \in \mathcal{C}\}$,
$\mathbf{H}(\mathcal{C}) = \{\mathbf{H}(C) \in R \ltimes M\text{-Mod: } C \in \mathcal{C}\}$,
$\mathbf{Z}(\mathcal{C}) = \{\mathbf{Z}(C) \in R \ltimes M\text{-Mod: } C \in \mathcal{C}\}$,
$\mathfrak{A}^{\mathcal{C}} = \{(X, \alpha) \in R \ltimes M\text{-Mod: } X \in \mathcal{C}\} = \{[Y, \beta] \in R \ltimes M\text{-Mod: } Y \in \mathcal{C}\}$,
$\mathfrak{L}^{\mathcal{C}} = \{(X, \alpha) \in R \ltimes M\text{-Mod: } \operatorname{coker}(\alpha) \in \mathcal{C}\}$,
$\mathfrak{K}^{\mathcal{C}} = \{[Y, \beta] \in R \ltimes M\text{-Mod: } \ker(\beta) \in \mathcal{C}\}$.

Clearly, $\mathcal{C} \subseteq {}^{\perp_0}(\mathcal{C}^{\perp_0})$, $\mathcal{C} \subseteq ({}^{\perp_0}\mathcal{C})^{\perp_0}$, $\mathcal{C}^{\perp_0} = ({}^{\perp_0}(\mathcal{C}^{\perp_0}))^{\perp_0}$, ${}^{\perp_0}\mathcal{C} = {}^{\perp_0}(({}^{\perp_0}\mathcal{C})^{\perp_0})$, $\mathbf{T}(\mathcal{C}) \subseteq \mathfrak{L}^{\mathcal{C}}$ and $\mathbf{H}(\mathcal{C}) \subseteq \mathfrak{K}^{\mathcal{C}}$.

**Lemma 3.1.** *Let $\mathcal{C}$ be a class of left $R$-modules. Then*

(1) ${}^{\perp_0}\boldsymbol{Z}(\mathcal{C}) = \mathfrak{L}^{{}^{\perp_0}\mathcal{C}}$.
(2) $\boldsymbol{Z}(\mathcal{C})^{\perp_0} = \mathfrak{K}^{\mathcal{C}^{\perp_0}}$.
(3) ${}^{\perp_0}\boldsymbol{H}(\mathcal{C}) = \mathfrak{A}^{{}^{\perp_0}\mathcal{C}}$.
(4) $\boldsymbol{T}(\mathcal{C})^{\perp_0} = \mathfrak{A}^{\mathcal{C}^{\perp_0}}$.

***Proof.*** (1) Let $(N,g) \in {}^{\perp_0}\mathbf{Z}(\mathcal{C})$ and $C \in \mathcal{C}$. Then

$$\mathrm{Hom}_R(\mathrm{coker}(g), C) \cong \mathrm{Hom}_{R\ltimes M}((N,g), \mathbf{Z}(C)) = 0.$$

Hence $\mathrm{coker}(g) \in {}^{\perp_0}\mathcal{C}$. Thus $(N,g) \in \mathfrak{L}^{{}^{\perp_0}\mathcal{C}}$ and so ${}^{\perp_0}\mathbf{Z}(\mathcal{C}) \subseteq \mathfrak{L}^{{}^{\perp_0}\mathcal{C}}$.

Conversely, let $(X,f) \in \mathfrak{L}^{{}^{\perp_0}\mathcal{C}}$ and $C \in \mathcal{C}$. Then

$$\mathrm{Hom}_{R\ltimes M}((X,f), \mathbf{Z}(C)) \cong \mathrm{Hom}_R(\mathrm{coker}(f), C) = 0.$$

Thus $(X,f) \in {}^{\perp_0}\mathbf{Z}(\mathcal{C})$ and so $\mathfrak{L}^{{}^{\perp_0}\mathcal{C}} \subseteq {}^{\perp_0}\mathbf{Z}(\mathcal{C})$. Hence ${}^{\perp_0}\mathbf{Z}(\mathcal{C}) = \mathfrak{L}^{{}^{\perp_0}\mathcal{C}}$.

(2) Let $[N,g] \in \mathbf{Z}(\mathcal{C})^{\perp_0}$ and $C \in \mathcal{C}$. Then

$$\mathrm{Hom}_R(C, \ker(g)) \cong \mathrm{Hom}_{R\ltimes M}(\mathbf{Z}(C), [N,g]) = 0.$$

Hence $\ker(g) \in \mathcal{C}^{\perp_0}$. Thus $[N,g] \in \mathfrak{K}^{\mathcal{C}^{\perp_0}}$ and so $\mathbf{Z}(\mathcal{C})^{\perp_0} \subseteq \mathfrak{K}^{\mathcal{C}^{\perp_0}}$.

Conversely, let $[Y,\beta] \in \mathfrak{K}^{\mathcal{C}^{\perp_0}}$ and $C \in \mathcal{C}$. Then

$$\mathrm{Hom}_{R\ltimes M}(\mathbf{Z}(C), [Y,\beta]) \cong \mathrm{Hom}_R(C, \ker(\beta)) = 0.$$

Thus $[Y,\beta] \in \mathbf{Z}(\mathcal{C})^{\perp_0}$ and so $\mathfrak{K}^{\mathcal{C}^{\perp_0}} \subseteq \mathbf{Z}(\mathcal{C})^{\perp_0}$. Hence $\mathbf{Z}(\mathcal{C})^{\perp_0} = \mathfrak{K}^{\mathcal{C}^{\perp_0}}$.

(3) Let $(Y,\beta) \in {}^{\perp_0}\mathbf{H}(\mathcal{C})$ and $C \in \mathcal{C}$. Then

$$\mathrm{Hom}_R(Y, C) \cong \mathrm{Hom}_{R\ltimes M}((Y,\beta), \mathbf{H}(C)) = 0.$$

So $Y \in {}^{\perp_0}\mathcal{C}$. Thus $(Y,\beta) \in \mathfrak{A}^{{}^{\perp_0}\mathcal{C}}$. Hence ${}^{\perp_0}\mathbf{H}(\mathcal{C}) \subseteq \mathfrak{A}^{{}^{\perp_0}\mathcal{C}}$.

Conversely, let $(X,\alpha) \in \mathfrak{A}^{{}^{\perp_0}\mathcal{C}}$ and $C \in \mathcal{C}$. Then

$$\mathrm{Hom}_{R\ltimes M}((X,\alpha), \mathbf{H}(C)) \cong \mathrm{Hom}_R(X, C) = 0.$$

Hence $(X,\alpha) \in {}^{\perp_0}\mathbf{H}(\mathcal{C})$. Thus $\mathfrak{A}^{{}^{\perp_0}\mathcal{C}} \subseteq {}^{\perp_0}\mathbf{H}(\mathcal{C})$. So ${}^{\perp_0}\mathbf{H}(\mathcal{C}) = \mathfrak{A}^{{}^{\perp_0}\mathcal{C}}$.

(4) Let $(N,g) \in \mathbf{T}(\mathcal{C})^{\perp_0}$ and $C \in \mathcal{C}$. Then

$$\mathrm{Hom}_R(C, N) \cong \mathrm{Hom}_{R\ltimes M}(\mathbf{T}(C), (N,g)) = 0.$$

Hence $N \in \mathcal{C}^{\perp_0}$. Thus $(N,g) \in \mathfrak{A}^{\mathcal{C}^{\perp_0}}$ and so $\mathbf{T}(\mathcal{C})^{\perp_0} \subseteq \mathfrak{A}^{\mathcal{C}^{\perp_0}}$.

Conversely, let $(X,\alpha) \in \mathfrak{A}^{\mathcal{C}^{\perp_0}}$ and $C \in \mathcal{C}$. Then

$$\mathrm{Hom}_{R\ltimes M}(\mathbf{T}(C), (X,\alpha)) \cong \mathrm{Hom}_R(C, X) = 0.$$

So $(X,\alpha) \in \mathbf{T}(\mathcal{C})^{\perp_0}$. Thus $\mathfrak{A}^{\mathcal{C}^{\perp_0}} \subseteq \mathbf{T}(\mathcal{C})^{\perp_0}$. Hence $\mathbf{T}(\mathcal{C})^{\perp_0} = \mathfrak{A}^{\mathcal{C}^{\perp_0}}$. $\square$

Recall that a pair $(\mathcal{C}_1, \mathcal{C}_2)$ of classes of left $R$-modules is a *torsion pair* (*torsion theory*) if $\mathcal{C}_1^{\perp_0} = \mathcal{C}_2$ and $\mathcal{C}_1 = {}^{\perp_0}\mathcal{C}_2$. In the situation, $\mathcal{C}_1$ is called the *torsion class* and $\mathcal{C}_2$ is called the *torsionfree class*. It is known that a class of left $R$-modules is a torsion class of some torsion pair if and only if it is closed under extensions, direct sums and quotients; a class of left $R$-modules is a torsionfree class of some torsion pair if and only if it is closed under extensions, direct products and submodules.

The following theorem shows that trivial ring extensions can produce rich torsion pairs.

**Theorem 3.2.** *Let $(\mathcal{C}_1, \mathcal{C}_2)$ be a torsion pair in $R$-Mod. Then*

(1) $(\mathfrak{L}^{\mathcal{C}_1}, (\mathfrak{L}^{\mathcal{C}_1})^{\perp_0})$ *is a torsion pair in $R \ltimes M$-Mod.*
(2) $({}^{\perp_0}(\mathfrak{K}^{\mathcal{C}_2}), \mathfrak{K}^{\mathcal{C}_2})$ *is a torsion pair in $R \ltimes M$-Mod.*
(3) $(\mathfrak{A}^{\mathcal{C}_1}, (\mathfrak{A}^{\mathcal{C}_1})^{\perp_0})$ *is a torsion pair in $R \ltimes M$-Mod.*
(4) $({}^{\perp_0}(\mathfrak{A}^{\mathcal{C}_2}), \mathfrak{A}^{\mathcal{C}_2})$ *is a torsion pair in $R \ltimes M$-Mod.*

***Proof.*** (1) By Lemma 3.1(1), we have

$${}^{\perp_0}((\mathfrak{L}^{\mathcal{C}_1})^{\perp_0}) = {}^{\perp_0}((\mathfrak{L}^{{}^{\perp_0}\mathcal{C}_2})^{\perp_0}) = {}^{\perp_0}(({}^{\perp_0}\mathbf{Z}(\mathcal{C}_2))^{\perp_0}) = {}^{\perp_0}\mathbf{Z}(\mathcal{C}_2) = \mathfrak{L}^{{}^{\perp_0}\mathcal{C}_2} = \mathfrak{L}^{\mathcal{C}_1}.$$

Therefore $(\mathfrak{L}^{\mathcal{C}_1}, (\mathfrak{L}^{\mathcal{C}_1})^{\perp_0})$ is a torsion pair in $R \ltimes M$-Mod.

(2) By Lemma 3.1(2), we have

$$({}^{\perp_0}(\mathfrak{K}^{\mathcal{C}_2}))^{\perp_0} = ({}^{\perp_0}(\mathfrak{K}^{\mathcal{C}_1^{\perp_0}}))^{\perp_0} = ({}^{\perp_0}(\mathbf{Z}(\mathcal{C}_1)^{\perp_0}))^{\perp_0} = \mathbf{Z}(\mathcal{C}_1)^{\perp_0} = \mathfrak{K}^{\mathcal{C}_1^{\perp_0}} = \mathfrak{K}^{\mathcal{C}_2}.$$

Therefore $({}^{\perp_0}(\mathfrak{K}^{\mathcal{C}_2}), \mathfrak{K}^{\mathcal{C}_2})$ is a torsion pair in $R \ltimes M$-Mod.

(3) By Lemma 3.1(3), we have

$$^{\perp_0}((\mathfrak{A}^{\mathcal{C}_1})^{\perp_0}) = {}^{\perp_0}((\mathfrak{A}^{{}^{\perp_0}\mathcal{C}_2})^{\perp_0}) = {}^{\perp_0}(({}^{\perp_0}\mathbf{H}(\mathcal{C}_2))^{\perp_0}) = {}^{\perp_0}\mathbf{H}(\mathcal{C}_2) = \mathfrak{A}^{{}^{\perp_0}\mathcal{C}_2} = \mathfrak{A}^{\mathcal{C}_1}.$$

Therefore $(\mathfrak{A}^{\mathcal{C}_1}, (\mathfrak{A}^{\mathcal{C}_1})^{\perp_0})$ is a torsion pair in $R \ltimes M$-Mod.

(4) By Lemma 3.1(4), we have

$$({}^{\perp_0}(\mathfrak{A}^{\mathcal{C}_2}))^{\perp_0} = ({}^{\perp_0}(\mathfrak{A}^{\mathcal{C}_1^{\perp_0}}))^{\perp_0} = ({}^{\perp_0}(\mathbf{T}(\mathcal{C}_1)^{\perp_0}))^{\perp_0} = \mathbf{T}(\mathcal{C}_1)^{\perp_0} = \mathfrak{A}^{\mathcal{C}_1^{\perp_0}} = \mathfrak{A}^{\mathcal{C}_2}.$$

Therefore $({}^{\perp_0}(\mathfrak{A}^{\mathcal{C}_2}), \mathfrak{A}^{\mathcal{C}_2})$ is a torsion pair in $R \ltimes M$-Mod. □

Recall that a torsion pair $(\mathcal{C}_1, \mathcal{C}_2)$ is *hereditary* (resp. *cohereditary*) if $\mathcal{C}_1$ is closed under submodules (resp. $\mathcal{C}_2$ is closed under quotients).

**Theorem 3.3.** *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two classes of left $R$-modules. Then*

(1) *$(\mathcal{C}_1, \mathcal{C}_2)$ is a hereditary torsion pair in $R$-Mod if and only if $(\mathfrak{A}^{\mathcal{C}_1}, \mathfrak{K}^{\mathcal{C}_2})$ is a hereditary torsion pair in $R \ltimes M$-Mod.*

(2) *$(\mathcal{C}_1, \mathcal{C}_2)$ is a cohereditary torsion pair in $R$-Mod if and only if $(\mathfrak{L}^{\mathcal{C}_1}, \mathfrak{A}^{\mathcal{C}_2})$ is a cohereditary torsion pair in $R \ltimes M$-Mod.*

***Proof.*** (1) " ⇒ " By Theorem 3.2(3), $(\mathfrak{A}^{\mathcal{C}_1}, (\mathfrak{A}^{\mathcal{C}_1})^{\perp_0})$ is a torsion pair in $R \ltimes M$-Mod.

Let $[Y, \beta] \in \mathfrak{A}^{\mathcal{C}_1}$ and $[X, \alpha] \in \mathfrak{K}^{\mathcal{C}_2}$. Then $Y \in \mathcal{C}_1$ and $\ker(\alpha) \in \mathcal{C}_2$. Since $(\mathcal{C}_1, \mathcal{C}_2)$ is a hereditary torsion pair, we have $\mathrm{im}(\beta) \in \mathcal{C}_1$ and $\ker(\beta) \in \mathcal{C}_1$.

There exists an exact sequence in $R \ltimes M$-Mod

$$0 \to \mathbf{Z}(\ker(\beta)) \to [Y, \beta] \to \mathbf{Z}(\mathrm{im}(\beta)) \to 0,$$

which induces the exact sequence

$$0 \to \mathrm{Hom}_{R\ltimes M}(\mathbf{Z}(\mathrm{im}(\beta)), [X, \alpha]) \to \mathrm{Hom}_{R\ltimes M}([Y, \beta], [X, \alpha]) \to \mathrm{Hom}_{R\ltimes M}(\mathbf{Z}(\ker(\beta)), [X, \alpha]).$$

Note that

$$\mathrm{Hom}_{R\ltimes M}(\mathbf{Z}(\mathrm{im}(\beta)), [X, \alpha]) \cong \mathrm{Hom}_R(\mathrm{im}(\beta), \ker(\alpha)) = 0$$

and

$$\mathrm{Hom}_{R\ltimes M}(\mathbf{Z}(\ker(\beta)), [X, \alpha]) \cong \mathrm{Hom}_R(\ker(\beta), \ker(\alpha)) = 0.$$

So $\mathrm{Hom}_{R\ltimes M}([Y, \beta], [X, \alpha]) = 0$. Thus $\mathfrak{K}^{\mathcal{C}_2} \subseteq (\mathfrak{A}^{\mathcal{C}_1})^{\perp_0}$.

Next let $[N, \gamma] \in (\mathfrak{A}^{\mathcal{C}_1})^{\perp_0}$ and $C_1 \in \mathcal{C}_1$. Then

$$\mathrm{Hom}_{R\ltimes M}(C_1, \ker(\gamma)) \cong \mathrm{Hom}_R(\mathbf{Z}(C_1), [N, \gamma]) = 0.$$

So $\ker(\gamma) \in \mathcal{C}_1^{\perp_0} = \mathcal{C}_2$. Thus $(\mathfrak{A}^{\mathcal{C}_1})^{\perp_0} \subseteq \mathfrak{K}^{\mathcal{C}_2}$. Therefore $(\mathfrak{A}^{\mathcal{C}_1})^{\perp_0} = \mathfrak{K}^{\mathcal{C}_2}$.

Since $\mathcal{C}_1$ is closed under submodules, $\mathfrak{A}^{\mathcal{C}_1}$ is closed under submodules. It follows that $(\mathfrak{A}^{\mathcal{C}_1}, \mathfrak{K}^{\mathcal{C}_2})$ is a hereditary torsion pair in $R \ltimes M$-Mod.

" ⇐ " Since $\mathfrak{A}^{\mathcal{C}_1}$ is closed under extensions, direct sums, quotients and submodules, we have that $\mathcal{C}_1$ is closed under extensions, direct sums, quotients and submodules.

Let $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$. Then

$$\mathrm{Hom}_R(C_1, C_2) \cong \mathrm{Hom}_{R\ltimes M}(\mathbf{Z}(C_1), \mathbf{H}(C_2)) = 0.$$

Hence $C_2 \in \mathcal{C}_1^{\perp_0}$ and so $\mathcal{C}_2 \subseteq \mathcal{C}_1^{\perp_0}$.

Let $W \in \mathcal{C}_1^{\perp_0}$ and $[Y, \beta] \in \mathfrak{A}^{\mathcal{C}_1}$. Then

$$\mathrm{Hom}_{R\ltimes M}([Y, \beta], \mathbf{H}(W)) \cong \mathrm{Hom}_R(Y, W) = 0.$$

Hence $\mathbf{H}(W) \in (\mathfrak{A}^{\mathcal{C}_1})^{\perp_0} = \mathfrak{K}^{\mathcal{C}_2}$ and so $W \in \mathcal{C}_2$. Thus $\mathcal{C}_1^{\perp_0} \subseteq \mathcal{C}_2$. Therefore $\mathcal{C}_1^{\perp_0} = \mathcal{C}_2$.

It follows that $(\mathcal{C}_1, \mathcal{C}_2)$ is a hereditary torsion pair in $R$-Mod.

(2) " ⇒ " By Theorem 3.2(4), $({}^{\perp_0}(\mathfrak{A}^{\mathcal{C}_2}), \mathfrak{A}^{\mathcal{C}_2})$ is a torsion pair in $R \ltimes M$-Mod.

Let $(Y, \beta) \in \mathfrak{A}^{\mathcal{C}_2}$ and $(X, \alpha) \in \mathfrak{L}^{\mathcal{C}_1}$. Then $\mathrm{coker}(\alpha) \in \mathcal{C}_1$ and $Y \in \mathcal{C}_2$. Since $(\mathcal{C}_1, \mathcal{C}_2)$ is a cohereditary torsion pair, we have $\mathrm{im}(\beta) \in \mathcal{C}_2$ and $\mathrm{coker}(\beta) \in \mathcal{C}_2$.

There exists an exact sequence in $R\ltimes M$-Mod

$$0\to \mathbf{Z}(\mathrm{im}(\beta))\to (Y,\beta)\to \mathbf{Z}(\mathrm{coker}(\beta))\to 0,$$

which induces the exact sequence

$$0\to \mathrm{Hom}_{R\ltimes M}((X,\alpha),\mathbf{Z}(\mathrm{im}(\beta)))\to \mathrm{Hom}_{R\ltimes M}((X,\alpha),(Y,\beta))\to \mathrm{Hom}_{R\ltimes M}((X,\alpha),\mathbf{Z}(\mathrm{coker}(\beta))).$$

Note that

$$\mathrm{Hom}_{R\ltimes M}((X,\alpha),\mathbf{Z}(\mathrm{im}(\beta)))\cong \mathrm{Hom}_R(\mathrm{coker}(\alpha),\mathrm{im}(\beta))=0$$

and

$$\mathrm{Hom}_{R\ltimes M}((X,\alpha),\mathbf{Z}(\mathrm{coker}(\beta)))\cong \mathrm{Hom}_R(\mathrm{coker}(\alpha),\mathrm{coker}(\beta))=0.$$

So $\mathrm{Hom}_{R\ltimes M}((X,\alpha),(Y,\beta))=0$. Thus $\mathfrak{L}^{\mathcal{C}_1}\subseteq {}^{\perp_0}(\mathfrak{A}^{\mathcal{C}_2})$.

Next let $(N,\gamma)\in {}^{\perp_0}(\mathfrak{A}^{\mathcal{C}_2})$ and $C_2\in\mathcal{C}_2$. Then

$$\mathrm{Hom}_R(\mathrm{coker}(\gamma),C_2)\cong \mathrm{Hom}_{R\ltimes M}((N,\gamma),\mathbf{Z}(C_2))=0.$$

So $\mathrm{coker}(\gamma)\in {}^{\perp_0}\mathcal{C}_2=\mathcal{C}_1$. Thus ${}^{\perp_0}(\mathfrak{A}^{\mathcal{C}_2})\subseteq\mathfrak{L}^{\mathcal{C}_1}$. Consequently ${}^{\perp_0}(\mathfrak{A}^{\mathcal{C}_2})=\mathfrak{L}^{\mathcal{C}_1}$.

Since $\mathcal{C}_2$ is closed under quotients, $\mathfrak{A}^{\mathcal{C}_2}$ is closed under quotients. So $(\mathfrak{L}^{\mathcal{C}_1},\mathfrak{A}^{\mathcal{C}_2})$ is a cohereditary torsion pair in $R\ltimes M$-Mod.

"$\Leftarrow$" Since $\mathfrak{A}^{\mathcal{C}_2}$ is closed under extensions, direct products, submodules and quotients, we have that $\mathcal{C}_2$ is closed under extensions, direct products, submodules and quotients.

Let $C_1\in\mathcal{C}_1$ and $C_2\in\mathcal{C}_2$. Then

$$\mathrm{Hom}_R(C_1,C_2)\cong \mathrm{Hom}_{R\ltimes M}(\mathbf{T}(C_1),\mathbf{Z}(C_2))=0.$$

Thus $C_1\in {}^{\perp_0}\mathcal{C}_2$ and so $\mathcal{C}_1\subseteq {}^{\perp_0}\mathcal{C}_2$.

Let $W\in {}^{\perp_0}\mathcal{C}_2$ and $(Y,\beta)\in\mathfrak{A}^{\mathcal{C}_2}$. Then

$$\mathrm{Hom}_{R\ltimes M}(\mathbf{T}(W),(Y,\beta))\cong \mathrm{Hom}_R(W,Y)=0.$$

Hence $\mathbf{T}(W)\in {}^{\perp_0}(\mathfrak{A}^{\mathcal{C}_2})=\mathfrak{L}^{\mathcal{C}_1}$ and so $W\in\mathcal{C}_1$. Thus ${}^{\perp_0}\mathcal{C}_2\subseteq\mathcal{C}_1$. Therefore ${}^{\perp_0}\mathcal{C}_2=\mathcal{C}_1$.

It follows that $(\mathcal{C}_1,\mathcal{C}_2)$ is a cohereditary torsion pair in $R$-Mod. $\square$

Next, we provide a way to construct torsion pairs over $R\ltimes M$ by some $R$-module.

Let $X$ be a left $R$-module. Write $X^{\perp_1}=\{L\in R\text{-Mod: }\mathrm{Ext}^1_R(X,L)=0\}$ and ${}^{\perp_1}X=\{L\in R\text{-Mod: }\mathrm{Ext}^1_R(L,X)=0\}$, $\mathrm{Gen}(X)=$ the class consisting of quotients of direct sums of copies of $X$ and $\mathrm{Cogen}(X)=$ the class consisting of submodules of direct products of copies of $X$.

According to [7], $X$ is called *tilting* if $\mathrm{Gen}(X)=X^{\perp_1}$, equivalently, if $pd(X)\leq 1$, $\mathrm{Ext}^1_R(X,X^{(\kappa)})=0$ for each cardinal $\kappa$ and $X^{\perp_1}\bigcap X^{\perp_0}=\{0\}$. $X$ is called *cotilting* if $\mathrm{Cogen}(X)={}^{\perp_1}X$, equivalently, if $id(X)\leq 1$, $\mathrm{Ext}^1_R(X^\kappa,X)=0$ for each cardinal $\kappa$ and ${}^{\perp_0}X\bigcap{}^{\perp_1}X=\{0\}$.

**Lemma 3.4.** *Let $X$ and $Y$ be left $R$-modules. Then*

(1) *$X$ is a tilting left $R$-module if and only if $(X^{\perp_1},X^{\perp_0})$ is a torsion pair in $R$-Mod.*
(2) *$Y$ is a cotilting left $R$-module if and only if $({}^{\perp_0}Y,\ {}^{\perp_1}Y)$ is a torsion pair in $R$-Mod.*

***Proof.*** (1) If $X$ is tilting, then it is clear that $(X^{\perp_1},X^{\perp_0})=(\mathrm{Gen}(X),X^{\perp_0})=(\mathrm{Gen}(X),\mathrm{Gen}(X)^{\perp_0})$ is a torsion pair in $R$-Mod (see [7, p. 226]).

Conversely, if $(X^{\perp_1},X^{\perp_0})$ is a torsion pair in $R$-Mod, then $pd(X)\leq 1$ since $X^{\perp_1}$ is closed under quotients. It is easy to see that $\mathrm{Ext}^1_R(X,X^{(\kappa)})=0$ for each cardinal $\kappa$ and $X^{\perp_1}\bigcap X^{\perp_0}=\{0\}$. So $X$ is a tilting left $R$-module.

The proof of (2) is dual. $\square$

**Corollary 3.5.** *Let $X$ and $Y$ be left $R$-modules.*

(1) *If $X$ is tilting, then $(\mathfrak{L}^{\mathrm{Gen}(X)},(\mathfrak{L}^{\mathrm{Gen}(X)})^{\perp_0})$ and $(\mathfrak{A}^{\mathrm{Gen}(X)},(\mathfrak{A}^{\mathrm{Gen}(X)})^{\perp_0})$ are torsion pairs in $R\ltimes M$-Mod.*

(2) *If $Y$ is cotilting, then $({}^{\perp_0}(\mathfrak{K}^{\mathrm{Cogen}(Y)}), \mathfrak{K}^{\mathrm{Cogen}(Y)})$ and $({}^{\perp_0}(\mathfrak{A}^{\mathrm{Cogen}(Y)}), \mathfrak{A}^{\mathrm{Cogen}(Y)})$ are torsion pairs in $R \ltimes M$-Mod.*

***Proof.*** It is an immediate consequence of Theorem 3.2 and Lemma 3.4. □

**Proposition 3.6.** *Let $X$ be a left $R$-module such that $\mathrm{Tor}_i^R(M, X) = 0$ for $i = 1, 2$, $Y$ a left $R$-module such that $\mathrm{Ext}_R^i(M, Y) = 0$ for $i = 1, 2$. Then*

(1) *$(X^{\perp_1}, X^{\perp_0})$ is a torsion pair in $R$-Mod and $M \otimes_R X \in \mathrm{Gen}(X)$ if and only if $(\boldsymbol{T}(X)^{\perp_1}, \boldsymbol{T}(X)^{\perp_0}) = (\mathfrak{A}^{\mathrm{Gen}(X)}, (\mathfrak{A}^{\mathrm{Gen}(X)})^{\perp_0})$ is a torsion pair in $R \ltimes M$-Mod.*

(2) *$({}^{\perp_0}Y, {}^{\perp_1}Y)$ is a torsion pair in $R$-Mod and $\mathrm{Hom}_R(M, Y) \in \mathrm{Cogen}(Y)$ if and only if $({}^{\perp_0}\boldsymbol{H}(Y), {}^{\perp_1}\boldsymbol{H}(Y)) = ({}^{\perp_0}(\mathfrak{A}^{\mathrm{Cogen}(Y)}), \mathfrak{A}^{\mathrm{Cogen}(Y)})$ is a torsion pair in $R \ltimes M$-Mod.*

***Proof.*** (1) "⇒" By Lemma 3.4(1), $X$ is a tilting left $R$-module. So $\mathbf{T}(X)$ is a tilting left $R \ltimes M$-module by [11, Corollary 3.5]. Thus $(\mathbf{T}(X)^{\perp_1}, \mathbf{T}(X)^{\perp_0})$ is a torsion pair in $R \ltimes M$-Mod by Lemma 3.4(1).

Next we prove that $\mathrm{Gen}(\mathbf{T}(X)) = \mathfrak{A}^{\mathrm{Gen}(X)}$.

Let $(Y, \alpha) \in \mathrm{Gen}(\mathbf{T}(X))$. Then there is an epimorphism

$$(X \oplus M \otimes_R X)^{(I)} \to Y$$

for some cardinal $I$. Since $M \otimes_R X \in \mathrm{Gen}(X)$, we have $Y \in \mathrm{Gen}(X)$. Thus $\mathrm{Gen}(\mathbf{T}(X)) \subseteq \mathfrak{A}^{\mathrm{Gen}(X)}$.

Conversely, let $(N, \beta) \in \mathfrak{A}^{\mathrm{Gen}(X)}$. Then there is an epimorphism $X^{(J)} \to N$ for some cardinal $J$, which induces the epimorphism $\mathbf{T}(X)^{(J)} \to \mathbf{T}(N)$.

Define $h : N \oplus (M \otimes_R N) \to N$ by

$$h(x, y) = x + \beta(y), x \in N, y \in M \otimes_R N.$$

It is easy to see that $h$ is an epimorphism. Let $z \otimes (x, y) \in M \otimes_R (N \oplus (M \otimes_R N))$. Then $\beta(M \otimes_R h)(z \otimes (x, y)) = \beta(z \otimes h(x, y)) = \beta(z \otimes (x + \beta(y))) = \beta(z \otimes x) + \beta(z \otimes \beta(y)) = \beta(z \otimes x)$. Also $h\mu(z \otimes (x, y)) = h(0, z \otimes x) = \beta(z \otimes x)$. So the following diagram commutes.

$$\begin{tikzcd}
M \otimes_R (N \oplus (M \otimes_R N)) \arrow[r, "M \otimes_R h"] \arrow[d, "\mu"] & M \otimes_R N \arrow[r] \arrow[d, "\beta"] & 0 \\
N \oplus (M \otimes_R N) \arrow[r, "h"] & N \arrow[r] & 0.
\end{tikzcd}$$

Hence we get an epimorphism $\mathbf{T}(N) \to (N, \beta)$, which means that $(N, \beta) \in \mathrm{Gen}(\mathbf{T}(X))$. Hence $\mathfrak{A}^{\mathrm{Gen}(X)} \subseteq \mathrm{Gen}(\mathbf{T}(X))$ and so $\mathrm{Gen}(\mathbf{T}(X)) = \mathfrak{A}^{\mathrm{Gen}(X)}$. Thus

$$(\mathbf{T}(X)^{\perp_1}, \mathbf{T}(X)^{\perp_0}) = (\mathrm{Gen}(\mathbf{T}(X)), \mathbf{T}(X)^{\perp_0}) = (\mathfrak{A}^{\mathrm{Gen}(X)}, (\mathfrak{A}^{\mathrm{Gen}(X)})^{\perp_0}).$$

"⇐" It holds by [11, Corollary 3.5] and Lemma 3.4(1).

The proof of (2) is dual to that of (1) by [11, Corollary 3.7] and Lemma 3.4(2). □

At the end of this section, we give an application of Theorem 3.2.

Given a torsion pair $\tau = (\mathcal{C}_1, \mathcal{C}_2)$ in $R$-Mod. Recall from Ding and Chen [3] that a left $R$-module $X$ is *$\tau$-finitely generated* if $X/K \in \mathcal{C}_1$ for some finitely generated submodule $K$ of $X$. $X$ is said to be *$\tau$-finitely presented* if there is an exact sequence $0 \to K \to P \to X \to 0$ in $R$-Mod, where $P$ is finitely generated projective and $K$ is $\tau$-finitely generated.

**Lemma 3.7.** *Let $\tau = (\mathcal{C}_1, \mathcal{C}_2)$ be a torsion pair in $R$-Mod. Then any quotient of a $\tau$-finitely generated left $R$-module is $\tau$-finitely generated.*

***Proof.*** Let $X$ be a $\tau$-finitely generated left $R$-module. Then there is a finitely generated submodule $K$ of $X$ such that $X/K \in \mathcal{C}_1$. For any epimorphism $\varphi : X \to Y$, we get the exact sequence in $R$-Mod

$$0 \to \varphi(K) \to Y \to Y/\varphi(K) \to 0.$$

It is clear that $\varphi(K)$ is finitely generated. Also, there is an obvious epimorphism $X/K \to Y/\varphi(K)$. Hence $X/K \in \mathcal{C}_1$ implies that $Y/\varphi(K) \in \mathcal{C}_1$. It follows that $Y$ is a $\tau$-finitely generated left $R$-module. □

By Theorem 3.2, if $\tau = (\mathcal{C}_1, \mathcal{C}_2)$ is a torsion pair in $R$-Mod, then $\widetilde{\tau} = (\mathfrak{L}^{\mathcal{C}_1}, (\mathfrak{L}^{\mathcal{C}_1})^{\perp_0})$ is a torsion pair in $R \ltimes M$-Mod.

**Lemma 3.8.** *Let $\tau = (\mathcal{C}_1, \mathcal{C}_2)$ be a torsion pair in $R$-Mod and $\widetilde{\tau} = (\mathfrak{L}^{\mathcal{C}_1}, (\mathfrak{L}^{\mathcal{C}_1})^{\perp_0})$ a torsion pair in $R \ltimes M$-Mod.*

(1) *If $(X, \alpha)$ is a $\widetilde{\tau}$-finitely generated left $R \ltimes M$-module, then* $\operatorname{coker}(\alpha)$ *is a $\tau$-finitely generated left $R$-module.*
(2) *If $(X, \alpha)$ is a $\widetilde{\tau}$-finitely presented left $R \ltimes M$-module, then* $\operatorname{coker}(\alpha)$ *is a $\tau$-finitely presented left $R$-module.*

***Proof.*** (1) There is a finitely generated submodule $(A, \beta)$ of $(X, \alpha)$ with $(X, \alpha)/(A, \beta) \in \mathfrak{L}^{\mathcal{C}_1}$. Then $\operatorname{coker}(\beta)$ is a finitely generated left $R$-module by [12, Theorem 2.8]. Since $\mathbf{C}$ is a right exact functor, the exact sequence in $R \ltimes M$-Mod

$$0 \to (A, \beta) \to (X, \alpha) \to (X, \alpha)/(A, \beta) \to 0$$

induces the exact sequence in $R$-Mod

$$\operatorname{coker}(\beta) \to \operatorname{coker}(\alpha) \to \operatorname{coker}((X, \alpha)/(A, \beta)) \to 0$$

with $\operatorname{coker}((X, \alpha)/(A, \beta)) \in \mathcal{C}_1$. Then we get the exact sequence in $R$-Mod

$$0 \to K \to \operatorname{coker}(\alpha) \to L \to 0$$

with $K$ finitely generated and $L \in \mathcal{C}_1$. So $\operatorname{coker}(\alpha)$ is $\tau$-finitely generated.

(2) There is an exact sequence in $R \ltimes M$-Mod

$$0 \to (B, \gamma) \to \mathbf{T}(P) \to (X, \alpha) \to 0,$$

where $P$ is a finitely generated projective left $R$-module by [6, Corollary 1.6(c)] and $(B, \gamma)$ is $\widetilde{\tau}$-finitely generated. Then $\operatorname{coker}(\gamma)$ is a $\tau$-finitely generated left $R$-module by (1). The above exact sequence induces the exact sequence in $R$-Mod

$$\operatorname{coker}(\gamma) \to P \to \operatorname{coker}(\alpha) \to 0.$$

Hence we get the exact sequence in $R$-Mod

$$0 \to A \to P \to \operatorname{coker}(\alpha) \to 0$$

with $A$ $\tau$-finitely generated by Lemma 3.7. Thus $X$ is $\tau$-finitely presented. □

Let $\tau = (\mathcal{C}_1, \mathcal{C}_2)$ be a torsion pair in $R$-Mod. According to Ding and Chen [3], a left $R$-module $X$ is called *$\tau$-flat* if every homomorphism from a $\tau$-finitely presented left $R$-module into $X$ factors through a finitely generated projective left $R$-module.

$R$ is called a *right $\tau$-coherent ring* [3] if any direct product of $R$ is a $\tau$-flat left $R$-module.

**Theorem 3.9.** *Let $\tau = (\mathcal{C}_1, \mathcal{C}_2)$ be a torsion pair in $R$-Mod and $\widetilde{\tau} = (\mathfrak{L}^{\mathcal{C}_1}, (\mathfrak{L}^{\mathcal{C}_1})^{\perp_0})$ a torsion pair in $R \ltimes M$-Mod.*

(1) *$X$ is a $\tau$-finitely generated left $R$-module if and only if $\boldsymbol{T}(X)$ is a $\widetilde{\tau}$-finitely generated left $R \ltimes M$-module.*
(2) *$X$ is a $\tau$-finitely presented left $R$-module if and only if $\boldsymbol{T}(X)$ is a $\widetilde{\tau}$-finitely presented left $R \ltimes M$-module.*
(3) *If $\boldsymbol{T}(X)$ is a $\widetilde{\tau}$-flat left $R \ltimes M$-module, then $X$ is a $\tau$-flat left $R$-module.*
(4) *If $M$ is a finitely presented right $R$-module and $R \ltimes M$ is a right $\widetilde{\tau}$-coherent ring, then $R$ is a right $\tau$-coherent ring.*

***Proof.*** (1) " $\Rightarrow$ " There is a finitely generated submodule $K$ of $X$ such that $X/K \in \mathcal{C}_1$. Since $\mathbf{T}$ is a right exact functor, the exact sequence, $0 \to K \to X \to X/K \to 0$ in $R$-Mod induces the exact sequence in $R \ltimes M$-Mod

$$\mathbf{T}(K) \to \mathbf{T}(X) \to \mathbf{T}(X/K) \to 0.$$

By [12, Corollary 2.9], $\mathbf{T}(K)$ is a finitely generated left $R \ltimes M$-module. Thus we get the exact sequence in $R \ltimes M$-Mod

$$0 \to (Y, \alpha) \to \mathbf{T}(X) \to \mathbf{T}(X/K) \to 0$$

with $(Y, \alpha)$ finitely generated. Since $\mathbf{T}(X/K) \in \mathfrak{L}^{\mathcal{C}_1}$, $\mathbf{T}(X)$ is a $\widetilde{\tau}$-finitely generated left $R \ltimes M$-module.

" $\Leftarrow$ " follows from Lemma 3.8(1).

(2) " $\Rightarrow$ " There is an exact sequence $0 \to K \to P \to X \to 0$ in $R$-Mod with $P$ finitely generated projective and $K$ $\tau$-finitely generated, which induces the exact sequence in $R \ltimes M$-Mod

$$\mathbf{T}(K) \to \mathbf{T}(P) \to \mathbf{T}(X) \to 0.$$

By [6, Corollary 1.6(c)], $\mathbf{T}(P)$ is a finitely generated projective left $R \ltimes M$-module. Note that $\mathbf{T}(K)$ is a $\widetilde{\tau}$-finitely generated left $R \ltimes M$-module by (1). So there is the exact sequence in $R \ltimes M$-Mod

$$0 \to (A, \beta) \to \mathbf{T}(P) \to \mathbf{T}(X) \to 0$$

with $(A, \beta)$ $\widetilde{\tau}$-finitely generated by Lemma 3.7. Thus $\mathbf{T}(X)$ is a $\widetilde{\tau}$-finitely presented left $R \ltimes M$-module.

" $\Leftarrow$ " follows from Lemma 3.8(2).

(3) Let $B$ be a $\tau$-finitely presented left $R$-module and $\psi : B \to X$ be any homomorphism. Then we get a homomorphism $\mathbf{T}(\psi) : \mathbf{T}(B) \to \mathbf{T}(X)$. By (2), $\mathbf{T}(B)$ is a $\widetilde{\tau}$-finitely presented left $R \ltimes M$-module. So there are finitely generated projective left $R \ltimes M$-module $\mathbf{T}(P)$, $\xi : \mathbf{T}(B) \to \mathbf{T}(P)$ and $\theta : \mathbf{T}(P) \to \mathbf{T}(X)$ such that $\mathbf{T}(\psi) = \theta\xi$. Hence we have

$$\psi = \mathbf{CT}(\psi) = \mathbf{C}(\theta\xi) = \mathbf{C}(\theta)\mathbf{C}(\xi).$$

Thus $\psi$ factors through the finitely generated projective left $R$-module $P$ by [6, Corollary 1.6(c)], i.e., $X$ is a $\tau$-flat left $R$-module.

(4) Since $M$ is a finitely presented right $R$-module, $M \otimes_R R^I \cong M^I$ for any index set $I$ by [18, Lemma 13.2, p.42]. Since $R \ltimes M$ is a right $\widetilde{\tau}$-coherent ring, $(\mathbf{T}(R))^I$ is a $\widetilde{\tau}$-flat left $R \ltimes M$-module. Note that $\mathbf{T}(R^I) \cong (\mathbf{T}(R))^I$ by [12, Lemma 2.1(1)]. Hence $\mathbf{T}(R^I)$ is a $\widetilde{\tau}$-flat left $R \ltimes M$-module. Thus $R^I$ is a $\tau$-flat left $R$-module by (3). So $R$ is a right $\tau$-coherent ring. $\square$

## 4. Transfers of torsion pairs over $R \ltimes M$ to $R$

Let $R \ltimes M$ be a trivial extension of a ring $R$ by an $R$-$R$-bimodule $M$ and $\mathcal{D}$ a class of left $R \ltimes M$-modules. We write

$\mathbf{U}(\mathcal{D}) = \{\mathbf{U}(D) \in R\text{-Mod}: D \in \mathcal{D}\}$,
$\mathbf{C}(\mathcal{D}) = \{\mathbf{C}(D) \in R\text{-Mod}: D \in \mathcal{D}\}$,
$\mathbf{K}(\mathcal{D}) = \{\mathbf{K}(D) \in R\text{-Mod}: D \in \mathcal{D}\}$,
$\mathfrak{Z}^{\mathcal{D}} = \{X \in R\text{-Mod}: \mathbf{Z}(X) \in \mathcal{D}\}$,
$\mathfrak{T}^{\mathcal{D}} = \{X \in R\text{-Mod}: \mathbf{T}(X) \in \mathcal{D}\}$,
$\mathfrak{H}^{\mathcal{D}} = \{X \in R\text{-Mod}: \mathbf{H}(X) \in \mathcal{D}\}$.

**Lemma 4.1.** *Let $\mathcal{D}$ be a class of left $R \ltimes M$-modules. Then*

(1) ${}^{\perp_0}\boldsymbol{K}(\mathcal{D}) = \mathfrak{Z}^{{}^{\perp_0}\mathcal{D}}$.
(2) $\boldsymbol{C}(\mathcal{D})^{\perp_0} = \mathfrak{Z}^{\mathcal{D}^{\perp_0}}$.
(3) ${}^{\perp_0}\boldsymbol{U}(\mathcal{D}) = \mathfrak{T}^{{}^{\perp_0}\mathcal{D}}$.

(4) $\boldsymbol{U}(\mathcal{D})^{\perp_0} = \mathfrak{H}^{\mathcal{D}^{\perp_0}}$.

***Proof.*** (1) Let $[N, g] \in \mathcal{D}$ and $X \in {}^{\perp_0}\mathbf{K}(\mathcal{D})$. Then

$$\mathrm{Hom}_{R \ltimes M}(\mathbf{Z}(X), [N, g]) \cong \mathrm{Hom}_R(X, \ker(g)) = 0.$$

Hence $X \in \mathfrak{Z}^{{}^{\perp_0}\mathcal{D}}$. Thus ${}^{\perp_0}\mathbf{K}(\mathcal{D}) \subseteq \mathfrak{Z}^{{}^{\perp_0}\mathcal{D}}$.

Conversely, let $[N, g] \in \mathcal{D}$ and $Y \in \mathfrak{Z}^{{}^{\perp_0}\mathcal{D}}$. Then

$$\mathrm{Hom}_R(Y, \ker(g)) \cong \mathrm{Hom}_{R \ltimes M}(\mathbf{Z}(Y), [N, g]) = 0.$$

Thus $Y \in {}^{\perp_0}\mathbf{K}(\mathcal{D})$ and so $\mathfrak{Z}^{{}^{\perp_0}\mathcal{D}} \subseteq {}^{\perp_0}\mathbf{K}(\mathcal{D})$. Hence ${}^{\perp_0}\mathbf{K}(\mathcal{D}) = \mathfrak{Z}^{{}^{\perp_0}\mathcal{D}}$.

(2) Let $[N, g] \in \mathcal{D}$ and $X \in \mathbf{C}(\mathcal{D})^{\perp_0}$. Then

$$\mathrm{Hom}_{R \ltimes M}([N, g], \mathbf{Z}(X)) \cong \mathrm{Hom}_R(\mathrm{coker}(g), X) = 0.$$

Hence $\mathbf{Z}(X) \in \mathcal{D}^{\perp_0}$. Thus $X \in \mathfrak{Z}^{\mathcal{D}^{\perp_0}}$ and so $\mathbf{C}(\mathcal{D})^{\perp_0} \subseteq \mathfrak{Z}^{\mathcal{D}^{\perp_0}}$.

Conversely, let $[N, g] \in \mathcal{D}$ and $Y \in \mathfrak{Z}^{\mathcal{D}^{\perp_0}}$. Then

$$\mathrm{Hom}_R(\mathrm{coker}(g), Y) \cong \mathrm{Hom}_{R \ltimes M}([N, g], \mathbf{Z}(Y)) = 0.$$

Thus $Y \in \mathbf{C}(\mathcal{D})^{\perp_0}$ and so $\mathfrak{Z}^{\mathcal{D}^{\perp_0}} \subseteq \mathbf{C}(\mathcal{D})^{\perp_0}$. Hence $\mathbf{C}(\mathcal{D})^{\perp_0} = \mathfrak{Z}^{\mathcal{D}^{\perp_0}}$.

(3) Let $(Y, \beta) \in \mathcal{D}$ and $X \in {}^{\perp_0}\mathbf{U}(\mathcal{D})$. Then

$$\mathrm{Hom}_{R \ltimes M}(\mathbf{T}(X), (Y, \beta)) \cong \mathrm{Hom}_R(X, Y) = 0.$$

Hence $\mathbf{T}(X) \in {}^{\perp_0}\mathcal{D}$. Thus $X \in \mathfrak{T}^{{}^{\perp_0}\mathcal{D}}$ and so ${}^{\perp_0}\mathbf{U}(\mathcal{D}) \subseteq \mathfrak{T}^{{}^{\perp_0}\mathcal{D}}$.

Conversely, let $(Y, \beta) \in \mathcal{D}$ and $N \in \mathfrak{T}^{{}^{\perp_0}\mathcal{D}}$. Then

$$\mathrm{Hom}_R(N, Y) \cong \mathrm{Hom}_{R \ltimes M}(\mathbf{T}(N), (Y, \beta)) = 0$$

and so $N \in {}^{\perp_0}\mathbf{U}(\mathcal{D})$. Thus $\mathfrak{T}^{{}^{\perp_0}\mathcal{D}} \subseteq {}^{\perp_0}\mathbf{U}(\mathcal{D})$. So ${}^{\perp_0}\mathbf{U}(\mathcal{D}) = \mathfrak{T}^{{}^{\perp_0}\mathcal{D}}$.

(4) Let $[N, g] \in \mathcal{D}$ and $X \in \mathbf{U}(\mathcal{D})^{\perp_0}$. Then

$$\mathrm{Hom}_{R \ltimes M}([N, g], \mathbf{H}(X)) \cong \mathrm{Hom}_R(N, X) = 0.$$

So $\mathbf{H}(X) \in \mathcal{D}^{\perp_0}$. Thus $X \in \mathfrak{H}^{\mathcal{D}^{\perp_0}}$. Hence $\mathbf{U}(\mathcal{D})^{\perp_0} \subseteq \mathfrak{H}^{\mathcal{D}^{\perp_0}}$.

Conversely, let $[N, g] \in \mathcal{D}$ and $Y \in \mathfrak{H}^{\mathcal{D}^{\perp_0}}$. Then

$$\mathrm{Hom}_R(N, Y) \cong \mathrm{Hom}_{R \ltimes M}([N, g], \mathbf{H}(Y)) = 0.$$

Thus $Y \in \mathbf{U}(\mathcal{D})^{\perp_0}$ and so $\mathfrak{H}^{\mathcal{D}^{\perp_0}} \subseteq \mathbf{U}(\mathcal{D})^{\perp_0}$. Hence $\mathbf{U}(\mathcal{D})^{\perp_0} = \mathfrak{H}^{\mathcal{D}^{\perp_0}}$. □

The following theorem shows that a torsion pair in $R \ltimes M$-Mod can produce rich torsion pairs in $R$-Mod.

**Theorem 4.2.** *Let $(\mathcal{D}_1, \mathcal{D}_2)$ be a torsion pair in $R \ltimes M$-Mod. Then*

(1) $(\mathfrak{Z}^{\mathcal{D}_1}, (\mathfrak{Z}^{\mathcal{D}_1})^{\perp_0})$ *is a torsion pair in R-Mod.*
(2) $({}^{\perp_0}(\mathfrak{Z}^{\mathcal{D}_2}), \mathfrak{Z}^{\mathcal{D}_2})$ *is a torsion pair in R-Mod.*
(3) $(\mathfrak{T}^{\mathcal{D}_1}, (\mathfrak{T}^{\mathcal{D}_1})^{\perp_0})$ *is a torsion pair in R-Mod.*
(4) $({}^{\perp_0}(\mathfrak{H}^{\mathcal{D}_2}), \mathfrak{H}^{\mathcal{D}_2})$ *is a torsion pair in R-Mod.*

***Proof.*** (1) By Lemma 4.1(1), we have

$${}^{\perp_0}((\mathfrak{Z}^{\mathcal{D}_1})^{\perp_0}) = {}^{\perp_0}((\mathfrak{Z}^{{}^{\perp_0}\mathcal{D}_2})^{\perp_0}) = {}^{\perp_0}(({}^{\perp_0}\mathbf{K}(\mathcal{D}_2))^{\perp_0}) = {}^{\perp_0}\mathbf{K}(\mathcal{D}_2) = \mathfrak{Z}^{{}^{\perp_0}\mathcal{D}_2} = \mathfrak{Z}^{\mathcal{D}_1}.$$

It follows that $(\mathfrak{Z}^{\mathcal{D}_1}, (\mathfrak{Z}^{\mathcal{D}_1})^{\perp_0})$ is a torsion pair in $R \ltimes M$-Mod.

(2) By Lemma 4.1(2), we have

$$({}^{\perp_0}(\mathfrak{Z}^{\mathcal{D}_2}))^{\perp_0} = ({}^{\perp_0}(\mathfrak{Z}^{\mathcal{D}_1^{\perp_0}}))^{\perp_0} = ({}^{\perp_0}(\mathbf{C}(\mathcal{D}_1)^{\perp_0}))^{\perp_0} = \mathbf{C}(\mathcal{D}_1)^{\perp_0} = \mathfrak{Z}^{\mathcal{D}_1^{\perp_0}} = \mathfrak{Z}^{\mathcal{D}_2}.$$

It follows that $({}^{\perp_0}(\mathfrak{Z}^{\mathcal{D}_2}), \mathfrak{Z}^{\mathcal{D}_2})$ is a torsion pair in $R \ltimes M$-Mod.

(3) By Lemma 4.1(3), we have

$${}^{\perp_0}((\mathfrak{T}^{\mathcal{D}_1})^{\perp_0}) = {}^{\perp_0}((\mathfrak{T}^{{}^{\perp_0}\mathcal{D}_2})^{\perp_0}) = {}^{\perp_0}(({}^{\perp_0}\mathbf{U}(\mathcal{D}_2))^{\perp_0}) = {}^{\perp_0}\mathbf{U}(\mathcal{D}_2) = \mathfrak{T}^{{}^{\perp_0}\mathcal{D}_2} = \mathfrak{T}^{\mathcal{D}_1}.$$

It follows that $(\mathfrak{T}^{\mathcal{D}_1}, (\mathfrak{T}^{\mathcal{D}_1})^{\perp_0})$ is a torsion pair in $R \ltimes M$-Mod.

(4) By Lemma 4.1(4), we have

$$({}^{\perp_0}(\mathfrak{H}^{\mathcal{D}_2}))^{\perp_0} = ({}^{\perp_0}(\mathfrak{H}^{\mathcal{D}_1^{\perp_0}}))^{\perp_0} = ({}^{\perp_0}(\mathbf{U}(\mathcal{D}_1)^{\perp_0}))^{\perp_0} = \mathbf{U}(\mathcal{D}_1)^{\perp_0} = \mathfrak{H}^{\mathcal{D}_1^{\perp_0}} = \mathfrak{H}^{\mathcal{D}_2}.$$

It follows that $({}^{\perp_0}(\mathfrak{H}^{\mathcal{D}_2}), \mathfrak{H}^{\mathcal{D}_2})$ is a torsion pair in $R \ltimes M$-Mod. □

**Theorem 4.3.** *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two classes in $R \ltimes M$-Mod.*

(1) *If $(\mathcal{D}_1, \mathcal{D}_2)$ is a hereditary torsion pair in $R \ltimes M$-Mod, then $(\mathfrak{Z}^{\mathcal{D}_1}, \mathfrak{H}^{\mathcal{D}_2})$ is a hereditary torsion pair in R-Mod.*

(2) *If $(\mathcal{D}_1, \mathcal{D}_2)$ is a cohereditary torsion pair in $R \ltimes M$-Mod, then $(\mathfrak{T}^{\mathcal{D}_1}, \mathfrak{Z}^{\mathcal{D}_2})$ is a cohereditary torsion pair in R-Mod.*

***Proof.*** (1) We first prove that $(\mathfrak{Z}^{\mathcal{D}_1})^{\perp_0} = \mathfrak{H}^{\mathcal{D}_2}$. Let $Y \in \mathfrak{Z}^{\mathcal{D}_1}$ and $X \in \mathfrak{H}^{\mathcal{D}_2}$. Then

$$\mathrm{Hom}_R(Y, X) \cong \mathrm{Hom}_{R\ltimes M}(\mathbf{Z}(Y), \mathbf{H}(X)) = 0.$$

Hence $X \in (\mathfrak{Z}^{\mathcal{D}_1})^{\perp_0}$ and so $\mathfrak{H}^{\mathcal{D}_2} \subseteq (\mathfrak{Z}^{\mathcal{D}_1})^{\perp_0}$.

Conversely, let $N \in (\mathfrak{Z}^{\mathcal{D}_1})^{\perp_0}$ and $[W, \beta] \in \mathcal{D}_1$. Then there exists an exact sequence

$$0 \to \mathbf{Z}(\ker(\beta)) \to [W, \beta] \to \mathbf{Z}(\mathrm{im}(\beta)) \to 0,$$

which induces the exact sequence

$$0 \to \mathrm{Hom}_{R\ltimes M}(\mathbf{Z}(\mathrm{im}(\beta)), \mathbf{H}(N)) \to \mathrm{Hom}_{R\ltimes M}([W, \beta], \mathbf{H}(N)) \to \mathrm{Hom}_{R\ltimes M}(\mathbf{Z}(\ker(\beta)), \mathbf{H}(N)).$$

Since $(\mathcal{D}_1, \mathcal{D}_2)$ is a hereditary torsion pair, we have $\mathbf{Z}(\mathrm{im}(\beta)) \in \mathcal{D}_1$ and $\mathbf{Z}(\ker(\beta)) \in \mathcal{D}_1$. Note that

$$\mathrm{Hom}_{R\ltimes M}(\mathbf{Z}(\mathrm{im}(\beta)), \mathbf{H}(N)) \cong \mathrm{Hom}_R(\mathrm{im}(\beta), N) = 0$$

and

$$\mathrm{Hom}_{R\ltimes M}(\mathbf{Z}(\ker(\beta)), \mathbf{H}(N)) \cong \mathrm{Hom}_R(\ker(\beta), N) = 0.$$

Thus $\mathrm{Hom}_{R\ltimes M}([W, \beta], \mathbf{H}(N)) = 0$, which implies that $\mathbf{H}(N) \in \mathcal{D}_1^{\perp_0} = \mathcal{D}_2$. Hence $(\mathfrak{Z}^{\mathcal{D}_1})^{\perp_0} \subseteq \mathfrak{H}^{\mathcal{D}_2}$. So $(\mathfrak{Z}^{\mathcal{D}_1})^{\perp_0} = \mathfrak{H}^{\mathcal{D}_2}$.

Since $\mathcal{D}_1$ is closed under submodules, we have that $\mathfrak{Z}^{\mathcal{D}_1}$ is closed under submodules. By Theorem 4.2(1), $(\mathfrak{Z}^{\mathcal{D}_1}, \mathfrak{H}^{\mathcal{D}_2})$ is a hereditary torsion pair in $R$-Mod.

(2) We first prove that ${}^{\perp_0}(\mathfrak{Z}^{\mathcal{D}_2}) = \mathfrak{T}^{\mathcal{D}_1}$. Let $Y \in \mathfrak{Z}^{\mathcal{D}_2}$ and $X \in \mathfrak{T}^{\mathcal{D}_1}$. Then

$$\mathrm{Hom}_R(X, Y) \cong \mathrm{Hom}_{R\ltimes M}(\mathbf{T}(X), \mathbf{Z}(Y)) = 0.$$

Hence $X \in {}^{\perp_0}(\mathfrak{Z}^{\mathcal{D}_2})$ and so $\mathfrak{T}^{\mathcal{D}_1} \subseteq {}^{\perp_0}(\mathfrak{Z}^{\mathcal{D}_2})$.

Conversely, let $N \in {}^{\perp_0}(\mathfrak{Z}^{\mathcal{D}_2})$ and $(W, \beta) \in \mathcal{D}_2$. Then there exists an exact sequence

$$0 \to \mathbf{Z}(\mathrm{im}(\beta)) \to (W, \beta) \to \mathbf{Z}(\mathrm{coker}(\beta)) \to 0,$$

which induces the exact sequence

$$0 \to \mathrm{Hom}_{R\ltimes M}(\mathbf{T}(N), \mathbf{Z}(\mathrm{im}(\beta))) \to \mathrm{Hom}_{R\ltimes M}(\mathbf{T}(N), (W, \beta)) \to \mathrm{Hom}_{R\ltimes M}(\mathbf{T}(N), \mathbf{Z}(\mathrm{coker}(\beta))).$$

Since $(\mathcal{D}_1, \mathcal{D}_2)$ is a cohereditary torsion pair, $\mathbf{Z}(\mathrm{im}(\beta)) \in \mathcal{D}_2$ and $\mathbf{Z}(\mathrm{coker}(\beta)) \in \mathcal{D}_2$.

Note that

$$\mathrm{Hom}_{R\ltimes M}(\mathbf{T}(N), \mathbf{Z}(\mathrm{im}(\beta))) \cong \mathrm{Hom}_R(N, \mathrm{im}(\beta)) = 0$$

and

$$\mathrm{Hom}(\mathbf{T}(N), \mathbf{Z}(\mathrm{coker}(\beta))) \cong \mathrm{Hom}_R(N, \mathrm{coker}(\beta)) = 0.$$

Therefore $\mathrm{Hom}_{R\ltimes M}(\mathbf{T}(N), (W, \beta)) = 0$. Thus $\mathbf{T}(N) \in {}^{\perp_0}\mathcal{D}_2 = \mathcal{D}_1$. Hence ${}^{\perp_0}(\mathfrak{Z}^{\mathcal{D}_2}) \subseteq \mathfrak{T}^{\mathcal{D}_1}$ and so ${}^{\perp_0}(\mathfrak{Z}^{\mathcal{D}_2}) = \mathfrak{T}^{\mathcal{D}_1}$.

Since $\mathcal{D}_2$ is closed under quotients, we have that $\mathfrak{Z}^{\mathcal{D}_2}$ is closed under quotients. By Theorem 4.2(2), $(\mathfrak{T}^{\mathcal{D}_1}, \mathfrak{Z}^{\mathcal{D}_2})$ is a cohereditary torsion pair in $R$-Mod. □

## 5. Torsion pairs over some Morita context rings

Morita context rings, originated from equivalences of module categories [13] and formulated by Bass [1], have been studied explicitly in numerous papers and books [1,6,8,9,13]. In this section, we apply the foregoing results to Morita context rings with zero bimodule homomorphisms since this kind of rings is one special case of trivial ring extensions.

Let $\Lambda_{(0,0)} = \begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(0,0)}$, where $A$ and $B$ are rings, ${}_BU_A$ and ${}_AV_B$ are bimodules, $\Lambda_{(0,0)}$ is called a *Morita context ring with zero bimodule homomorphisms* or *formal matrix ring* [9,13], where the addition of elements of $\Lambda_{(0,0)}$ is componentwise and multiplication is given by

$$\begin{pmatrix} a_1 & v_1 \\ u_1 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & v_2 \\ u_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1v_2 + v_1b_2 \\ u_1a_2 + b_1u_2 & b_1b_2 \end{pmatrix}.$$

Green [8, Theorem 1.5] proved that the category $\Lambda_{(0,0)}$-Mod is equivalent to the category $\Omega$ whose objects are tuples $(X, Y, f, g)$, where $X \in A$-Mod, $Y \in B$-Mod, $f \in \mathrm{Hom}_B(U \otimes_A X, Y)$ and $g \in \mathrm{Hom}_A(V \otimes_B Y, X)$ such that $g(V \otimes_B f) = 0, f(U \otimes_A g) = 0$ and whose morphisms from $(X_1, Y_1, f_1, g_1)$ to $(X_2, Y_2, f_2, g_2)$ are pairs $(\alpha, \beta)$ such that $\alpha \in \mathrm{Hom}_A(X_1, X_2), \beta \in \mathrm{Hom}_B(Y_1, Y_2)$ and $f_2(U \otimes_A \alpha) = \beta f_1, g_2(V \otimes_B \beta) = \alpha g_1$. In view of the well-known adjointness relation, the category $\Lambda_{(0,0)}$-Mod is also equivalent to the category $\Gamma$ whose objects are tuples $[X, Y, f, g]$, where $X \in A$-Mod, $Y \in B$-Mod, $f \in \mathrm{Hom}_A(X, \mathrm{Hom}_B(U, Y))$ and $g \in \mathrm{Hom}_B(Y, \mathrm{Hom}_A(V, X))$ such that $\mathrm{Hom}_B(U, g)f = 0, \mathrm{Hom}_A(U, f)g = 0$ and whose morphisms from $[X_1, Y_1, f_1, g_1]$ to $[X_2, Y_2, f_2, g_2]$ are pairs $[\alpha, \beta]$ such that $\alpha \in \mathrm{Hom}_A(X_1, X_2), \beta \in \mathrm{Hom}_B(Y_1, Y_2)$ and $f_2\alpha = \mathrm{Hom}_B(U, \beta)f_1$, $g_2\beta = \mathrm{Hom}_A(V, \alpha)g_1$.

In this section, we will identify $\Lambda_{(0,0)}$-Mod with $\Omega$ and $\Gamma$.

It is known that $\Lambda_{(0,0)} = \begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(0,0)}$ is isomorphic to the trivial ring extension $(A \times B) \ltimes (U \oplus V)$ [6] under the correspondence:

$$\begin{pmatrix} a & v \\ u & b \end{pmatrix} \to ((a, b), (u, v)).$$

So $\Lambda_{(0,0)}$-Mod is isomorphic to $(A \times B) \ltimes (U \oplus V)$-Mod by the functor $\Theta : \Lambda_{(0,0)}$-Mod $\to (A \times B) \ltimes (U \oplus V)$-Mod given by $\Theta(X, Y, f, g) = ((X, Y), (g, f))$ or by $\Theta[X, Y, f, g] = [(X, Y), (f, g)]$.

Let $\mathcal{C}$ be a class of left $A$-modules and $\mathcal{D}$ a class of left $B$-modules. We write

$\mathfrak{A}^{\mathcal{C}}_{\mathcal{D}} = \{(X, Y, f, g) \in \Lambda_{(0,0)}\text{-Mod}\colon\ X \in \mathcal{C} \text{ and } Y \in \mathcal{D}\}$,

$\mathfrak{L}^{\mathcal{C}}_{\mathcal{D}} = \{(X, Y, f, g) \in \Lambda_{(0,0)}\text{-Mod}\colon\ \mathrm{coker}(f) \in \mathcal{D}, \mathrm{coker}(g) \in \mathcal{C}\}$,

$\mathfrak{K}^{\mathcal{C}}_{\mathcal{D}} = \{[X, Y, f, g] \in \Lambda_{(0,0)}\text{-Mod}\colon\ \ker(f) \in \mathcal{C}, \ker(g) \in \mathcal{D}\}$.

It is clear that $\Theta(\mathfrak{A}^{\mathcal{C}}_{\mathcal{D}}) = \mathfrak{A}^{(\mathcal{C},\mathcal{D})}$, $\Theta(\mathfrak{L}^{\mathcal{C}}_{\mathcal{D}}) = \mathfrak{L}^{(\mathcal{C},\mathcal{D})}$ and $\Theta(\mathfrak{K}^{\mathcal{C}}_{\mathcal{D}}) = \mathfrak{K}^{(\mathcal{C},\mathcal{D})}$.

**Theorem 5.1.** *Let* $\Lambda_{(0,0)} = \begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(0,0)}$ *be a Morita context ring.*

(1) *If* $(\mathcal{C}_1, \mathcal{C}_2)$ *and* $(\mathcal{D}_1, \mathcal{D}_2)$ *are torsion pairs in A-Mod and B-Mod respectively, then* $(\mathfrak{L}^{\mathcal{C}_1}_{\mathcal{D}_1}, (\mathfrak{L}^{\mathcal{C}_1}_{\mathcal{D}_1})^{\perp_0})$, $({}^{\perp_0}(\mathfrak{K}^{\mathcal{C}_2}_{\mathcal{D}_2}), \mathfrak{K}^{\mathcal{C}_2}_{\mathcal{D}_2})$, $(\mathfrak{A}^{\mathcal{C}_1}_{\mathcal{D}_1}, (\mathfrak{A}^{\mathcal{C}_1}_{\mathcal{D}_1})^{\perp_0})$ *and* $({}^{\perp_0}(\mathfrak{A}^{\mathcal{C}_2}_{\mathcal{D}_2}), \mathfrak{A}^{\mathcal{C}_2}_{\mathcal{D}_2})$ *are torsion pairs in* $\Lambda_{(0,0)}$*-Mod.*

(2) $(\mathcal{C}_1, \mathcal{C}_2)$ *and* $(\mathcal{D}_1, \mathcal{D}_2)$ *are hereditary torsion pairs in A-Mod and B-Mod respectively if and only if* $(\mathfrak{A}^{\mathcal{C}_1}_{\mathcal{D}_1}, \mathfrak{K}^{\mathcal{C}_2}_{\mathcal{D}_2})$ *is a hereditary torsion pair in* $\Lambda_{(0,0)}$*-Mod.*

(3) $(\mathcal{C}_1, \mathcal{C}_2)$ *and* $(\mathcal{D}_1, \mathcal{D}_2)$ *are cohereditary torsion pairs in A-Mod and B-Mod respectively if and only if* $(\mathfrak{L}^{\mathcal{C}_1}_{\mathcal{D}_1}, \mathfrak{A}^{\mathcal{C}_2}_{\mathcal{D}_2})$ *is a cohereditary torsion pair in* $\Lambda_{(0,0)}$*-Mod.*

***Proof.*** (1) By Theorem 3.2, one gets that $(\mathfrak{L}^{(\mathcal{C}_1,\mathcal{D}_1)},(\mathfrak{L}^{(\mathcal{C}_1,\mathcal{D}_1)})^{\perp_0})$, $({}^{\perp_0}(\mathfrak{K}^{(\mathcal{C}_2,\mathcal{D}_2)}),\mathfrak{K}^{(\mathcal{C}_2,\mathcal{D}_2)})$, $(\mathfrak{A}^{(\mathcal{C}_1,\mathcal{D}_1)},(\mathfrak{A}^{(\mathcal{C}_1,\mathcal{D}_1)})^{\perp_0})$ and $({}^{\perp_0}(\mathfrak{A}^{(\mathcal{C}_2,\mathcal{D}_2)}),\mathfrak{A}^{(\mathcal{C}_2,\mathcal{D}_2)})$ are torsion pairs in $(A\times B)\ltimes(U\oplus V)$-Mod. Therefore $(\mathfrak{L}^{\mathcal{C}_1}_{\mathcal{D}_1},(\mathfrak{L}^{\mathcal{C}_1}_{\mathcal{D}_1})^{\perp_0})$, $({}^{\perp_0}(\mathfrak{K}^{\mathcal{C}_2}_{\mathcal{D}_2}),\mathfrak{K}^{\mathcal{C}_2}_{\mathcal{D}_2})$, $(\mathfrak{A}^{\mathcal{C}_1}_{\mathcal{D}_1},(\mathfrak{A}^{\mathcal{C}_1}_{\mathcal{D}_1})^{\perp_0})$ and $({}^{\perp_0}(\mathfrak{A}^{\mathcal{C}_2}_{\mathcal{D}_2}),\mathfrak{A}^{\mathcal{C}_2}_{\mathcal{D}_2})$ are torsion pairs in $\Lambda_{(0,0)}$-Mod.

(2) $(\mathcal{C}_1,\mathcal{C}_2)$ and $(\mathcal{D}_1,\mathcal{D}_2)$ are hereditary torsion pairs in $A$-Mod and $B$-Mod respectively if and only if $((\mathcal{C}_1,\mathcal{D}_1),(\mathcal{C}_2,\mathcal{D}_2))$ is a hereditary torsion pair in $A\times B$-Mod if and only if $(\mathfrak{A}^{(\mathcal{C}_1,\mathcal{D}_1)},\mathfrak{K}^{(\mathcal{C}_2,\mathcal{D}_2)})$ is a hereditary torsion pair in $(A\times B)\ltimes(U\oplus V)$-Mod by Theorem 3.3(1) if and only if $(\mathfrak{A}^{\mathcal{C}_1}_{\mathcal{D}_1},\mathfrak{K}^{\mathcal{C}_2}_{\mathcal{D}_2})$ is a hereditary torsion pair in $\Lambda_{(0,0)}$-Mod.

(3) $(\mathcal{C}_1,\mathcal{C}_2)$ and $(\mathcal{D}_1,\mathcal{D}_2)$ are cohereditary torsion pairs in $A$-Mod and $B$-Mod respectively if and only if $((\mathcal{C}_1,\mathcal{D}_1),(\mathcal{C}_2,\mathcal{D}_2))$ is a cohereditary torsion pair in $A\times B$-Mod if and only if $(\mathfrak{L}^{(\mathcal{C}_1,\mathcal{D}_1)},\mathfrak{A}^{(\mathcal{C}_2,\mathcal{D}_2)})$ is a cohereditary torsion pair in $(A\times B)\ltimes(U\oplus V)$-Mod by Theorem 3.3(2) if and only if $(\mathfrak{L}^{\mathcal{C}_1}_{\mathcal{D}_1},\mathfrak{A}^{\mathcal{C}_2}_{\mathcal{D}_2})$ is a cohereditary torsion pair in $\Lambda_{(0,0)}$-Mod. □

Let $\mathcal{W}$ be a class of left $\Lambda_{(0,0)}$-modules. Write
$\mathfrak{Z}_1^{\mathcal{W}}=\{X\in A\text{-Mod}\colon (X,0,0,0)\in\mathcal{W}\}$,
$\mathfrak{Z}_2^{\mathcal{W}}=\{Y\in B\text{-Mod}\colon (0,Y,0,0)\in\mathcal{W}\}$,
$\mathfrak{T}_1^{\mathcal{W}}=\{X\in A\text{-Mod}\colon (X,U\otimes_A X,1,0)\in\mathcal{W}\}$,
$\mathfrak{T}_2^{\mathcal{W}}=\{Y\in B\text{-Mod}\colon (V\otimes_B Y,Y,0,1)\in\mathcal{W}\}$,
$\mathfrak{H}_1^{\mathcal{W}}=\{X\in A\text{-Mod}\colon [X,\mathrm{Hom}_A(V,X),0,1]\in\mathcal{W}\}$,
$\mathfrak{H}_2^{\mathcal{W}}=\{Y\in B\text{-Mod}\colon [\mathrm{Hom}_B(U,Y),Y,1,0]\in\mathcal{W}\}$.

**Theorem 5.2.** *Let* $\Lambda_{(0,0)}=\begin{pmatrix} A & {}_AV_B\\ {}_BU_A & B\end{pmatrix}_{(0,0)}$ *be a Morita context ring.*

1. *If $(\mathcal{W}_1,\mathcal{W}_2)$ is a torsion pair in $\Lambda_{(0,0)}$-Mod, then $(\mathfrak{Z}_1^{\mathcal{W}_1},(\mathfrak{Z}_1^{\mathcal{W}_1})^{\perp_0})$, $({}^{\perp_0}(\mathfrak{Z}_1^{\mathcal{W}_2}),\mathfrak{Z}_1^{\mathcal{W}_2})$, $(\mathfrak{T}_1^{\mathcal{W}_1},(\mathfrak{T}_1^{\mathcal{W}_1})^{\perp_0})$ and $({}^{\perp_0}(\mathfrak{H}_1^{\mathcal{W}_2}),\mathfrak{H}_1^{\mathcal{W}_2})$ are torsion pairs in $A$-Mod, $(\mathfrak{Z}_2^{\mathcal{W}_1},(\mathfrak{Z}_2^{\mathcal{W}_1})^{\perp_0})$, $({}^{\perp_0}(\mathfrak{Z}_2^{\mathcal{W}_2}),\mathfrak{Z}_2^{\mathcal{W}_2})$, $(\mathfrak{T}_2^{\mathcal{W}_1},(\mathfrak{T}_2^{\mathcal{W}_1})^{\perp_0})$ and $({}^{\perp_0}(\mathfrak{H}_2^{\mathcal{W}_2}),\mathfrak{H}_2^{\mathcal{W}_2})$ are torsion pairs in $B$-Mod.*
2. *If $(\mathcal{W}_1,\mathcal{W}_2)$ is a hereditary torsion pair in $\Lambda_{(0,0)}$-Mod, then $(\mathfrak{Z}_1^{\mathcal{W}_1},\mathfrak{H}_1^{\mathcal{W}_2})$ is a hereditary torsion pair in $A$-Mod and $(\mathfrak{Z}_2^{\mathcal{W}_1},\mathfrak{H}_2^{\mathcal{W}_2})$ is a hereditary torsion pair in $B$-Mod.*
3. *If $(\mathcal{W}_1,\mathcal{W}_2)$ is a cohereditary torsion pair in $\Lambda_{(0,0)}$-Mod, then $(\mathfrak{T}_1^{\mathcal{W}_1},\mathfrak{Z}_1^{\mathcal{W}_2})$ is a cohereditary torsion pair in $A$-Mod and $(\mathfrak{T}_2^{\mathcal{W}_1},\mathfrak{Z}_2^{\mathcal{W}_2})$ is a cohereditary torsion pair in $B$-Mod.*

***Proof.*** (1) Since $(\mathcal{W}_1,\mathcal{W}_2)$ is a torsion pair in $\Lambda_{(0,0)}$-Mod, $(\Theta(\mathcal{W}_1),\Theta(\mathcal{W}_2))$ is a torsion pair in $(A\times B)\ltimes(U\oplus V)$-Mod. By Theorem 4.2, we have $(\mathfrak{Z}^{\Theta(\mathcal{W}_1)},(\mathfrak{Z}^{\Theta(\mathcal{W}_1)})^{\perp_0})$, $({}^{\perp_0}(\mathfrak{Z}^{\Theta(\mathcal{W}_2)}),\mathfrak{Z}^{\Theta(\mathcal{W}_2)})$, $(\mathfrak{T}^{\Theta(\mathcal{W}_1)},(\mathfrak{T}^{\Theta(\mathcal{W}_1)})^{\perp_0})$ and $({}^{\perp_0}(\mathfrak{H}^{\Theta(\mathcal{W}_2)}),\mathfrak{H}^{\Theta(\mathcal{W}_2)})$ are torsion pairs in $A\times B$-Mod. Thus one gets that $((\mathfrak{Z}_1^{\mathcal{W}_1},\mathfrak{Z}_2^{\mathcal{W}_1}),(\mathfrak{Z}_1^{\mathcal{W}_1},\mathfrak{Z}_2^{\mathcal{W}_1})^{\perp_0})$, $({}^{\perp_0}(\mathfrak{Z}_1^{\mathcal{W}_2},\mathfrak{Z}_2^{\mathcal{W}_2}),(\mathfrak{Z}_1^{\mathcal{W}_2},\mathfrak{Z}_2^{\mathcal{W}_2}))$, $((\mathfrak{T}_1^{\mathcal{W}_1},\mathfrak{T}_2^{\mathcal{W}_1}),(\mathfrak{T}_1^{\mathcal{W}_1},\mathfrak{T}_2^{\mathcal{W}_1})^{\perp_0})$ and $({}^{\perp_0}(\mathfrak{H}_1^{W_2},\mathfrak{H}_2^{\mathcal{W}_2}),(\mathfrak{H}_1^{W_2},\mathfrak{H}_2^{\mathcal{W}_2}))$ are torsion pairs in $A\times B$-Mod. So $(\mathfrak{Z}_1^{\mathcal{W}_1},(\mathfrak{Z}_1^{\mathcal{W}_1})^{\perp_0})$, $({}^{\perp_0}(\mathfrak{Z}_1^{\mathcal{W}_2}),\mathfrak{Z}_1^{\mathcal{W}_2})$, $(\mathfrak{T}_1^{\mathcal{W}_1},(\mathfrak{T}_1^{\mathcal{W}_1})^{\perp_0})$ and $({}^{\perp_0}(\mathfrak{H}_1^{\mathcal{W}_2}),\mathfrak{H}_1^{\mathcal{W}_2})$ are torsion pairs in $A$-Mod, and $(\mathfrak{Z}_2^{\mathcal{W}_1},(\mathfrak{Z}_2^{\mathcal{W}_1})^{\perp_0})$, $({}^{\perp_0}(\mathfrak{Z}_2^{\mathcal{W}_2}),\mathfrak{Z}_2^{\mathcal{W}_2})$, $(\mathfrak{T}_2^{\mathcal{W}_1},(\mathfrak{T}_2^{\mathcal{W}_1})^{\perp_0})$ and $({}^{\perp_0}(\mathfrak{H}_2^{\mathcal{W}_2}),\mathfrak{H}_2^{\mathcal{W}_2})$ are torsion pairs in $B$-Mod.

(2) Since $(\mathcal{W}_1,\mathcal{W}_2)$ is a hereditary torsion pair in $\Lambda_{(0,0)}$-Mod, $(\Theta(\mathcal{W}_1),\Theta(\mathcal{W}_2))$ is a hereditary torsion pair in $(A\times B)\ltimes(U\oplus V)$-Mod. By Theorem 4.3(1), $(\mathfrak{Z}^{\Theta(\mathcal{W}_1)},\mathfrak{H}^{\Theta(\mathcal{W}_2)})=((\mathfrak{Z}_1^{\mathcal{W}_1},\mathfrak{Z}_2^{\mathcal{W}_1}),(\mathfrak{H}_1^{\mathcal{W}_2},\mathfrak{H}_2^{\mathcal{W}_2}))$ is a hereditary torsion pair in $A\times B$-Mod. Thus $(\mathfrak{Z}_1^{\mathcal{W}_1},\mathfrak{H}_1^{\mathcal{W}_2})$ is a hereditary torsion pair in $A$-Mod and $(\mathfrak{Z}_2^{\mathcal{W}_1},\mathfrak{H}_2^{\mathcal{W}_2})$ is a hereditary torsion pair in $B$-Mod.

(3) Since $(\mathcal{W}_1,\mathcal{W}_2)$ is a cohereditary torsion pair in $\Lambda_{(0,0)}$-Mod, $(\Theta(\mathcal{W}_1),\Theta(\mathcal{W}_2))$ is a cohereditary torsion pair in $(A\times B)\ltimes(U\oplus V)$-Mod. By Theorem 4.3(2), we have $(\mathfrak{T}^{\Theta(\mathcal{W}_1)},\mathfrak{Z}^{\Theta(\mathcal{W}_2)})=((\mathfrak{T}_1^{\mathcal{W}_1},\mathfrak{T}_2^{\mathcal{W}_1}),(\mathfrak{Z}_1^{\mathcal{W}_2},\mathfrak{Z}_2^{\mathcal{W}_2}))$ is a cohereditary torsion pair in $A\times B$-Mod. Hence $(\mathfrak{T}_1^{\mathcal{W}_1},\mathfrak{Z}_1^{\mathcal{W}_2})$ is a cohereditary torsion pair in $A$-Mod and $(\mathfrak{T}_2^{\mathcal{W}_1},\mathfrak{Z}_2^{\mathcal{W}_2})$ is a cohereditary torsion pair in $B$-Mod. □

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