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MISCELLANEOUS PROPERTIES OF STURM-LIOUVILLE PROBLEMS IN MULTIPLICATIVE CALCULUS

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Abstract. The purpose of this paper is to investigate some properties of multiplicative regular and periodic Sturm-Liouville problems given in general form. We first introduce regular and periodic Sturm-Liouville (S-L) problems in multiplicative analysis by using some algebraic structures. Then, we discuss the main properties such as orthogonality of different eigenfunctions of the given problems. We show that the eigenfunctions corresponding to same eigenvalues are unique modulo a constant multiplicative factor and reality of the eigenvalues of multiplicative regular S-L problems. Finally, we present some examples to illustrate our main results.

1. Introduction

Grossman and Katz established a new part of analysis by giving definitions of new kinds of derivatives and integrals in the period between 1967 and 1970, which is called non-Newtonian calculus [12, 13]. This calculus provides alternative approaches to the classical calculus developed by Newton and Leibniz. Non-Newtonian calculus has many subbranches as multiplicative, anageometric, biogeometric, quadratic, and harmonic calculus. One of the most popular of them is multiplicative calculus. Arithmetics, which are a complete ordered field on a subset of real numbers, play a substantial role in the construction of non-Newtonian calculus. It is well known that the system of real numbers is a classical arithmetic. Each

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arithmetic yields one generator, the opposite of this is also true, i.e., each generator yields one arithmetic. For instance, usual arithmetic and geometric arithmetic are produced by the generators I (unit operator) and exp, respectively. Also, the function $\sigma(x) = \frac{e^x-1}{e^x+1}$ is a generator for sigmoidal arithmetic which characterizes sigmoidal curves that appear in the research of biological growth and population. There is a useful relationship providing advantages to each other between ordinary Newtonian calculus and multiplicative calculus. There are actually many reasons to investigate multiplicative analysis. For instance, it is not easy to find solutions of nonlinear differential equations in general, but this theory provides more advantages to get this kind of solutions [25]. The main difference of multiplicative calculus from the classical analysis is that it moves the roles of subtraction and addition in ordinary Newtonian calculus to division and multiplication, respectively. Since several events in the real world such as the magnitudes of earthquakes, the levels of sound signals and the acidities of chemicals change exponentially, geometric calculus which is defined as multiplicative calculus provides a great benefit. Multiplicative calculus is convenient for some problems, e.g., in applied mathematics [1, 3, 4, 6, 19, 28, 29], mathematical analysis [6, 15, 19, 21, 24, 30], spectral analysis [11, 14, 31], physics [10, 22], biology [16, 17], economics and finance [7, 8], medicine [9], pattern recognition in images [18] and signal processing [20]. In recent years, multiplicative calculus has received a lot of attention, and most of the published research has been interested in some problems of differential equation, integral equation, spectral analysis, mathematical analysis. Sturm-Liouville equations lead to the development of many problems in mathematics and physics [32]. Important results have been obtained on Sturm-Liouville equations by many researchers over the years. Recently, some spectral properties of Sturm-Liouville problems in multiplicative calculus have been studied by many authors [11, 14, 31]. In [11], the author has moved a special S-L problem in the usual case to multiplicative calculus in the aspect of spectral analysis. He has investigated asymptotic behaviors of eigenvalues and eigenfunctions of the given S-L problem.

General properties of multiplicative Sturm-Liouville problems which arise in many problems of mathematics, physics, engineering have not been studied in multiplicative analysis yet. In this paper, we deal with multiplicative Sturm-Liouville problems in general form. We give some general properties of multiplicative regular and periodic Sturm-Liouville problems.

The paper is organized as follows. In Section 2, we recall some main definitions and concepts in multiplicative analysis. In Section 3, we present orthogonality of different eigenfuctions corresponding to different eigenvalues of multiplicative Sturm-Liouville problems and we discuss the uniqueness with a constant factor difference of eigenfunctions corresponding to same eigenvalues. Also, we find that the eigenvalues of multiplicative regular S-L problems are real. Finally, we give some applications of our main problems in the last section.

2. Preliminaries

In this section, we will recall some well-known fundamental definitions and theorems of the multiplicative calculus given in [2, 12, 13, 23].
Non-Newtonian calculus use different types of arithmetic and their generators. Let α be a bijection between subsets X and Y of the set of real numbers $\mathbb{R}$, with $\alpha : X \to Y \subset \mathbb{R}$. α defines an arithmetic if the following operators are satisfied:

$$\begin{aligned} x \oplus y &= \alpha\left(\alpha^{-1}(x) + \alpha^{-1}(y)\right) \\ x \ominus y &= \alpha\left(\alpha^{-1}(x) - \alpha^{-1}(y)\right) \\ x \odot y &= \alpha\left(\alpha^{-1}(x).\alpha^{-1}(y)\right) \\ x \oslash y &= \alpha\left(\alpha^{-1}(x)/\alpha^{-1}(y)\right). \end{aligned} \tag{1}$$

If we choose α as identity function and $X = \mathbb{R}$, then (1) reduces to standard arithmetic and we get the ordinary Newtonian calculus.
Throughout the paper, we fix $\alpha(x) = e^x$, $\alpha^{-1}(x) = \ln(x)$, and $X = \mathbb{R}^+$. Then, it follows from (1)

$$\begin{aligned} x \oplus y &= x.y, \\ x \ominus y &= \frac{x}{y}, \\ x \odot y &= x^{\ln(y)}, \\ x \oslash y &= x^{\ln(\frac{1}{y})}. \end{aligned}$$

Let $a, b, c \in \mathbb{R}^+$. The operation $\odot$ satisfies the following properties (cf. Proposition 2.1 of [5, 24])

i) $a \odot b = b \odot a$ (commutativity)
ii) $a \odot (b \odot c) = (a \odot b) \odot c$ (associativity)
iii) $a \odot e = a$ (Euler's number e is the neutral element for $\odot$)
iv) If $a^{\{-1\}} = e \oslash a$, $a \neq 1$, then $a \odot a^{\{-1\}} = e$ (inverse element)
v) $b \odot a^{\{-1\}} = b \oslash a$
vi) $\left(a^{\{-1\}}\right)^{\{-1\}} = a$
vii) $\ln(a \odot b) = \ln(a) \oplus \ln(b)$
viii) $(a \odot b)^{\{-1\}} = a^{\{-1\}} \odot b^{\{-1\}}$.

In view of the mentioned properties, $(\mathbb{R}^+, \oplus, \odot)$ is a field (see [5]).
Let A be a set of positive functions defined on a subset of $\mathbb{R}$ and let $\oplus : A\text{x}A \to A$ be an operation satisfying the following properties:

$$\begin{aligned} f \oplus g &= fg \\ f \ominus g &= \frac{f}{g} \\ f \odot g &= f^{\ln g} = g^{\ln f}. \end{aligned} \tag{2}$$

Then, the algebraic structure $(A, \oplus)$ is called a multiplicative group and $(A, \oplus, \odot)$ is a multiplicative ring [2]. This situation allows us to define different structures.

Definition 1. *Let $S \subset A \neq \emptyset$ and $<,>_*: S \times S \to \mathbb{R}^+$ be a mapping such that the following axioms are satisfied for each $f, g, h \in S$:*

i) $< f, f >_* \geq 1$,
ii) $< f, f >_* = 1$ *if* $f = 1$,
iii) $< f \oplus g, h >_* = < f, h >_* \oplus < g, h >_*$,
iv) $< e^{\alpha} \odot f, g >_* = e^{\alpha} \odot < f, g >_*$, $\alpha \in \mathbb{R}$,
v) $< f, g >_* = < g, f >_*$.

This mapping is called multiplicative inner product on S and is denoted by $<,>_*$. Also, the space $(S, <,>_*)$ is called the *inner product space [11].

Definition 2 (see [2])**.** *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}^+$ be a positive function. The multiplicative derivative of the function f, which is denoted by f^*, is defined as*

$$f^*(x) = \lim_{h \to 0} \left[\frac{f(x+h)}{f(x)} \right]^{\frac{1}{h}},$$

if the above limit exists. Note that the multiplicative derivative is also called geometric derivative.

Since f is a positive function, we can write the multiplicative derivative in the following form

$$f^*(x) = e^{(\ln \circ f)'(x)}$$

by using the properties of the classical derivative. It is seen that there exists the following relation between the classical derivative and multiplicative derivative

$$f'(x) = f(x) \ln f^*(x),$$

where f is a positive function. Moreover, the second order multiplicative derivative of f is obtained by taking multiplicative derivative of the function f^* and it is represented by f^{**}. By taking n-times multiplicative derivative of the function f consecutively, we get n-th order multiplicative derivative of the function f at the point x as

$$f^{*(n)}(x) = e^{(\ln \circ f)^{(n)}(x)}.$$

Theorem 1 (see [2])**.** *Assume that f,g are multiplicative differentiable functions and h is a classical differentiable function at the point x. Then, it follows*

i) $(cf)^*(x) = f^*(x)$,
ii) $(fg)^*(x) = f^*(x)\, g^*(x)$,
iii) $\left(\frac{f}{g}\right)^*(x) = \frac{f^*(x)}{g^*(x)}$,
iv) $\left(f^h\right)^*(x) = f^*(x)^{h(x)}\, f(x)^{h'(x)}$,

v) $(f \circ h)^{*}(x) = f^{*}(h(x))^{h'(x)}$,

vi) $(f+g)^{*}(x) = f^{*}(x)^{f(x)/(f(x)+g(x))}\, g^{*}(x)^{g(x)/(f(x)+g(x))}$,

where c is a positive constant.

Definition 3 (see [2])**.** *Let f be a positive bounded function on $[a,b]$ where $-\infty < a < b < \infty$. A multiplicative integral of the function f is defined by*

$$\int_a^b f(x)^{dx} = e^{\int_a^b (\ln f(x))dx}$$

if f is Riemann integrable on $[a,b]$.

On the other hand, the multiplicative integral of f on $[a,b]$ shows that

$$\int_a^b f(x)\,dx = \ln \int_a^b \left(e^{f(x)}\right)^{dx}.$$

This multiplicative integral has the following properties:

i) $\displaystyle\int_a^b \left[f(x)^k\right]^{dx} = \left[\int_a^b f(x)^{dx}\right]^k$,

ii) $\displaystyle\int_a^b [f(x)\,g(x)]^{dx} = \int_a^b f(x)^{dx} \int_a^b g(x)^{dx}$,

iii) $\displaystyle\int_a^b \left[\frac{f(x)}{g(x)}\right]^{dx} = \frac{\int_a^b f(x)^{dx}}{\int_a^b g(x)^{dx}}$,

iv) $\displaystyle\int_a^b f(x)^{dx} = \int_a^c f(x)^{dx} \int_c^b f(x)^{dx}$,

where f, g are multiplicative integrable functions, $k \in \mathbb{R}$ is a constant and $c \in [a,b]$.

Definition 4. *Assume that $y_1, y_2, \ldots, y_n$ functions are positive functions which are multiplicative differentiable at least (n-1) times and a matrix M with dimension $n \times n$ is defined as*

$$M = \begin{pmatrix} \ln y_1 & \ln y_2 & \ldots & \ln y_n \\ \ln y_1^{*} & \ln y_2^{*} & \ldots & \ln y_n^{*} \\ \vdots & \vdots & \ldots & \vdots \\ \ln y_1^{*(n-1)} & \ln y_2^{*(n-1)} & \ldots & \ln y_n^{*(n-1)} \end{pmatrix}.$$

Then, the determinant W_n defined as

$$W_n\left(y_1, y_2, \ldots, y_n\right) = \det M$$

is called the multiplicative Wronskian determinant of the functions $\{y_i\}_{i=1}^{n}$ [26].

Note that the space $L_2^*[a,b] = \left\{ f : \int_a^b \left[f(x) \odot f(x)\right]^{dx} < \infty \right\}$ is an *inner product space with multiplicative inner product

$$<,>_*: L_2^*[a,b] \times L_2^*[a,b] \to \mathbb{R}^+, \quad < f, g >_* = \int_a^b \left[f(x) \odot g(x)\right]^{dx},$$

where $f, g \in L_2^*[a,b]$ are positive functions. It is clear that the space $L_2^*[a,b]$ is the multiplicative analogue of the well-known $L_2[a,b]$. Since this space is a linear space and the field that we study is a special field whose scalars are real numbers, it helps us to find the properties of eigenvalues of the problems. Hence, it is important to study in the field $(\mathbb{R}^+, \oplus, \odot)$ for our results.

Definition 5. i) *The n-th order multiplicative linear differential expression is given by*

$$T(y) = \left[y^{*(n)}\right]^{a_n(x)} \left[y^{*(n-1)}\right]^{a_{n-1}(x)} \ldots y^{a_0(x)}.$$

Here $a_n(x),\ a_{n-1}(x), \ldots, a_0(x)$ are continuous exponents on $[a,b]$ and $y(x) \in C^{*(n)}$, where $C^{*(n)}$ is the set of the functions which are n-th order multiplicative differentiable and continuous.

ii) *A solution of $T(y) = y^{\lambda}$ which satisfies $y \neq 1$ and $y \in L_2^*[a,b]$ is called a multiplicative eigenfunction of the operator T and the corresponding value of λ is called a multiplicative eigenvalue of the operator T [11].*

3. Main Results

Let us start our discussion with the boundary value problem

$$L[y] = \left(\left(y^*\right)^{p(x)}\right)^* y^{q(x)} = y^{-\lambda s(x)}; \ \ p(x) > 0, s(x) > 0 \tag{3}$$

defined on (a,b), which has the boundary conditions

$$\begin{aligned} &\left(y(a)\right)^{a_1} \left(y^*(a)\right)^{a_2} = 1 \\ &\left(y(b)\right)^{b_1} \left(y^*(b)\right)^{b_2} = 1, \end{aligned} \tag{4}$$

where $a_i,\ b_i;\ i = 1, 2$ are given constants, $a_1^2 + a_2^2 \neq 0,\ b_1^2 + b_2^2 \neq 0,\ p(x),\ p^*(x),$ $q(x)$ and $s(x)$ are to be assumed continuous for $x \in [a,b]$. This problem is called a multiplicative regular Sturm-Liouville problem.

Definition 6. *Let $\{\psi_k\}$ be a sequence of multiplicative integrable functions and s be a positive function on $[a,b]$. If the following equation holds for $k \neq j$*

$$\int_a^b \left(\psi_k(x)^{s(x)\ln\psi_j(x)}\right)^{dx} = 1, \tag{5}$$

then the sequence of functions $\{\psi_k\}$ is orthogonal with respect to the weight function s on $[a,b]$. In particular, the special case of this definition for $s=1$ was given in [11].

Theorem 2. *Let $\psi_j(x,\lambda_j)$ and $\psi_k(x,\lambda_k)$ be multiplicative eigenfunctions of the regular Sturm-Liouville problem* (3)-(4) *corresponding to different multiplicative eigenvalues λ_j and λ_k, respectively. Then, $\psi_j(x,\lambda_j)$ and $\psi_k(x,\lambda_k)$ are orthogonal with respect to the weight function s.*

Proof. Since $\psi_j(x,\lambda_j)$ and $\psi_k(x,\lambda_k)$ are the solutions of the equation (3), we can write

$$\left(\psi_j^*(x)^{p(x)}\right)^* \psi_j(x)^{q(x)} = \psi_j(x)^{-\lambda_j s(x)} \tag{6}$$

$$\left(\psi_k^*(x)^{p(x)}\right)^* \psi_k(x)^{q(x)} = \psi_k(x)^{-\lambda_k s(x)}. \tag{7}$$

By using (6) and (7), we obtain

$$\frac{\left(\psi_j^*(x)^{p(x)}\right)^{*\ln\psi_k(x)}}{\left(\psi_k^*(x)^{p(x)}\right)^{*\ln\psi_j(x)}} = \frac{\psi_j(x)^{-\lambda_j s(x)\ln\psi_k(x)}}{\psi_k(x)^{-\lambda_k s(x)\ln\psi_j(x)}}. \tag{8}$$

On the other hand, we have

$$\frac{\left(\psi_j^*(x)^{p(x)}\right)^{*\ln\psi_k(x)}}{\left(\psi_k^*(x)^{p(x)}\right)^{*\ln\psi_j(x)}} = \frac{\left(\psi_j^*(x)^{p(x)\ln\psi_k(x)}\right)^*}{\left(\psi_k^*(x)^{p(x)\ln\psi_j(x)}\right)^*} = \left(\frac{\psi_j^*(x)^{p(x)\ln\psi_k(x)}}{\psi_k^*(x)^{p(x)\ln\psi_j(x)}}\right)^*. \tag{9}$$

From (8) and (9), it follows

$$\left(\frac{\psi_j^*(x)^{p(x)\ln\psi_k(x)}}{\psi_k^*(x)^{p(x)\ln\psi_j(x)}}\right)^* = \frac{\psi_j(x)^{-\lambda_j s(x)\ln\psi_k(x)}}{\psi_k(x)^{-\lambda_k s(x)\ln\psi_j(x)}}. \tag{10}$$

By taking the multiplicative integral from a to b in (10) and using the properties of multiplicative integrals given in previous section, the following equation is obtained

$$\frac{\psi_j^*(b)^{p(b)\ln\psi_k(b)}}{\psi_k^*(b)^{p(b)\ln\psi_j(b)}}\frac{\psi_k^*(a)^{p(a)\ln\psi_j(a)}}{\psi_j^*(a)^{p(a)\ln\psi_k(a)}} = \left(\int_a^b \left(\psi_k(x)^{s(x)\ln\psi_j(x)}\right)^{dx}\right)^{\lambda_k-\lambda_j}. \tag{11}$$

By the help of the boundary conditions in (4) and the equation (11), we find

$$\left(\int_a^b \left(\psi_k(x)^{s(x)\ln\psi_j(x)}\right)^{dx}\right)^{\lambda_k-\lambda_j} = 1.$$

Since $\lambda_j \neq \lambda_k$, the proof of theorem is completed. □

Now, consider a multiplicative periodic Sturm-Liouville problem

$$L[y] = \left((y^*)^{p(x)}\right)^* y^{q(x)} = y^{-\lambda s(x)}, \ \ x \in [a,b]$$

with the periodic boundary conditions

$$\begin{aligned} y(a) &= y(b) \\ y^*(a) &= y^*(b), \end{aligned} \tag{12}$$

where $p(a) = p(b)$.

Theorem 3. *The multiplicative eigenfunctions of the multiplicative periodic Sturm-Liouville problem* (3)-(12) *are orthogonal with respect to the weight function* s *on* $[a,b]$.

Proof. Let $\psi_j(x,\lambda_j)$ and $\psi_k(x,\lambda_k)$ be multiplicative eigenfunctions corresponding to distinct multiplicative eigenvalues λ_j and λ_k, respectively. Since ψ_j and ψ_k satisfy the periodic boundary conditions, we have

$$\begin{array}{lcl} \psi_j(a) = \psi_j(b) & , & \psi_j^*(a) = \psi_j^*(b) \\ \psi_k(a) = \psi_k(b) & , & \psi_k^*(a) = \psi_k^*(b). \end{array} \tag{13}$$

By using (3), we can easily find

$$\left(\frac{\psi_j^*(b)^{\ln \psi_k(b)}}{\psi_k^*(b)^{\ln \psi_j(b)}}\right)^{p(b)} \left(\frac{\psi_k^*(a)^{\ln \psi_j(a)}}{\psi_j^*(a)^{\ln \psi_k(a)}}\right)^{p(a)} = \left(\int_a^b \left(\psi_k(x)^{s(x)\ln \psi_j(x)}\right)^{dx}\right)^{\lambda_k - \lambda_j}.$$

Since $p(a) = p(b)$ is in the periodic Sturm-Liouville problem and by the help of (13), by taking into account multiplicative algebraic operations given by (2) it follows

$$\int_a^b \left(\psi_k(x)^{s(x)\ln \psi_j(x)}\right)^{dx} = 1$$

for $\lambda_j \neq \lambda_k$. □

Lemma 1. *All multiplicative eigenvalues of the multiplicative regular Sturm-Liouville problem (3)-(4) are real.*

Proof. Let $\lambda_j = \alpha + i\beta$ be a complex multiplicative eigenvalue of the problem (3)-(4) corresponding the eigenfunction $\psi_j(x,\lambda_j)$. Then, $\lambda_k = \alpha - i\beta$, which is the conjugate of the multiplicative eigenvalue of λ_j, is also the multiplicative eigenvalue for (3)-(4) corresponding to eigenfunction $\psi_k(x,\lambda_k)$. By means of Theorem 2

$$\left(\int_a^b \left(\psi_k(x,\lambda_k)^{s(x)\ln \overline{\psi_k(x,\lambda_k)}}\right)^{dx}\right)^{\lambda_k - \lambda_j} = 1, \tag{14}$$

from rule (i) in Definition 3, it follows

$$\int_a^b \left(\left(\psi_k\left(x,\lambda_k\right)^{s(x)\ln\overline{\psi_k(x,\lambda_k)}}\right)^{\lambda_k-\lambda_j}\right)^{dx} = 1.$$

By definition of the multiplicative integral, we obtain

$$2i\beta\int_a^b s\left(x\right)\left|\ln\psi_k\left(x,\lambda_k\right)\right|^2 dx = 0.$$

The last equation holds if and only if when $\beta = 0$. Because, $s\left(x\right)$ is a positive function and $\psi_k\left(x,\lambda_k\right)$ can not be equal to 1 . This is a contradiction, i.e., all multiplicative eigenvalues of (3)-(4) are real. □

Theorem 4. *If the functions $\psi_j\left(x\right)$ and $\psi_k\left(x\right)$ are any two solutions of (3) on $[a,b]$, then the following equation is verified*

$$p\left(x\right)W\left(x;\psi_j,\psi_k\right) = \mu,$$

where W is Wronskian and μ is a constant.

Proof. Since the functions $\psi_j\left(x,\lambda\right)$ and $\psi_k\left(x,\lambda\right)$ are solutions of the following equation on $[a,b]$

$$L\left[y\right] = y^{-\lambda s(x)},$$

it follows from (10)

$$\left(\frac{\psi_j^*\left(x\right)^{p(x)\ln\psi_k(x)}}{\psi_k^*\left(x\right)^{p(x)\ln\psi_j(x)}}\right)^* = 1. \tag{15}$$

By taking the multiplicative integral from a to x in (15), we find

$$\left(\frac{\psi_j^*\left(x\right)^{\ln\psi_k(x)}}{\psi_k^*\left(x\right)^{\ln\psi_j(x)}}\right)^{p(x)}\left(\frac{\psi_k^*\left(a\right)^{\ln\psi_j(a)}}{\psi_j^*\left(a\right)^{\ln\psi_k(a)}}\right)^{p(a)} = 1. \tag{16}$$

From the definition of Wronskian, we get

$$\begin{aligned} W\left(x;\psi_j,\psi_k\right) &= \ln\psi_k^*\left(x\right)^{\ln\psi_j(x)} - \ln\psi_j^*\left(x\right)^{\ln\psi_k(x)} \\ &= \ln\left(\frac{\psi_k^*\left(x\right)^{\ln\psi_j(x)}}{\psi_j^*\left(x\right)^{\ln\psi_k(x)}}\right). \end{aligned} \tag{17}$$

By using (16) and (17) the following can be easily seen

$$e^{-W\left(x;\psi_j,\psi_k\right)p(x)}e^{W\left(a;\psi_j,\psi_k\right)p(a)} = 1.$$

It is seen that

$$W\left(x;\psi_j,\psi_k\right)p\left(x\right) = W\left(a;\psi_j,\psi_k\right)p\left(a\right).$$

By the help of the last equality, the proof is completed. □

Theorem 5. *The multiplicative eigenfunction corresponding to any multiplicative eigenvalue of the regular Sturm-Liouville problem given by (3)-(4) is unique with a constant factor difference.*

Proof. Let $\psi_j(x,\lambda)$ and $\psi_k(x,\lambda)$ be multiplicative eigenfunctions of (3) corresponding to multiplicative eigenvalue λ. From Theorem 4, we have

$$p(x)\,W\left(x;\psi_j,\psi_k\right)=\mu,$$

where $p>0$. It is clear from this equation that for each any point $x_0\in[a,b]$ if $W\left(x_0;\psi_j,\psi_k\right)=0$, then for all $x_0\in[a,b]$ it should be $W\left(x;\psi_j,\psi_k\right)\equiv 0$. On the other hand, by using the boundary condition (4), we obtain

$$\psi_j^*(a)^{a_2}=\frac{1}{\psi_j(a)^{a_1}},\quad \psi_k^*(a)^{a_2}=\frac{1}{\psi_k(a)^{a_1}}\ . \tag{18}$$

Since a_1 and a_2 should not be zero at once, it follows from the definition of Wronskian

$$e^{-W\left(a;\psi_j,\psi_k\right)}=\frac{\psi_k^*(a)^{\ln\psi_j(a)}}{\psi_j^*(a)^{\ln\psi_k(a)}}.$$

By using (18) and the last equality, we get

$$e^{-W\left(a;\psi_j,\psi_k\right)}=\frac{\psi_j(a)^{\frac{a_1}{a_2}\ln\psi_k(a)}}{\psi_k(a)^{\frac{a_1}{a_2}\ln\psi_j(a)}}=1,$$

which gives $W=0$ at the point $x_0=a\in[a,b]$. So, $W\equiv 0$ on $[a,b]$. This is a sufficient condition for ψ_j and ψ_k to be linear dependent. Therefore, one of these solution is a constant multiple of the other. $\square$

4. Applications

In this section, we will give some examples of multiplicative Sturm-Liouville problems defined by (3)-(4) and (3)-(12).

Example 1. *Consider the multiplicative eigenvalue problem*

$$\begin{gathered} y^{**}y^{\lambda}=1,\quad 0\le x\le\pi \\ y(0)=y^*(\pi)=1. \end{gathered} \tag{19}$$

Assume that $\lambda\le 0$. We get the solution of (19) as follow

$$y(x)=e^{c_1e^{\sqrt{-\lambda}x}+c_2e^{-\sqrt{-\lambda}x}},$$

where c_1 and c_2 are real numbers. Since $y(0)=y^*(\pi)=1$, $c_1=c_2=0$ is found. Since for $\lambda\le 0$ we have the trivial multiplicative eigenfunction $y(x,\lambda)=1$ of the problem (19)*, there is no eigenvalue for $\lambda\le 0$. Now, assume that $\lambda>0$. We get the solution of* (19)

$$y(x)=e^{l_1\cos\sqrt{\lambda}x+l_2\sin\sqrt{\lambda}x},$$

where l_1 and l_2 are real numbers. Using the condition $y(0) = y^*(\pi) = 1$, we obtain the multiplicative eigenfunctions

$$y_n(x) = e^{l_2 \sin\left(\frac{2n-1}{2}\right)x}$$

of (19) corresponding to eigenvalues

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2; \quad n = 1, 2, 3 \ldots$$

Without loss of generality, by taking $l_2 = 1$, we get the family of eigenfunctions

$$y_n(x) = e^{\sin\left(\frac{2n-1}{2}\right)x}.$$

Remark 1. *Since different multiplicative eigenfunctions corresponding to different eigenvalues of (3)-(4) are orthogonal as a consequence of Theorem 2, the following result holds*

$$\int_0^\pi \left(y_n(x)^{\ln y_m(x)}\right)^{dx} = \int_0^\pi \left(e^{\sin\left(\frac{2n-1}{2}x\right)\sin\left(\frac{2m-1}{2}x\right)}\right)^{dx} = 1.$$

Example 2. *Let us consider the following multiplicative periodic eigenvalue problem*

$$y^{**}y^{\lambda} = 1, \quad 0 \le x \le \pi \tag{20}$$
$$y(0) = y(\pi), \quad y^*(0) = y^*(\pi).$$

Since we have trivial eigenfunction $y(x) = 1$ for $\lambda < 0$, there is no eigenfunction of S-L problem for $\lambda < 0$. For $\lambda = 0$, the nontrivial solution of the problem is obtained as $y(x) = e$. Now, suppose that $\lambda > 0$. Then, we find the solution of (20) as follow

$$y(x) = e^{k_1 \cos\sqrt{\lambda}x + k_2 \sin\sqrt{\lambda}x},$$

where k_1 and k_2 are real numbers. From $y(0) = y(\pi)$, $y^*(0) = y^*(\pi)$, we find

$$k_2 = -k_1 \sin\sqrt{\lambda}\pi + k_2 \cos\sqrt{\lambda}\pi, \quad k_1 = k_1 \cos\sqrt{\lambda}\pi + k_2 \sin\sqrt{\lambda}\pi,$$

from which it gives $k_1 = k_2 = 0$. So, we get only a trivial eigenfunction of (20) for $\lambda > 0$. Thus, it is seen that the nontrivial solution of the given multiplicative periodic eigenvalue problem is $y(x) = e$ corresponding to eigenvalue $\lambda = 0$.

Example 3. *Consider the following multiplicative Sturm-Liouville problem*

$$(y^{**})^{x^2}(y^*)^{x}y^{\lambda} = 1 \tag{21}$$
$$y(1) = 1, \quad y(e) = 1.$$

It is known that the following equalities are provided when the substitution $x = e^t$ is applied [27]

$$\left(\widetilde{D}y\right)^{x} = \widetilde{D}_1 y \quad \text{and} \quad \left(\widetilde{D}^{(2)}y\right)^{x^2} = \left(\widetilde{D}_1^{(2)}y\right)\left(\widetilde{D}_1 y\right)^{-1}, \tag{22}$$

where $\widetilde{D}$ is the multiplicative derivative operator of y with respect to x and $\widetilde{D}_1$ is the multiplicative derivative operator of y with respect to t. By the help of (22), we write

$$(y^*)^x = \left(e^{D \ln y}\right)^x = e^{D_1 \ln y} = \widetilde{D}_1 y$$

$$(y^{**})^{x^2} = \left(e^{D^2(\ln y)}\right)^{x^2} = e^{D_1(D_1-1)\ln y} = \left(\widetilde{D}_1^{(2)} y\right)\left(\widetilde{D}_1 y\right)^{-1}, \tag{23}$$

where D is the derivative operator of y with respect to x and D_1 is the derivative operator of y with respect to t. Then, from (21) *and (23), we get*

$$e^{(D_1(D_1-1)+D_1+\lambda)\ln y} = 1,$$

from which it follows $(D_1^2 + \lambda)\ln y = 0$. If $\lambda \leq 0$, then we get

$$y(x) = e^{m_1 e^{\sqrt{-\lambda}\ln x} + m_2 e^{-\sqrt{-\lambda}\ln x}},$$

where m_1 and m_2 are real numbers. By using the condition $y(1) = 1$, $y(e) = 1$, it is clear that $m_1 = m_2 = 0$. Since for $\lambda \leq 0$ we have the trivial multiplicative eigenfunction $y(x, \lambda) = 1$ of this example, there is no eigenvalue for $\lambda \leq 0$. If $\lambda > 0$, we find the solution of (21) *as follow*

$$y(x) = e^{v_1 \cos\left(\sqrt{\lambda}\ln x\right) + v_2 \sin\left(\sqrt{\lambda}\ln x\right)},$$

where v_1 and v_2 are real numbers. From the boundary condition $y(1) = 1$,
$y(e) = 1$, we get the multiplicative eigenfunctions

$$y_n(x) = e^{v_2 \sin(n\pi \ln x)}$$

of (21) *corresponding to eigenvalues*

$$\lambda_n = (n\pi)^2,$$

where $n = 1, 2, 3, \ldots$.

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