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A Note on Some Generalized Curvature Tensor

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

Abstract

For any semi-Riemannian manifold (M, g) we define some generalized curvature tensor E as a linear combination of Kulkarni-Nomizu products formed by the metric tensor, the Ricci tensor and its square of given manifold. That tensor is closely related to quasi-Einstein spaces, Roter spaces and some Roter type spaces.

Keywords: *warped product manifold; Einstein, quasi-Einstein and partially Einstein manifold; generalized Einstein metric condition; pseudosymmetry type curvature condition; Roter and generalized Roter space; hypersurface.*

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1. Introduction

Let (M, g) be a semi-Riemannian manifold. We denote by g, R, S, κ and C, the metric tensor, the Riemann-Christoffel curvature tensor, the Ricci tensor, the scalar curvature and the Weyl conformal curvature tensor of (M, g), respectively. Further, let $A \wedge B$ be the Kulkarni-Nomizu product of symmetric $(0, 2)$-tensors A and B. Now we can define the $(0, 2)$-tensors S^2 and S^3, the $(0, 4)$-tensors $R \cdot S$, $C \cdot S$ and $Q(A, B)$, and the $(0, 6)$-tensors $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$ and $Q(A, T)$, where T is a generalized curvature tensor. For precise definitions of the symbols used, we refer to Section 2 of this paper, as well as to [34, Section 1], [37, Section 1], [38, Chapter 6] and [45, Sections 1 and 2].

A semi-Riemannian manifold (M, g), $\dim M = n \geq 2$, is said to be an *Einstein manifold* [2], or an *Einstein space*, if at every point of M its Ricci tensor S is proportional to g, i.e.,

$$S = \frac{\kappa}{n}\, g \tag{1.1}$$

on M, assuming that the scalar curvature κ is constant when $n = 2$. According to [2, p. 432] this condition is called the *Einstein metric condition*.

Let (M, g) be a semi-Riemannian manifold of dimension $\dim M = n \geq 3$. We set

$$E = g \wedge S^2 + \frac{n-2}{2}\, S \wedge S - \kappa\, g \wedge S + \frac{\kappa^2 - \mathrm{tr}_g(S^2)}{2(n-1)}\, g \wedge g. \tag{1.2}$$

It is easy to check that the tensor E is a generalized curvature tensor. Further, we define the subsets $\mathcal{U}_R$ and $\mathcal{U}_S$ of M by $\mathcal{U}_R = \{x \in M \mid R - \frac{\kappa}{(n-1)n}\, G \neq 0 \text{ at } x\}$ and $\mathcal{U}_S = \{x \in M \mid S - \frac{\kappa}{n}\, g \neq 0 \text{ at } x\}$, respectively, where $G = \frac{1}{2}\, g \wedge g$. If $n \geq 4$ then we define the set $\mathcal{U}_C \subset M$ as the set of all points of (M, g) at which $C \neq 0$. We note that if $n \geq 4$ then $\mathcal{U}_S \cup \mathcal{U}_C = \mathcal{U}_R$ (see, e.g., [24]).

An extension of the class of Einstein manifolds form quasi-Einstein, 2-quasi-Einstein and partially Einstein manifolds. A semi-Riemannian manifold (M, g), $\dim M = n \geq 3$, is said to be a *quasi-Einstein manifold*, or a

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quasi-Einstein space, if

$$\operatorname{rank}(S - \alpha\, g) = 1 \tag{1.3}$$

on $\mathcal{U}_S \subset M$, where α is some function on $\mathcal{U}_S$. It is known that every non-Einstein warped product manifold $\overline{M} \times_F \widetilde{N}$ with a 1-dimensional $(\overline{M}, \overline{g})$ base manifold and a 2-dimensional manifold $(\widetilde{N}, \widetilde{g})$ or an $(n-1)$-dimensional Einstein manifold $(\widetilde{N}, \widetilde{g})$, $\dim M = n \geq 4$, and a warping function F, is a quasi-Einstein manifold (see, e.g., [7, 34]). A Riemannian manifold (M, g), $\dim M = n \geq 3$, whose Ricci tensor has an eigenvalue of multiplicity $n-1$ is a non-Einstein quasi-Einstein manifold (cf. [23, Introduction]). We mention that quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and the investigation on quasi-umbilical hypersurfaces of conformally flat spaces (see, e.g., [27, 34] and references therein). Quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature were studied among others in [29, 40, 43, 61] (see also [27] and references therein). Quasi-Einstein manifolds satisfying some pseudosymmetry type curvature conditions were investigated recently in [1, 7, 24, 31, 42]. Quasi-Einstein hypersurfaces in conformally flat semi-Riemannian manifolds were studied in [55, 78]. In those papers quasi-Einstein hypersurfaces were called *pseudo-Einstein hypersurfaces* (see also [66, 71]). Similarly, in [50, 86, 87] quasi-Einstein semi-Riemannian manifolds (hypersurfaces) were called *pseudo-Einstein manifolds (hypersurfaces)*.

Let (M, g), $\dim M = n \geq 3$, be a semi-Riemannian manifold. We note that (1.3) holds at a point $x \in \mathcal{U}_S \subset M$ if and only if $(S - \alpha\, g) \wedge (S - \alpha\, g) = 0$ at x, i.e.,

$$\frac{1}{2}\, S \wedge S - \alpha\, g \wedge S + \frac{\alpha^2}{2}\, g \wedge g = 0 \tag{1.4}$$

at x (cf. [61, Proposition 2.1]). From (1.4), by a suitable contraction, we get immediately

$$S^2 = (\kappa - (n-2)\alpha)\, S + \alpha((n-1)\alpha - \kappa)\, g. \tag{1.5}$$

Using (1.1) we can easily check that the following equation is satisfied on any Einstein manifold (M, g)

$$g \wedge S^2 + \frac{n-2}{2}\, S \wedge S - \kappa\, g \wedge S + \frac{\kappa^2 - \operatorname{tr}_g(S^2)}{2(n-1)}\, g \wedge g = 0, \tag{1.6}$$

i.e., $E = 0$ on M, where the tensor E is defined by (1.2). Moreover, as it was stated in [28, Lemma 2.1], (1.6) is satisfied on every quasi-Einstein manifold (M, g), $n \geq 3$. The converse statement also is true. Precisely, from Proposition 2.1 it follows that if (M, g), $n \geq 4$, is a semi-Riemannian manifold satisfying (1.6) on $\mathcal{U}_S \subset M$ then a condition of the form (1.3) holds on $\mathcal{U}_S$, where α is some function on this set. In Section 2 we also present another result related to the tensor E (see Proposition 2.2). Namely, we prove that if a generalized curvature tensor T is a linear combination of the tensors R, $S \wedge S$, $g \wedge S$, $g \wedge S^2$, and $g \wedge g$ then the Weyl tensor of T is a linear combination of the tensors C and E. The tensor E is determined by some Kulkarni-Nomizu products formed by g, S and S^2, i.e., E is defined by (1.2). In the same way, we can define the $(0,4)$-tensor $E(A)$ corresponding to a symmetric $(0,2)$-tensor A

$$E(A) = g \wedge A^2 + \frac{n-2}{2}\, A \wedge A - \operatorname{tr}_g(A)\, g \wedge A + \frac{(\operatorname{tr}_g(A))^2 - \operatorname{tr}_g(A^2)}{2(n-1)}\, g \wedge g. \tag{1.7}$$

The semi-Riemannian manifold (M, g), $\dim M = n \geq 3$, will be called a *partially Einstein manifold*, or a *partially Einstein space* (cf. [5, Foreword], [82, p. 20]), if at every point $x \in \mathcal{U}_S \subset M$ its Ricci operator $\mathcal{S}$ satisfies $\mathcal{S}^2 = \lambda \mathcal{S} + \mu Id_x$, or equivalently,

$$S^2 = \lambda\, S + \mu\, g, \tag{1.8}$$

where $\lambda, \mu \in \mathbb{R}$ and Id_x is the identity transformation of $T_x M$. Evidently, (1.5) is a special case of (1.8). Thus every quasi-Einstein manifold is a partially Einstein manifold. The converse statement is not true. Contracting (1.8) we get $\operatorname{tr}_g(S^2) = \lambda\, \kappa + n\, \mu$. This together with (1.8) yields (cf. [25, Section 5])

$$S^2 - \frac{\operatorname{tr}_g(S^2)}{n}\, g = \lambda \left(S - \frac{\kappa}{n}\, g \right).$$

In particular, a Riemannian manifold (M, g), $\dim M = n \geq 3$, is a partially Einstein space if at every point $x \in \mathcal{U}_S \subset M$ its Ricci operator $\mathcal{S}$ has exactly two distinct eigenvalues κ_1 and κ_2 with multiplicities p and $n-p$, respectively, where $1 \leq p \leq n-1$. Evidently, if $p = 1$, or $p = n-1$, then (M, g) is a quasi-Einstein manifold.

In Section 3 we present definitions of some classes of semi-Riemannian manifolds determined by curvature conditions of pseudosymmetry type. Investigations of semi-Riemannian manifolds satisfying some particular curvature conditions of pseudosymmetry type lead to Roter spaces (see Propositon 4.1). Roter spaces form an important subclass of the class of non-conformally flat and non-quasi-Einstein partially Einstein manifolds of dimension ≥ 4. Namely, a non-quasi-Einstein and non-conformally flat semi-Riemannian manifold (M, g), $\dim M = n \geq 4$, satisfying on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ the following equation

$$R = \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g, \tag{1.9}$$

where ϕ, μ and η are some functions on this set, is called a *Roter type manifold*, or a *Roter manifold*, or a *Roter space* (see, e.g., [6, Section 15.5], [22, 34, 35, 38]). Equation (1.9) is called a *Roter equation* (see, e.g., [28, Section 1]). In Section 4 we present results on such manifolds. For instance, every Roter space (M, g), $\dim M = n \geq 4$, satisfies among others the following pseudosymmetry type curvature condition on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ (see Theorem 4.1)

$$C \cdot R - R \cdot C = Q(S, C) - \frac{\kappa}{n-1}\, Q(g, C). \tag{1.10}$$

Let (M, g), $\dim M = n \geq 4$, be a non-partially-Einstein and non-conformally flat semi-Riemannian manifold. If its Riemann-Christoffel curvature R is at every point of $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ a linear combination of the Kulkarni-Nomizu products formed by the tensors $S^0 = g$ and $S^1 = S, \ldots, S^{p-1}, S^p$, where p is some natural number ≥ 2, then (M, g) is called a *generalized Roter type manifold*, or a *generalized Roter manifold*, or a *generalized Roter type space*, or a *generalized Roter space*. For instance, when $p = 2$, we have

$$R = \frac{\phi_2}{2}\, S^2 \wedge S^2 + \phi_1\, S \wedge S^2 + \frac{\phi}{2}\, S \wedge S + \mu_1\, g \wedge S^2 + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g, \tag{1.11}$$

where ϕ, ϕ_1, ϕ_2, μ_1, μ and η are functions on $\mathcal{U}_S \cap \mathcal{U}_C$. Because (M, g) is a non-partially Einstein manifold, at least one of the functions μ_1, ϕ_1 and ϕ_2 is a non-zero function. Equation (1.11) is called a *Roter type equation* (see, e.g., [28, Section 1]). We refer to [28, 33, 34, 35, 42, 73, 74, 75, 76, 77] for results on manifolds (hypersurfaces) satisfying (1.11).

As it was stated in [28, Lemma 2.2] (see Proposition 4.2), if (M, g), $\dim M = n \geq 4$, is a Roter space satisfying (1.9) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then on this set

$$C = \frac{\phi}{n-2} \left(g \wedge S^2 + \frac{n-2}{2}\, S \wedge S - \kappa\, g \wedge S + \frac{\kappa^2 - \mathrm{tr}_g(S^2)}{2(n-1)}\, g \wedge g \right). \tag{1.12}$$

In Section 5 we recall results on some warped product manifolds with 2-dimensional base manifold obtained in [28].

In Section 6 we state that on every essentially conformally symmetric manifold the following equation is satisfied

$$F\, C = \frac{n-2}{2(n-2)}\, S \wedge S \;=\; \frac{1}{n-2} \left(g \wedge S^2 + \frac{n-2}{2}\, S \wedge S - \kappa\, g \wedge S + \frac{\kappa^2 - \mathrm{tr}_g(S^2)}{2(n-1)}\, g \wedge g \right). \tag{1.13}$$

In Section 7 we recall some known results on hypersurfaces M, $\dim M \geq 4$, isometrically immersed in a conformally flat spaces. In particular, we mention that at every point of M its Weyl conformal curvature tensor C and the $(0, 4)$-tensor $E(H)$, formed for the second fundamental tensor H of M, are linearly dependent (see (7.3)).

In the last section we investigate non-Einstein and non-conformally flat hypersurfaces M, $\dim M \geq 4$, isometrically immersed in semi-Riemannian spaces of constant curvature satisfying some curvature conditions of pseudosymmetry type. Under some additional assumptions imposed on the second fundamental tensor H of M we obtain equations involved with the tensor E.

2. Preliminaries.

Throughout this paper, all manifolds are assumed to be connected paracompact manifolds of class C^∞. Let (M, g), $\dim M = n \geq 3$, be a semi-Riemannian manifold, and let ∇ be its Levi-Civita connection and $\Xi(M)$

the Lie algebra of vector fields on M. We define on M the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X,Y)$ of $\Xi(M)$ by $(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y$ and

$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

respectively, where $X, Y, Z \in \Xi(M)$ and A is a symmetric $(0,2)$-tensor on M. The Ricci tensor S, the Ricci operator $\mathcal{S}$ and the scalar curvature κ of (M,g) are defined by

$$S(X,Y) = \operatorname{tr}\{Z \rightarrow \mathcal{R}(Z,X)Y\}, \quad g(\mathcal{S}X,Y) \;=\; S(X,Y), \quad \kappa = \operatorname{tr}\mathcal{S},$$

respectively. The endomorphism $\mathcal{C}(X,Y)$ is defined by

$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y)Z.$$

Now the $(0,4)$-tensor G, the Riemann-Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M,g) are defined by $G(X_1,X_2,X_3,X_4) = g((X_1 \wedge_g X_2)X_3, X_4)$ and

$$R(X_1,X_2,X_3,X_4) = g(\mathcal{R}(X_1,X_2)X_3,X_4), \quad C(X_1,X_2,X_3,X_4) = g(\mathcal{C}(X_1,X_2)X_3,X_4),$$

respectively, where $X_1, X_2, \ldots \in \Xi(M)$. For a symmetric $(0,2)$-tensor A we denote by $\mathcal{A}$ the endomorphism related to A by $g(\mathcal{A}X,Y) = A(X,Y)$. The $(0,2)$-tensors A^p, $p = 2,3,\ldots$, are defined by $A^p(X,Y) = A^{p-1}(\mathcal{A}X,Y)$, assuming that $A^1 = A$. In this way, for $A = S$ and $\mathcal{A} = \mathcal{S}$ we get the tensors S^p, $p = 2,3,\ldots$, assuming that $S^1 = S$.

Let $\mathcal{B}$ be a tensor field sending any $X, Y \in \Xi(M)$ to a skew-symmetric endomorphism $\mathcal{B}(X,Y)$, and let B be the $(0,4)$-tensor associated with $\mathcal{B}$ by

$$B(X_1,X_2,X_3,X_4) = g(\mathcal{B}(X_1,X_2)X_3,X_4). \tag{2.1}$$

The tensor B is said to be a *generalized curvature tensor* if the following two conditions are fulfilled: $B(X_1,X_2,X_3,X_4) = B(X_3,X_4,X_1,X_2)$ and

$$B(X_1,X_2,X_3,X_4) + B(X_2,X_3,X_1,X_4) + B(X_3,X_1,X_2,X_4) = 0.$$

For $\mathcal{B}$ as above, let B be again defined by (2.1). We extend the endomorphism $\mathcal{B}(X,Y)$ to a derivation $\mathcal{B}(X,Y)\cdot$ of the algebra of tensor fields on M, assuming that it commutes with contractions and $\mathcal{B}(X,Y) \cdot f = 0$ for any smooth function f on M. Now for a $(0,k)$-tensor field T, $k \geq 1$, we can define the $(0,k+2)$-tensor $B \cdot T$ by

$$\begin{aligned} (B \cdot T)(X_1,\ldots,X_k,X,Y) \;&=\; (\mathcal{B}(X,Y) \cdot T)(X_1,\ldots,X_k) \\ =\; & -T(\mathcal{B}(X,Y)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},\mathcal{B}(X,Y)X_k). \end{aligned}$$

If A is a symmetric $(0,2)$-tensor then we define the $(0,k+2)$-tensor $Q(A,T)$ by

$$\begin{aligned} Q(A,T)(X_1,\ldots,X_k,X,Y) &= (X \wedge_A Y \cdot T)(X_1,\ldots,X_k) \\ =\; & -T((X \wedge_A Y)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},(X \wedge_A Y)X_k). \end{aligned}$$

In this manner we obtain the $(0,6)$-tensors $B \cdot B$ and $Q(A,B)$.

Substituting in the above formulas $\mathcal{B} = \mathcal{R}$ or $\mathcal{B} = \mathcal{C}$, $T = R$ or $T = C$ or $T = S$, $A = g$ or $A = S$ we get the tensors $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$, $R \cdot S$, $Q(g,R)$, $Q(S,R)$, $Q(g,C)$, $Q(S,C)$, and $Q(g,S)$, $Q(g,S^2)$, $Q(S,S^2)$.

For a symmetric $(0,2)$-tensor A and a $(0,k)$-tensor T, $k \geq 2$, we define their *Kulkarni-Nomizu tensor* $A \wedge T$ by (see, e.g., [24, Section 2])

$$\begin{aligned} & (A \wedge T)(X_1,X_2,X_3,X_4;Y_3,\ldots,Y_k) \\ =\; & A(X_1,X_4)T(X_2,X_3,Y_3,\ldots,Y_k) + A(X_2,X_3)T(X_1,X_4,Y_3,\ldots,Y_k) \\ & -A(X_1,X_3)T(X_2,X_4,Y_3,\ldots,Y_k) - A(X_2,X_4)T(X_1,X_3,Y_3,\ldots,Y_k). \end{aligned}$$

It is obvious that the following tensors are generalized curvature tensors: R, C and $A \wedge B$, where A and $B = T$ are symmetric $(0,2)$-tensors. We have

$$C = R - \frac{1}{n-2}\, g \wedge S + \frac{\kappa}{(n-2)(n-1)}\, G, \tag{2.2}$$

$$G = \frac{1}{2}\, g \wedge g, \tag{2.3}$$

and (see, e.g., [24, Lemma 2.2(i)])

$$
\begin{aligned}
&(a)\ \ Q(A, A\wedge B) = -\frac{1}{2}\, Q(B, A\wedge A),\\
&(b)\ \ A\wedge Q(A,B) = -\frac{1}{2}\, Q(B, A\wedge A).
\end{aligned} \tag{2.4}
$$

By an application of (2.4)(a) we obtain on M the identities

$$
Q(g, g\wedge S) = -Q(S, G), \quad Q(S, g\wedge S) = -\frac{1}{2}\, Q(g, S\wedge S). \tag{2.5}
$$

Further, by making use of (2.2), (2.3) and (2.5), we get immediately

$$
\begin{aligned}
Q(g,C) &= Q(g,R) - \frac{1}{n-2}\, Q(g, g\wedge S) + \frac{\kappa}{(n-2)(n-1)}\, Q(g,G)\\
&= Q(g,R) - \frac{1}{n-2}\, Q(g, g\wedge S) \ = \ Q(g,R) + \frac{1}{n-2}\, Q(S, \frac{1}{2}\, g\wedge g),
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
Q(S,C) &= Q(S,R) - \frac{1}{n-2}\, Q(S, g\wedge S) + \frac{\kappa}{(n-2)(n-1)}\, Q(S,G)\\
&= Q(S,R) + \frac{1}{2(n-2)}\, Q(g, S\wedge S) - \frac{\kappa}{(n-2)(n-1)}\, Q(g, g\wedge S).
\end{aligned} \tag{2.7}
$$

From (2.4) (a) it follows immmediately that $Q(g, g\wedge g) = 0$. Thus we have

$$
Q(g,E) = Q(g, g\wedge S^2 + \frac{n-2}{2}\, S\wedge S - \kappa\, g\wedge S), \tag{2.8}
$$

where the tensor E is defined by (1.2).

Let A_1, A_2 and B be symmetric $(0,2)$-tensors. We have (see, e.g., [7, Lemma 2.1(i)] and references therein)

$$
A_1\wedge Q(A_2, B) + A_2\wedge Q(A_1, B) + Q(B, A_1\wedge A_2) = 0. \tag{2.9}
$$

From (2.9) we get easily (see also [24, Lemma 2.2(iii)] and references therein)

$$
Q(B, A_1\wedge A_2) + Q(A_1, A_2\wedge B) + Q(A_2, B\wedge A_1) = 0.
$$

Let A be a symmetric $(0,2)$-tensor and T a $(0,k)$-tensor, $k = 2, 3, \ldots$. The tensor $Q(A,T)$ is called the *Tachibana tensor* of A and T, or the Tachibana tensor for short (see, e.g., [36]). Using the tensors g, R and S we can define the following $(0,6)$-Tachibana tensors: $Q(S,R)$, $Q(g,R)$, $Q(g, g\wedge S)$ and $Q(S, g\wedge S)$. We can check, by making use of (2.4)(a) and (2.5), that other $(0,6)$-Tachibana tensors constructed from g, R and S may be expressed by the four Tachibana tensors mentioned above or vanish identically on M.

Let T be a generalized curvature tensor on a semi-Riemannian manifold (M,g), $\dim M = n \geq 4$. We denote by $\mathrm{Ric}(T)$, $\kappa(T)$ and $\mathrm{Weyl}(T)$ the Ricci tensor, the scalar curvature and the Weyl tensor of the tensor T, respectively. We refer to [24, Section 2], [25, Section 3] or [31, Section 3] for definitions of the considered tensors. In particular, we have

$$
\mathrm{Weyl}(T) = T - \frac{1}{n-2}\, g\wedge \mathrm{Ric}(T) + \frac{\kappa(T)}{2(n-2)(n-1)}\, g\wedge g. \tag{2.10}
$$

Let A be a symmetric $(0,2)$-tensor on a semi-Riemannian manifold (M,g), $\dim M = n \geq 3$. Let $E(A)$ be the tensor defined by (1.7). It is easy to check that $\mathrm{Ric}(E(A))$ is a zero tensor. Therefore, we also have $\kappa(E(A)) = 0$. Any generalized curvature tensor T defined on a 3-dimensional semi-Riemannian manifold (M,g) is expressed by $T = g\wedge \mathrm{Ric}(T) - (\kappa(T)/4) g\wedge g$ [56, p. 48] (see also [21, Lemma 2 (ii)]). Thus we see that the tensor $T = E(A)$ on any 3-dimensional semi-Riemannian manifold (M,g) is a zero tensor. In particular, on any 3-dimensional semi-Riemannian manifold (M,g) we have $E = 0$.

Let A be a symmetric $(0,2)$-tensor on a semi-Riemannian manifold (M,g), $\dim M = n \geq 3$. We denote by $\mathcal{U}_A$ the set of points of M at which $A \neq \frac{\mathrm{tr}_g(A)}{n}\, g$.

Proposition 2.1. *Let A be a symmetric $(0, 2)$-tensor on a semi-Riemannian manifold (M, g), $\dim M = n \geq 4$.*
(i) (cf. [28, Lemma 2.1]) If the following condition is satisfied on $\mathcal{U}_A \subset M$

$$\mathrm{rank}(A - \alpha\, g) = 1 \tag{2.11}$$

then

$$g \wedge A^2 + \frac{n-2}{2}\, A \wedge A - \mathrm{tr}_g(A)\, g \wedge A + \frac{(\mathrm{tr}_g(A))^2 - \mathrm{tr}_g(A^2)}{2(n-1)}\, g \wedge g = 0 \tag{2.12}$$

and

$$A^2 - \frac{\mathrm{tr}_g(A^2)}{n} = (\mathrm{tr}_g(A) - (n-2)\alpha) \left(A - \frac{\mathrm{tr}_g(A)}{n}\, g \right) \tag{2.13}$$

on $\mathcal{U}_A$, where α is some function on $\mathcal{U}_A$.
(ii) If (2.12) is satisfied on $\mathcal{U}_A \subset M$ then

$$A^2 - \frac{\mathrm{tr}_g(A^2)}{n}\, g = \rho \left(A - \frac{\mathrm{tr}_g(A)}{n}\, g \right) \tag{2.14}$$

and

$$\left(A - \frac{\mathrm{tr}_g(A) - \rho}{n-2}\, g \right) \wedge \left(A - \frac{\mathrm{tr}_g(A) - \rho}{n-2}\, g \right) = 0 \tag{2.15}$$

on $\mathcal{U}_A$, where ρ is some function on $\mathcal{U}_A$.

Proof. (i) (cf. the proof of [28, Lemma 2.1]) Equation (2.11) yields [61, Proposition 2.2]

$$\frac{1}{2}\, A \wedge A = \alpha\, g \wedge A - \frac{\alpha^2}{2}\, g \wedge g. \tag{2.16}$$

This by suitable contractions yields

$$\begin{aligned} A^2 - \mathrm{tr}_g(A)\, A &= -(n-2)\alpha\, A - \alpha \mathrm{tr}_g(A)\, g + (n-1)\alpha^2\, g, && (2.17)\\ \mathrm{tr}_g(A^2) - (\mathrm{tr}_g(A))^2 &= -2(n-1)\alpha \mathrm{tr}_g(A) + n(n-1)\alpha^2. && (2.18) \end{aligned}$$

Now using (2.16), (2.17) and (2.18) we can easily check that (2.12) and (2.13) hold on $\mathcal{U}_A$.
(ii) (cf. the proof of [20, Lemma 3.4]) In the local coordinates (2.12) reads

$$\begin{aligned} & g_{hk} A^2_{ij} + g_{ij} A^2_{hk} - g_{hj} A^2_{ik} - g_{ik} A^2_{hj} + (n-2)\, (A_{hk} A_{ij} - A_{hj} A_{ik}) \\ & - \mathrm{tr}_g(A)\, (g_{hk} A_{ij} + g_{ij} A_{hk} - g_{hj} A_{ik} - g_{ik} A_{hj}) \\ & + \frac{(\mathrm{tr}_g(A))^2 - \mathrm{tr}_g(A^2)}{n-1}\, (g_{hk} g_{ij} - g_{hj} g_{ik}) = 0. \end{aligned} \tag{2.19}$$

Contracting (2.19) with $A^{ij} = A_{rs} g^{ri} g^{sj}$ and $A^k_{\ l} = A_{rl} g^{rk}$ we find

$$\begin{aligned} A^3 \;=\; & \frac{3\mathrm{tr}_g(A)}{n}\, A^2 + \left(\frac{(n^2 - 3n + 3)\mathrm{tr}_g(A^2)}{(n-1)n} - \frac{(\mathrm{tr}_g(A))^2}{n-1} \right) A \\ & + \left(\frac{(\mathrm{tr}_g(A))^3}{(n-1)n} - \frac{\mathrm{tr}_g(A)\, \mathrm{tr}_g(A^2)}{n-1} + \frac{\mathrm{tr}_g(A^3)}{n} \right) g, \end{aligned} \tag{2.20}$$

$$\begin{aligned} & A_{hl} A^2_{ij} - A_{il} A^2_{hj} + g_{ij} A^3_{hl} - g_{hj} A^3_{il} + (n-2)\, (A_{ij} A^2_{hl} - A_{hj} A^2_{il}) \\ & - \mathrm{tr}_g(A)\, (A_{hl} A_{ij} - A_{il} A_{hj} + g_{ij} A^2_{hl} - g_{hj} A^2_{il}) \\ & + \frac{(\mathrm{tr}_g(A))^2 - \mathrm{tr}_g(A^2)}{n-1}\, (g_{ij} A_{hl} - g_{hj} A_{il}) = 0, \end{aligned} \tag{2.21}$$

respectively. From (2.21), by symmetrization in l, j, we obtain

$$Q(g, A^3) + (n-3) Q(A, A^2) - \mathrm{tr}_g(A)\, Q(g, A^2) + \frac{(\mathrm{tr}_g(A))^2 - \mathrm{tr}_g(A^2)}{n-1}\, Q(g, A) = 0. \tag{2.22}$$

Applying (2.20) into (2.22) we get

$$(n-3)\left(Q(A,A^2) - \frac{\mathrm{tr}_g(A)}{n}\, Q(g,A^2) + \frac{\mathrm{tr}_g(A^2)}{n}\, Q(g,A)\right) = 0,$$

which yields

$$Q\left(A - \frac{\mathrm{tr}_g(A)}{n}\, g, A^2 - \frac{\mathrm{tr}_g(A^2)}{n}\, g\right) = 0.$$

From this, in view of [48, Lemma 2.4 (ii)], it follows that (2.14) holds on $\mathcal{U}_A$. Now (2.12) and (2.14), by an application of [58, Lemma 3.1], lead to (2.15), completing the proof of (ii).

Proposition 2.2. *Let T be a generalized curvature tensor on a semi-Riemannian manifold (M, g), $\dim M = n \geq 4$. If the following condition is satisfied at a point $x \in M$*

$$T = \alpha_1\, R + \frac{\alpha_2}{2}\, S \wedge S + \alpha_3\, g \wedge S + \alpha_4\, g \wedge S^2 + \frac{\alpha_5}{2}\, g \wedge g \tag{2.23}$$

then

$$\mathrm{Weyl}(T) = \alpha_1\, C + \frac{\alpha_2}{n-2}\, E \tag{2.24}$$

at this point, where the tensor E is defined by (1.2) and $\alpha_1, \ldots, \alpha_5 \in \mathbb{R}$.

Proof. From (2.23), by a suitable contraction, we get immediately

$$\mathrm{Ric}(T) = (\alpha_1 + \alpha_2\kappa + (n-2)\,\alpha_3)\, S + ((n-2)\alpha_4 - \alpha_2)\, S^2 + \alpha_6\, g, \tag{2.25}$$

where α_6 is some real number. Now using (1.2), (2.2), (2.3), (2.10), (2.23) and (2.25) we get

$$\begin{aligned}
\mathrm{Weyl}(T) &= T - \frac{1}{n-2}\, g \wedge \mathrm{Ric}(T) + \frac{\kappa(T)}{2(n-2)(n-1)}\, g \wedge g \\
&= \alpha_1\, R + \frac{\alpha_2}{2}\, S \wedge S + \alpha_3\, g \wedge S + \alpha_4\, g \wedge S^2 + \frac{\alpha_7}{2}\, g \wedge g \\
&\quad - \frac{1}{n-2}(\alpha_1 + \alpha_2\kappa + (n-2)\,\alpha_3)\, g \wedge S - \frac{1}{n-2}((n-2)\alpha_4 - \alpha_2)\, g \wedge S^2 \\
&= \alpha_1\, R + \frac{\alpha_2}{2}\, S \wedge S - \frac{\alpha_1 + \alpha_2\kappa}{n-2}\, g \wedge S + \frac{\alpha_2}{n-2}\, g \wedge S^2 + \frac{\alpha_7}{2}\, g \wedge g \\
&= \alpha_1\, (R - \frac{1}{n-2}\, g \wedge S) + \frac{\alpha_2}{n-2}\, (g \wedge S^2 + \frac{n-2}{2}\, S \wedge S - \kappa\, g \wedge S) + \frac{\alpha_7}{2}\, g \wedge g \\
&= \alpha_1\, C + \frac{\alpha_2}{n-2}\, E + \frac{\alpha_8}{2}\, g \wedge g,
\end{aligned}$$

i.e.,

$$\mathrm{Weyl}(T) = \alpha_1\, C + \frac{\alpha_2}{n-2}\, E + \frac{\alpha_8}{2}\, g \wedge g, \tag{2.26}$$

where α_7 and α_8 are some real numbers. From (2.26), by suitable contraction, we get immediately $\alpha_8 = 0$, and in a consequence (2.24), completing the proof.

3. Pseudosymmetry type curvature conditions

It is well-known that if a semi-Riemannian manifold (M, g), $\dim M = n \geq 3$, is locally symmetric then $\nabla R = 0$ on M (see, e.g., [70, Chapter 1.5]). This implies the following integrability condition $\mathcal{R}(X, Y) \cdot R = 0$ in short $R \cdot R = 0$. Semi-Riemannian manifold satisfying the last condition is called *semisymmetric* (see, e.g., [3, Chapter 8.5.3], [4, Chapter 20.7], [70, Chapter 1.6], [80, 84]). Semisymmetric manifolds form a subclass of the class of pseudosymmetric manifolds. A semi-Riemannian manifold (M, g), $\dim M = n \geq 3$, is said to be *pseudosymmetric* if the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent at every point of M (see, e.g., [3, Chapter 8.5.3], [4, Chapter 20.7], [6, Section 15.1], [38, Chapter 6], [70, Chapter 12.4], [24, 27, 38, 49, 63, 64, 75, 81, 83, 84, 85] and references therein). This is equivalent to

$$R \cdot R = L_R \, Q(g, R) \tag{3.1}$$

on $\mathcal{U}_R \subset M$, where L_R is some function on $\mathcal{U}_R$. Every semisymmetric manifold is pseudosymmetric. The converse statement is not true (see, e.g., [49]). We note that (3.1) implies

$$R \cdot S = L_R \, Q(g, S) \tag{3.2}$$

and

$$R \cdot C = L_R \, Q(g, C). \tag{3.3}$$

Conditions (3.1), (3.2) and (3.3) are equivalent on the set $\mathcal{U}_S \cap \mathcal{U}_C$ of any warped product manifold $\overline{M} \times_F \widetilde{N}$, with $\dim \overline{M} = \dim \widetilde{N} = 2$ (see, e.g., [34] and references therein).

A semi-Riemannian manifold (M, g), $\dim M = n \geq 3$, is called *Ricci-pseudosymmetric* if the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent at every point of M (see, e.g., [3, Chapter 8.5.3], [6, Section 15.1], [27]). This is equivalent on $\mathcal{U}_S$ to

$$R \cdot S = L_S \, Q(g, S), \tag{3.4}$$

where L_S is some function on $\mathcal{U}_S$. Every warped product manifold $\overline{M} \times_F \widetilde{N}$ with a 1-dimensional manifold $(\overline{M}, \overline{g})$ and an $(n-1)$-dimensional Einstein semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$, $n \geq 3$, and a warping function F, is a Ricci-pseudosymmetric manifold, see, e.g., [7, Section 1] and [34, Example 4.1].

A semi-Riemannian manifold (M, g), $\dim M = n \geq 4$, is said to be *Weyl-pseudosymmetric* if the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent at every point of M [24, 27]. This is equivalent on $\mathcal{U}_C$ to

$$R \cdot C = L_1 \, Q(g, C), \tag{3.5}$$

where L_1 is some function on $\mathcal{U}_C$. We can easily check that on every Einstein manifold (M, g), $\dim M \geq 4$, (3.5) turns into

$$R \cdot R = L_1 \, Q(g, R).$$

For a presentation of results on the problem of the equivalence of pseudosymmetry, Ricci-pseudosymmetry and Weyl-pseudosymmetry we refer to [27, Section 4].

A semi-Riemannian manifold (M, g), $\dim M = n \geq 4$, is said to be a *manifold with pseudosymmetric Weyl tensor* (*to have a pseudosymmetric conformal Weyl tensor*) if the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent at every point of M (see, e.g., [6, Section 15.1], [24, 27, 34]). This is equivalent on $\mathcal{U}_C$ to

$$C \cdot C = L_C \, Q(g, C), \tag{3.6}$$

where L_C is some function on $\mathcal{U}_C$. Every warped product manifold $\overline{M} \times_F \widetilde{N}$, with $\dim \overline{M} = \dim \widetilde{N} = 2$, satisfies (3.6) (see, e.g., [24, 27, 34] and references therein). Thus in particular, the Schwarzschild spacetime, the Kottler spacetime and the Reissner-Nordström spacetime satisfy (3.6). Semi-Riemannian manifolds with pseudosymmetric Weyl tensor were studied among others in [24, 42, 50].

Warped product manifolds $\overline{M} \times_F \widetilde{N}$, of dimension ≥ 4, satisfying on $\mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$, the condition

$$R \cdot R - Q(S, R) = L \, Q(g, C), \tag{3.7}$$

where L is some function on $\mathcal{U}_C$, were studied among others in [10]. In that paper necessary and sufficient conditions for $\overline{M} \times_F \widetilde{N}$ to be a manifold satisfying (3.7) are given. Moreover, in that paper it was proved that any 4-dimensional warped product manifold $\overline{M} \times_F \widetilde{N}$, with a 1-dimensional base $(\overline{M}, \overline{g})$, satisfies (3.7) [10, Theorem 4.1].

We refer to [7, 21, 24, 27, 31, 34, 38, 42, 75] for details on semi-Riemannian manifolds satisfying (3.1) and (3.4)-(3.7), as well other conditions of this kind, named pseudosymmetry type curvature conditions. We also refer to [42, Section 3] for a recent survey on manifolds satisfying such curvature conditions. It seems that the condition (3.1) is the most important condition of that family of curvature conditions (see, e.g., [34]). The Schwarzschild spacetime, the Kottler spacetime, the Reissner-Nordström spacetime, as well as the Friedmann-Lemaître-Robertson-Walker spacetimes are the "oldest" examples of pseudosymmetric warped product manifolds (see, e.g., [34, 38, 49, 75]). We finish this section with the following remarks.

Remark 3.1. (i) In view of [20, Lemma 3.2 (ii)], we can state that the following identity is satisfied on every semi-Riemannian manifold (M,g), $\dim M = n \geq 3$, with vanishing Weyl conformal curvature tensor C

$$R \cdot R - Q(S,R) = \frac{1}{(n-2)^2}\, Q(g, g \wedge S^2 + \frac{n-2}{2}\, S \wedge S - \kappa\, g \wedge S). \tag{3.8}$$

From (3.8), by (2.8), we get

$$R \cdot R - Q(S,R) = \frac{1}{(n-2)^2}\, Q(g,E),$$

where the tensor E is defined by (1.2). In particular, if $n = 3$ then $E = 0$ on M.
(ii) As it was stated in [20, Theorem 3.1] on every 3-dimensional semi-Riemannian manifold (M,g) the identity $R \cdot R = Q(S,R)$ is satisfied.
(iii) From (i) it follows that on every semi-Riemannian conformally flat manifold (M,g), $\dim M = n \geq 4$, the conditions: $R \cdot R = Q(S,R)$ and (1.6) are equivalent.

Remark 3.2. Let (M,g), $\dim M = n \geq 4$, be a semi-Riemannian manifold.
(i) [34, Theorem 3.4 (i)] The following identity is satisfied on $\mathcal{U}_C \subset M$

$$C \cdot R + R \cdot C = R \cdot R + C \cdot C - \frac{1}{(n-2)^2} Q(g, g \wedge S^2 - \frac{\kappa}{n-1}\, g \wedge S). \tag{3.9}$$

(ii) If (3.7) holds on $\mathcal{U}_C \subset M$ then (3.9) turns into

$$C \cdot R + R \cdot C = C \cdot C + Q(S,R) + L\, Q(g,C) - \frac{1}{(n-2)^2}\, Q(g, g \wedge S^2 + \frac{n-2}{2}\, S \wedge S - \frac{\kappa}{n-1}\, g \wedge S). \tag{3.10}$$

Moreover, from (3.10), by an application of (2.7) and (2.8), we get on $\mathcal{U}_C \subset M$

$$C \cdot R + R \cdot C = C \cdot C + Q(S,C) + L\, Q(g,C) - \frac{1}{(n-2)^2}\, Q(g,E), \tag{3.11}$$

where the tensor E is defined by (1.2).
(iii) (cf. [34, Theorem 3.4 (iii)]) If (3.6) and (3.7) hold on $\mathcal{U}_C \subset M$ then (3.11) turns into

$$C \cdot R + R \cdot C = Q(S,C) + (L_C + L)\, Q(g,C) - \frac{1}{(n-2)^2}\, Q(g,E).$$

4. Roter spaces

Some results of [24, 39, 54] (cf. [34, Section 1]) we can present in the following proposition.

Proposition 4.1. *Let (M,g), $\dim M = n \geq 4$, be a non-conformally flat and non-Einstein semi-Riemannian manifold.*
(i) [54, Theorem 3.1, Teorem 3.2 (ii)] If (3.1) and (3.6) hold on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then at every point $x \in \mathcal{U}_S \cap \mathcal{U}_C$ (1.3) or (1.9) is satisfied.
(ii) [39, Theorem 3.1, Teorem 3.2 (ii)] If (3.1) and (3.7) hold on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then at every point $x \in \mathcal{U}_S \cap \mathcal{U}_C$ (1.3) or (1.9) is satisfied.
(iii) (cf. [24, Proposition 3.2, Theorem 3.3, Theorem 4.4]) If (3.6), (3.7) and $R \cdot S = Q(g,D)$, for some symmetric $(0,2)$-tensor D, hold on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then at every point $x \in \mathcal{U}_S \cap \mathcal{U}_C$ (1.3) or (1.9) is satisfied.

We recall that a non-quasi-Einstein and non-conformally flat semi-Riemannian manifold (M,g), $\dim M = n \geq 4$, satisfying (1.9) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ is called a *Roter type manifold*, or a *Roter manifold*, or a *Roter space* (see, e.g., [6, Section 15.5], [22, 34, 35, 38]).

Roter spaces and in particular Roter hypersurfaces in semi-Riemannian spaces of constant curvature were studied in: [8, 22, 24, 31, 40, 44, 46, 47, 60, 67, 68]. In particular, (3.1) and (3.4)-(3.7) are satisfied on such manifolds. More precisely, we have

Theorem 4.1. *(see, e.g., [27, 34], [45, eq. (28)]) If (M, g), $\dim M = n \geq 4$, is a semi-Riemannian Roter space satisfying (1.9) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then on this set we have: (1.10) and*

$$\begin{aligned}
S^2 &= \alpha_1 S + \alpha_2 g, \quad \alpha_1 = \kappa + \frac{(n-2)\mu - 1}{\phi}, \quad \alpha_2 = \frac{\mu\kappa + (n-1)\eta}{\phi}, \\
R \cdot C &= L_R \, Q(g, C), \quad L_R = \frac{1}{\phi}\left((n-2)(\mu^2 - \phi\eta) - \mu\right), \\
R \cdot R &= L_R \, Q(g, R), \\
R \cdot S &= L_R \, Q(g, S),
\end{aligned}$$

$$\begin{aligned}
R \cdot R &= Q(S, R) + L \, Q(g, C), \quad L = L_R + \frac{\mu}{\phi} = \frac{n-2}{\phi}(\mu^2 - \phi\eta), \\
C \cdot C &= L_C \, Q(g, C), \quad L_C = L_R + \frac{1}{n-2}\left(\frac{\kappa}{n-1} - \alpha_1\right), \\
C \cdot R &= L_C \, Q(g, R), \\
C \cdot S &= L_C \, Q(g, S),
\end{aligned}$$

$$\begin{aligned}
C \cdot R + R \cdot C &= Q(S, C) + \left(L + L_C - \frac{1}{(n-2)\phi}\right) Q(g, C), \\
R \cdot C - C \cdot R &= \left(\frac{1}{\phi}\left(\mu - \frac{1}{n-2}\right) + \frac{\kappa}{n-1}\right) Q(g, R) + \left(\frac{\mu}{\phi}\left(\mu - \frac{1}{n-2}\right) - \eta\right) Q(S, G), \\
R \cdot C - C \cdot R &= Q\left(\left(\frac{\mu\kappa}{n-1} + \eta\right) g + \left(\frac{1}{n-2} - \mu - \frac{\phi\kappa}{n-1}\right) S, g \wedge S\right).
\end{aligned}$$

Remark 4.1. (i) In the standard Schwarzschild coordinates $(t; r; \theta; \phi)$, and the physical units $(c = G = 1)$, the Reissner-Nordström-de Sitter $(\Lambda > 0)$, and Reissner-Nordström-anti-de Sitter $(\Lambda < 0)$ spacetimes are given by the line element (see, e.g., [79])

$$\begin{aligned}
ds^2 &= -h(r)\, dt^2 + h(r)^{-1}\, dr^2 + r^2\, (d\theta^2 + \sin^2\theta\, d\phi^2), \\
h(r) &= 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^3,
\end{aligned} \tag{4.1}$$

where M, Q and Λ are non-zero constants.
(ii) [26, Section 6] (see also [8, Remark 2 (ii)], [41, Remark 2.1 (ii)]) The metric (4.1) satisfies (1.9) with

$$\begin{aligned}
\phi &= \frac{3}{2}(Q^2 - Mr) r^4 2 Q^{-4}, \quad \mu = \frac{1}{2}(Q^4 + 3Q^2\Lambda r^4 - 3\Lambda M r^5) Q^{-4}, \\
\eta &= \frac{1}{12}(3Q^6 + 4Q^4\Lambda r^4 - 3Q^4 M r + 9Q^2\Lambda^2 r^8 - 9\Lambda^2 M r^9) r^{-4} Q^{-4}.
\end{aligned}$$

If we set $\Lambda = 0$ in (4.1) then we obtain the line element of the Reissner-Nordström spacetime, see, e.g., [62, Section 9.2] and references therein. It seems that the Reissner-Nordström spacetime is the oldest example of the Roter warped product space.
(iii) In [41] a particular class of Roter warped product spaces was determined such that every manifold of that class admits a non-trivial geodesic mapping onto some Roter warped product space. Moreover, both geodesically related manifolds are pseudosymmetric of constant type.
(iv) An algebraic classification of the Roter type 4-dimensional spacetimes is given in [8].
(v) Some comments on pseudosymmetric manifolds (also called Deszcz symmetric spaces), as well as Roter spaces, are given in [9, Section 1] (see also [8, Remark 2 (iii)], [41, Remark 2.1 (iii)]): "*From a geometric point of view, the Deszcz symmetric spaces may well be considered to be the simplest Riemannian manifolds next to the real space forms.*" and "*From an algebraic point of view, Roter spaces may well be considered to be the simplest Riemannian manifolds next to the real space forms.*" For further comments we refer to [84].

We finish this section with the following results.

Proposition 4.2. *[28, Lemma 2.2] If (M, g), $\dim M = n \geq 4$, is a Roter space satisfying (1.9) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then (1.12) holds on this set, i.e., $(n-2)\, C = \phi\, E$, where the tensor E is defined by (1.2).*

Propositions 2.1, 4.1 and 4.2 lead to the following

Proposition 4.3. *Let (M, g), $\dim M = n \geq 4$, be a non-conformally flat and non-Einstein semi-Riemannian manifold. If (3.1) and (3.6), or (3.1) and (3.7), or (3.6), (3.7) and $R \cdot S = Q(g, D)$, for some symmetric $(0, 2)$-tensor D, hold on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then $E = \lambda\, C$ on $\mathcal{U}_S \cap \mathcal{U}_C$, where the tensor E is defined by (1.2) and λ is some function on this set.*

5. Warped product manifolds with 2-dimensional base manifold

Proposition 2.1 (i) and Proposition 4.2 imply

Proposition 5.1. *[28, Proposition 2.3] If (M, g), $\dim M = n \geq 4$, is a semi-Riemannian manifold satisfying (1.3) or (1.9) at every point of $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then the following equation is satisfied on this set*

$$\tau\, C = g \wedge S^2 + \frac{n-2}{2}\, S \wedge S - \kappa\, g \wedge S + \frac{\kappa^2 - \mathrm{tr}_g(S^2)}{2(n-1)}\, g \wedge g, \tag{5.1}$$

where τ is some function on $\mathcal{U}_S \cap \mathcal{U}_C$.

Proposition 5.1, [34, Theorem 7.1 (ii)] and [44, Theorem 4.1] imply

Theorem 5.1. *[28, Theorem 2.4] Let $\overline{M} \times_F \widetilde{N}$ be the warped product manifold with a 2-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$, an $(n-2)$-dimensional semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$, $n \geq 4$, a warping function F, and let $(\widetilde{N}, \widetilde{g})$ be a space of constant curvature when $n \geq 5$. Then (5.1) holds on $\mathcal{U}_S \cap \mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$.*

Example 5.1. [28, Example 2.1] (i) Let $\mathbb{S}^k(r)$ be a k-dimensional standard sphere of radius r in $\mathbb{E}^{k+1}$, $k \geq 1$. It is well-known that the Cartesian product $\mathbb{S}^1(r_1) \times \mathbb{S}^{n-1}(r_2)$ of spheres $\mathbb{S}^1(r_1)$ and $\mathbb{S}^{n-1}(r_2)$, $n \geq 4$, and more generally, the warped product manifold $\mathbb{S}^1(r_1) \times_F \mathbb{S}^{n-1}(r_2)$ of spheres $\mathbb{S}^1(r_1)$ and $\mathbb{S}^{n-1}(r_2)$, $n \geq 4$, with a warping function F, is a conformally flat manifold.
(ii) As it was stated in [60, Example 3.2], the Cartesian product $\mathbb{S}^p(r_1) \times \mathbb{S}^{n-p}(r_2)$ of spheres $\mathbb{S}^p(r_1)$ and $\mathbb{S}^k(r_2)$ such that $2 \leq p \leq n-2$ and $(n-p-1)r_1^2 \neq (p-1)r_2^2$ is a non-conformally flat and non-Einstein manifold satisfying the Roter equation (1.9) on $\mathcal{U}_S \cap \mathcal{U}_C = \mathbb{S}^p(r_1) \times \mathbb{S}^{n-p}(r_2)$.
(iii) [44, Example 4.1] The warped product manifold $\mathbb{S}^p(r_1) \times_F \mathbb{S}^{n-p}(r_2)$, $2 \leq p \leq n-2$, with some special warping function F, satisfies on $\mathcal{U}_S \cap \mathcal{U}_C \subset \mathbb{S}^p(r_1) \times \mathbb{S}^{n-p}(r_2)$ the Roter equation (1.9). Thus some warped product manifolds $\mathbb{S}^2(r_1) \times_F \mathbb{S}^{n-2}(r_2)$ are Roter spaces.
(iv) Properties of pseudosymmetry type of warped products with 2-dimensional base manifold, a warping function F, and an $(n-2)$-dimensional fibre, $n \geq 4$, assumed to be of constant curvature when $n \geq 5$, were determined in [34, Sections 6 and 7]. Evidently, warped product manifolds $\mathbb{S}^2(r_1) \times_F \mathbb{S}^{n-2}(r_2)$, $n \geq 4$, are such manifolds. Let g, R, S, κ and C denote the metric tensor, the Riemann-Christoffel curvature tensor, the Ricci tensor, the scalar curvature and the Weyl conformal curvature tensor of $\mathbb{S}^2(r_1) \times_F \mathbb{S}^{n-2}(r_2)$, respectively. From [34, Theorem 7.1] it follows that on set V of all points of $\mathcal{U}_S \cap \mathcal{U}_C \subset \mathbb{S}^2(r_1) \times_F \mathbb{S}^{n-2}(r_2)$ at which the tensor S^2 is not a linear combination of the tensors S and g, the Weyl tensor C is expressed by

$$C = \lambda \left(g \wedge S^2 + \frac{n-2}{2}\, S \wedge S - \kappa\, g \wedge S + \frac{\kappa^2 - \mathrm{tr}_g(S^2)}{2(n-1)}\, g \wedge g \right), \tag{5.2}$$

where λ is some function on V. This, by (2.2), turns into

$$R = \lambda\, g \wedge S^2 + \frac{n-2}{2} \lambda\, S \wedge S + \left(\frac{1}{n-2} - \kappa\lambda \right) g \wedge S + \frac{1}{2(n-1)} \left((\kappa^2 - \mathrm{tr}_g(S^2))\lambda - \frac{\kappa}{n-2} \right) g \wedge g.$$

Thus (1.11) is satisfied on V. Moreover, (1.9) holds at all points of $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus V$, at which (1.3) is not satisfied. From Proposition 4.2 it follows that (5.2) holds at all points of $\mathcal{U}_S \cap \mathcal{U}_C \subset \mathbb{S}^2(r_1) \times_F \mathbb{S}^{n-2}(r_2)$, $n \geq 4$, at which (1.3) is not satisfied. Finally, in view of Theorem 5.1, we can state that (5.1) holds on $\mathcal{U}_S \cap \mathcal{U}_C$.

6. Essentially conformally symmetric manifolds

Let (M, g), $\dim M = n \geq 4$, be a semi-Riemannian manifold with parallel Weyl conformal curvature tensor, i.e. $\nabla C = 0$ on M. It is obvious that the last condition implies $R \cdot C = 0$. Moreover, let the manifold (M, g) be neither conformally flat nor locally symmetric. Such manifolds are called *essentially conformally symmetric manifolds*, e.c.s. manifolds/metrics, or ECS manifolds/metrics, in short (see, e.g., [11, 12, 14, 18, 19, 65]). E.c.s. manifolds are semisymmetric manifolds ($R \cdot R = 0$, [11, Theorem 9]) satisfying $\kappa = 0$ and $Q(S, C) = 0$ ([11, Theorems 7 and 8]). In addition,

$$F\, C = \frac{1}{2}\, S \wedge S \tag{6.1}$$

holds on M, where F is some function on M, called the *fundamental function* [12]. At every point of M we also have rank $S \leq 2$ [12, Theorem 5]. We mention that the local structure of e.c.s. manifolds is already determined. We refer to [13, 16] for the final results related to this subject. We also mention that certain e.c.s. metrics are realized on compact manifolds [15, 17, 18, 19].

Equation (6.1), by suitable contraction, leads immediately to $S^2 = \kappa\, S$, which by $\kappa = 0$, reduces to $S^2 = 0$. Evidently, $\mathrm{tr}_g(S^2) = 0$. Now using (6.1) we get (1.13). Thus we have

Theorem 6.1. *Condition (5.1), with $\tau = (n-2)F$, is satisfied on every essentially conformally symmetric manifold (M, g).*

7. Hypersurfaces in semi-Riemannian conformally flat spaces

Let M, $\dim M = n \geq 4$, be a hypersurface isometrically immersed in a semi-Riemannian conformally flat manifold N, $\dim N = n + 1$. Let g_{ad}, H_{ad}, $G_{abcd} = g_{ad}g_{bc} - g_{ac}g_{bd}$ and C_{abcd} be the local components of the metric tensor g, the second fundamental tensor H, the $(0, 4)$-tensor G and the Weyl conformal curvature tensor C of M, respectively. As it was stated in [48, eq. (20)] (see also [52, eq. (11)]) we have

$$\begin{aligned} C_{abcd} &= \varepsilon\,(H_{ad}H_{bc} - H_{ac}H_{bd}) - \frac{\varepsilon\,\mathrm{tr}(H)}{n-2}\,(g_{ad}H_{bc} + g_{bc}H_{bd} - g_{ac}H_{bd} - g_{bd}H_{ac}) \\ &\quad + \frac{\varepsilon}{n-2}\,(g_{ad}H^2_{bc} + g_{bc}H^2_{bd} - g_{ac}H^2_{bd} - g_{bd}H^2_{ac}) + \mu\, G_{abcd}, \end{aligned} \tag{7.1}$$

where $\varepsilon = \pm 1$, $\mathrm{tr}(H) = g^{ad}H_{ad}$, $H^2_{ad} = g^{bc}H_{ab}H_{cd}$ and μ is some function on M. From (7.1), by contraction we get easily

$$\mu = \frac{\varepsilon}{(n-2)(n-1)}\,((\mathrm{tr}(H))^2 - \mathrm{tr}(H^2)), \tag{7.2}$$

where $\mathrm{tr}(H^2) = g^{ad}H^2_{ad}$. Now (7.1) and (7.2) yield

$$C = \frac{\varepsilon}{n-2}\left(g \wedge H^2 + \frac{n-2}{2}\, H \wedge H - \mathrm{tr}(H)\, g \wedge H + \frac{(\mathrm{tr}(H))^2 - \mathrm{tr}(H^2)}{2(n-1)}\, g \wedge g \right). \tag{7.3}$$

If $H = \frac{\mathrm{tr}(H)}{n} g$ at a point $x \in M$, i.e., M is umbilical at x, then from (7.3) it follows immediately that the tensor C vanishes at x. If at a non-umbilical point $x \in M$, we have $\mathrm{rank}(H - \alpha g) = 1$, for some $\alpha \in \mathbb{R}$, i.e., M is quasi-umbilical at x, then in view of Proposition 2.1 (i), the tensor C vanishes at x. Conversely, if at a non-umbilical point $x \in M$ the tensor C vanishes then in view of Proposition 2.1 (ii) we have $\mathrm{rank}(H - \alpha g) = 1$, for some $\alpha \in \mathbb{R}$. Thus we can present [48, Theorem 4.1] in the folowing form.

Theorem 7.1. *Let M, $\dim M = n \geq 4$, be a hypersurface isometrically immersed in a semi-Riemannian conformally flat manifold N, $\dim N = n + 1$. At every non-umbilical point $x \in M$ the tensor C vanishes at x if and only if at x we have $\mathrm{rank}(H - \alpha g) = 1$, for some $\alpha \in \mathbb{R}$.*

Remark 7.1. Let M, $\dim M = n \geq 4$, be a hypersurface isometrically immersed in a semi-Riemannian conformally flat manifold N, $\dim N = n + 1$.
(i) We assume that at all points of $\mathcal{U}_C \subset M$ the tensor H^2 is a linear combination of H and g, i.e.,

$$H^2 = \alpha_1\, H + \alpha_2\, g \tag{7.4}$$

on $\mathcal{U}_C$, where α_1 and α_2 are some functions on this set. Now (7.3) turns into

$$\begin{aligned} C &= \frac{\varepsilon}{2}\, H \wedge H + \frac{\varepsilon(\alpha_1 - \mathrm{tr}(H))}{n-2}\, g \wedge H + \frac{\varepsilon}{n-2}\left(\alpha_2 + \frac{(\mathrm{tr}(H))^2 - \mathrm{tr}(H^2)}{2(n-1)} \right) g \wedge g \\ &= \frac{\alpha}{2}\, H \wedge H + \beta\, g \wedge H + \frac{\gamma}{2}\, g \wedge g, \end{aligned} \tag{7.5}$$

where

$$\alpha = \varepsilon, \quad \beta = \frac{\varepsilon(\alpha_1 - \mathrm{tr}(H))}{n-2}, \quad \gamma = \frac{\varepsilon}{n-2}\left(2\alpha_2 + \frac{(\mathrm{tr}(H))^2 - \mathrm{tr}(H^2)}{n-1}\right). \tag{7.6}$$

From (7.5) and (7.6), in view of [68, Theorem 3.1 (i)], we get

$$C \cdot C = (n-2)\left(\frac{\beta^2}{\alpha} - \gamma\right) Q(g, C) = (n-2)(\varepsilon\beta^2 - \gamma)\, Q(g, C)$$

on $\mathcal{U}_C$, with α, β and γ defined by (7.6). Thus M is a hypersurface with pseudosymmetric Weyl tensor. We also note that from (7.4) we get immediately $\alpha_2 = \frac{1}{n}(\mathrm{tr}(H^2) - \alpha_1 \mathrm{tr}(H))$ and

$$H^2 - \frac{\mathrm{tr}(H^2)}{n}\, g = \alpha_1\left(H - \frac{\mathrm{tr}(H)}{n}\, g\right).$$

(ii) The above presented result, i.e., if (7.4) is satisfied at every point of $\mathcal{U}_C \subset M$ then (3.6) holds on this set, was already obtained in [53, Proposition 3.1]. We mention that Proposition 3.1 of [53] was proved without application of [68, Theorem 3.1 (i)].
(iii) We assume that the tensor H satisfies on $\mathcal{U}_C \subset M$

$$H^3 = \mathrm{tr}(H)\, H^2 + \psi\, H, \tag{7.7}$$

where ψ is some function on this set and the $(0, 2)$-tensor H^3 is defined by $H^3_{ad} = g^{bc} H^2_{ab} H_{cd}$. Then

$$C \cdot C = \left(\frac{\varepsilon}{(n-2)(n-1)}((\mathrm{tr}(H))^2 - \mathrm{tr}(H^2)) + \frac{\varepsilon\psi}{n-2}\right) Q(g, C) - \frac{n-3}{n-2}\, Q\left(H^2, \frac{1}{2}\, H \wedge H\right) \tag{7.8}$$

on $\mathcal{U}_C$ [51, eq. (10)], see also [53, the proof of Lemma 4.1]. We refer to [51, 53] for further results on hypersurfaces M in conformally flat manifold N satisfying (7.7).
(iv) Recently curvature properties of pseudosymmetry type of hypersurfaces isometrically immersed in a semi-Riemannian conformally flat manifold were investigated in [57] and [69].

8. Hypersurfaces in semi-Riemannian space forms

Let now $N^{n+1}_s(c)$, $n \geq 4$, be a semi-Riemannian space of constant curvature with signature $(s, n+1-s)$, where $c = \frac{\widetilde{\kappa}}{n(n+1)}$ and $\widetilde{\kappa}$ is its scalar curvature. Let M, $\dim M = n \geq 4$, be a connected hypersurface isometrically immersed in $N^{n+1}_s(c)$. We denote by g, R, S, κ and C, the metric tensor, the Riemann-Christoffel curvature tensor, the Ricci tensor, the scalar curvature and the Weyl conformal curvature tensor of the hypersurface M, respectively. The Gauss equation of M in $N^{n+1}_s(c)$ reads (see, e.g., [32, 35, 36, 37, 74])

$$R - \frac{\widetilde{\kappa}}{2n(n+1)}\, g \wedge g = \frac{\varepsilon}{2}\, H \wedge H, \quad \varepsilon \ = \ \pm 1. \tag{8.1}$$

From (8.1), by suitable contractions, we obtain

$$S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g = \varepsilon\,(\mathrm{tr}(H)\, H - H^2), \tag{8.2}$$

$$\frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n+1} = \frac{\varepsilon}{n-1}\,((\mathrm{tr}(H))^2 - \mathrm{tr}(H^2)). \tag{8.3}$$

Now using (8.1), (8.2) and (8.3) we get immediately

$$\begin{aligned} Q\left(H^2, \frac{1}{2}\, H \wedge H\right) &= -Q\left(\mathrm{tr}(H)\, H - H^2, \frac{1}{2}\, H \wedge H\right) \\ &= -Q\left(\varepsilon(\mathrm{tr}(H)\, H - H^2), \frac{\varepsilon}{2}\, H \wedge H\right) = -Q\left(S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g, R - \frac{\widetilde{\kappa}}{2n(n+1)}\, g \wedge g\right). \end{aligned} \tag{8.4}$$

We also recall that the curvatere condition of pseudosymmetry type (3.7) is satisfied on M. Precisely, we have on M [48, Proposition 3.1] (see also [37, eqs. (3.3) and (3.4)])

$$R \cdot R - Q(S,R) = -\frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, Q(g,C). \tag{8.5}$$

Now (3.11), by (8.5), turns into ([37, Propopsition 4.7, eq. (4.36)])

$$C \cdot R + R \cdot C = C \cdot C + Q(S,C) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, Q(g,C) - \frac{1}{(n-2)^2}\, Q(g,E).$$

Let $\mathcal{U}_H \subset M$ be the set of all points at which the tensor H^2 is not a linear combination of the metric tensor g and the second fundamental tensor H of M. We have $\mathcal{U}_H \subset \mathcal{U}_S \cap \mathcal{U}_C \subset M$ (see, e.g., [30, 36, 37, 59, 72]).

We assume that the following conditions are satisfied on $\mathcal{U}_H \subset M$

$$H^3 = \operatorname{tr}(H)\, H^2 + \psi\, H + \rho\, g \tag{8.6}$$

and

$$C \cdot C = Q(g,T), \tag{8.7}$$

where T is a generalized curvature tensor and ψ and ρ some functions on $\mathcal{U}_H$. Now, in view of [37, Theorem 4.5], we obtain

$$T \;=\; \left(\frac{\kappa + 2\varepsilon\psi}{n-1} - \frac{\widetilde{\kappa}}{n+1}\right) C + \frac{\lambda_1}{2}\, g \wedge g - \frac{n-3}{(n-2)^2(n-1)}\left(g \wedge S^2 + \frac{n-2}{2}\, S \wedge S - \kappa\, g \wedge S\right) \tag{8.8}$$

on $\mathcal{U}_H$, where λ_1 is some function on this set. Using (1.2), (8.7) and (8.8) we get immediately

$$T = \left(\frac{\kappa + 2\varepsilon\psi}{n-1} - \frac{\widetilde{\kappa}}{n+1}\right) C - \frac{n-3}{(n-2)^2(n-1)}\, E + \frac{\lambda}{2}\, g \wedge g$$

and

$$C \cdot C = \left(\frac{\kappa + 2\varepsilon\psi}{n-1} - \frac{\widetilde{\kappa}}{n+1}\right) Q(g,C) - \frac{n-3}{(n-2)^2(n-1)}\, Q(g,E) \tag{8.9}$$

on $\mathcal{U}_H$, where λ is some function on this set. In addition, if we assume that (3.6) holds on $\mathcal{U}_H$ then from (7.8) it follows that

$$\left(\frac{\kappa + 2\varepsilon\psi}{n-1} - \frac{\widetilde{\kappa}}{n+1} - L_C\right) C = \frac{n-3}{(n-2)^2(n-1)}\, E + \frac{\lambda_2}{2}\, g \wedge g \tag{8.10}$$

on $\mathcal{U}_H$, where λ_2 is some function on this set. We note that (8.10), by a suitable contraction, yields $\lambda_2 = 0$, and in a consequence we obtain

$$\left(\frac{\kappa + 2\varepsilon\psi}{n-1} - \frac{\widetilde{\kappa}}{n+1} - L_C\right) C = \frac{n-3}{(n-2)^2(n-1)}\, E. \tag{8.11}$$

From the above presented considerations it follows immediately the following result.

Theorem 8.1. *Let M be a non-Einstein and non-conformally flat hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. If (3.6) and (8.6) are satisfied on $\mathcal{U}_H \subset M$ then (8.11) holds on $\mathcal{U}_H$.*

According to [30, Corollary 4.1], if on the subset $\mathcal{U}_H$ of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, one of the tensors $R \cdot C$, $C \cdot R$ or $R \cdot C - R \cdot C$ is a linear combination of $R \cdot R$ and of a finite sum of tensors of the form $Q(A,T)$, where A is a symmetric $(0,2)$-tensor and T a generalized curvature tensor, then (8.6) holds on $\mathcal{U}_H$. In particular if one of the following conditions is satisfied on $\mathcal{U}_H \subset M$: $R \cdot C = Q(g,T_1)$, $C \cdot R = Q(g,T_2)$ or $R \cdot C - C \cdot R = Q(g,T_3)$, where T_1, T_2 and T_3 are generalized curvature tensors, then (8.6) holds on $\mathcal{U}_H$. Now from Theorems 5.2, 5.3 and 5.4 of [36], in view of Proposition 2.2, it follows that

$$\operatorname{Weyl}(T_1) = \left(\frac{\kappa + \varepsilon\psi}{n-1} - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right) C - \frac{1}{(n-2)(n-1)}\, E, \tag{8.12}$$

$$\operatorname{Weyl}(T_2) = \left(\frac{\kappa + 2\varepsilon\psi}{n-1} - \frac{\widetilde{\kappa}}{n+1}\right) C - \frac{n-3}{(n-2)^2(n-1)}\, E, \tag{8.13}$$

$$\operatorname{Weyl}(T_3) = \left(\frac{\widetilde{\kappa}}{n(n+1)} - \frac{\varepsilon\psi}{n-1}\right) C - \frac{1}{(n-2)^2(n-1)}\, E. \tag{8.14}$$

Thus we have

Theorem 8.2. *Let M be a non-Einstein and non-conformally flat hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (8.6) on $\mathcal{U}_H \subset M$, and let T_1, T_2 and T_3 be generalized curvature tensors defined on $\mathcal{U}_H$. If one of the following conditions: $R \cdot C = Q(g, T_1)$, respectively, $C \cdot R = Q(g, T_2)$ and $R \cdot C - C \cdot R = Q(g, T_3)$, is satisfied on $\mathcal{U}_H$ then (8.12), respectively (8.13) and (8.14), holds on $\mathcal{U}_H$.*

Finally, we assume that the tensor H satisfies (7.7) on $\mathcal{U}_H \subset M$. Now (7.8), by making use of (8.3) and (8.4), turns into

$$\frac{n-2}{n-3}\, C \cdot C = \rho\, Q(g, C) + Q\left(S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g, R - \frac{\widetilde{\kappa}}{2n(n+1)}\, g \wedge g\right), \tag{8.15}$$

where

$$\rho = \frac{1}{n-3}\left(\frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n+1} + \varepsilon\psi\right). \tag{8.16}$$

From (8.15), by an application of (2.6), we obtain

$$\begin{aligned}
\frac{n-2}{n-3}\, C \cdot C &= \rho\, Q(g, R) + \frac{\rho}{2(n-2)}\, Q(S, g \wedge g) + Q\left(S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g, R - \frac{\widetilde{\kappa}}{2n(n+1)}\, g \wedge g\right) \\
&= \frac{\rho}{2(n-2)}\, Q(S, g \wedge g) + \rho\, Q\left(g, R - \frac{\widetilde{\kappa}}{2n(n+1)}\, g \wedge g\right) \\
&\quad + Q\left(S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g, R - \frac{\widetilde{\kappa}}{2n(n+1)}\, g \wedge g\right) \\
&= Q\left(S, \frac{\rho}{2(n-2)}\, g \wedge g\right) + Q\left(S - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \rho\right) g, R - \frac{\widetilde{\kappa}}{2n(n+1)}\, g \wedge g\right) \\
&= Q\left(S - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \rho\right) g, \frac{\rho}{2(n-2)}\, g \wedge g\right) \\
&\quad + Q\left(S - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \rho\right) g, R - \frac{\widetilde{\kappa}}{2n(n+1)}\, g \wedge g\right) \\
&= Q\left(S - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \rho\right) g, R - \left(\frac{\widetilde{\kappa}}{n(n+1)} - \frac{\rho}{n-2}\right) \frac{1}{2}\, g \wedge g\right).
\end{aligned}$$

Thus we see that if the tensor H satisfies (7.7) on $\mathcal{U}_H \subset M$ then

$$C \cdot C = \frac{n-3}{n-2}\, Q\left(S - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \rho\right) g, R - \left(\frac{\widetilde{\kappa}}{n(n+1)} - \frac{\rho}{n-2}\right) \frac{1}{2}\, g \wedge g\right) \tag{8.17}$$

on $\mathcal{U}_H$, where the function ρ is defined by (8.16).

In addition, we assume that (3.6) holds on $\mathcal{U}_H \subset M$. Now (8.15) turns into

$$\tau\, Q(g, C) - Q\left(S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g, R - \frac{\widetilde{\kappa}}{2n(n+1)}\, g \wedge g\right) = 0, \tag{8.18}$$

where

$$\tau = \rho - \frac{n-2}{n-3}\, L_C. \tag{8.19}$$

From the presented above calculations it follows that (8.18) yields

$$Q\left(S - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \tau\right) g, R - \left(\frac{\widetilde{\kappa}}{n(n+1)} - \frac{\tau}{n-2}\right) \frac{1}{2}\, g \wedge g\right) = 0. \tag{8.20}$$

If

$$\operatorname{rank}\left(S - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \tau\right) g\right) = 1$$

at a point $x \in \mathcal{U}_H$ then in view of Proposition 2.1 $E = 0$ at x, where the tensor E is defined by (1.2). If

$$\operatorname{rank}\left(S - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \tau \right) g \right) > 1$$

at a point $x \in \mathcal{U}_H$ then by an application of [24, Proposition 2.4] (or, [31, Proposition 2.1]) it follows that the following equation is satisfied at x

$$R - \left(\frac{\widetilde{\kappa}}{n(n+1)} - \frac{\tau}{n-2} \right) \frac{1}{2}\, g \wedge g = \phi \left(S - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \tau \right) g \right) \wedge \left(S - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \tau \right) g \right), \quad \phi \in \mathbb{R}.$$

This by Proposition 4.2 implies $(n-2)\, C = \phi\, E$. Thus we have

Theorem 8.3. *Let M be a non-Einstein and non-conformally flat hypersurface in $N_s^{n+1}(c)$, $n \geq 4$.*
(i) If (7.7) is satisfied on $\mathcal{U}_H \subset M$ then (8.17) holds on $\mathcal{U}_H$, where the function ρ is defined by (8.16) on this set.
(ii) If (3.6) and (7.7) are satisfied on $\mathcal{U}_H \subset M$ then (8.20) holds on $\mathcal{U}_H$, where the function τ is defined by (8.19) on this set. Moreover, $\lambda\, C = E$ on $\mathcal{U}_H$, where λ is some function on this set.

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