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RESEARCH PAPER

A three-component prey-predator system with interval number

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Abstract

This paper presents a three-component model consisting of one prey and two predator species using imprecise biological parameters as interval numbers and applied functional parametric form in the proposed prey-predator system. The positivity and boundedness of the model are checked, and a stability analysis of the five equilibrium points is performed. Numerical simulations are performed to study the effect of the interval number and to illustrate analytical studies.

Keywords: Prey-predator; interval number; stability; competition; uncertainty

AMS 2020 Classification: 34D20; 34D23; 37C25; 37D45; 39A28

1 Introduction

The dynamics of predator-prey relationships [1–5] relationships is an essential aspect of any ecosystem where plants, animals, and other living organisms coexist in a delicate balance. Prey-predator model dynamics [6–10] are influenced by a variety of factors, including environmental conditions, competition from the predator population, and mortality rates. Mathematical ecology is an area of study that investigates the dynamic relationships between prey and predators. In this research, we analyze a three-species prey-predator model [11–15] with competition in predator populations to study dynamics with imprecise parameters.

Most previous research on prey-predator models has been based on the assumption of exact biological parameters [1–15]. However, in reality, biological parameters may not be fixed and can change due to various reasons, making the exact estimation of these parameters difficult. To address this issue, we consider interval number biological parameters in our study. Interval

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numbers allow us to incorporate uncertainty into our models and make more realistic predictions about the behavior of the system.

Furthermore, using interval numbers for parameters can capture the complex dynamics of predator-prey systems, such as boom-bust cycles, where the populations of both species oscillate over time. Interval numbers can also provide greater flexibility in modeling a prey-predator system, enabling us to simulate the effect of different environmental factors on the system and explore different scenarios.

In this paper, we focus on studying the advantages of using interval number biological parameters for prey-predator systems and the effect of certain model parameters on the system. Although some studies have explored the use of interval parameters in prey-predator models [16–22], our research offers new insights into the potential benefits of this approach. Ultimately, our findings can contribute to a better understanding and management of these vital ecosystems.

2 Prey-predator imprecise model with interval number

In this proposed model, we consider one prey species and two predator species. Let $X(t)$ denote the prey density, $Y(t)$ the 1^{st} predator and $Z(t)$ the 2^{nd} predator density at any time t.

The biological environment of populations is not completely predictable, so the biological parameters of modeling the prey-predator system should be considered imprecise. The proposed prey-predator system is developed on the following assumptions:

Assumption 1. The prey population grows according to the logistic curve with carrying capacity $k(k\epsilon\mathbb{R}_+)$ and with an intrinsic growth rate $r(r\epsilon\mathbb{R}_+)$, in the absence of both predator species. The logistic equation is a mathematical model that describes the growth of a population over time. It is represented by the following differential equation:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),$$

where N is the population size, t is time, r is the intrinsic growth rate of the population, and K is the carrying capacity, which is the maximum number of individuals that the environment can support.

The logistic equation incorporates the concept of density dependence, which means that the growth rate of the population decreases as it approaches the carrying capacity. This results in an S-shaped growth curve, where the population initially grows rapidly, slows down as it approaches the carrying capacity, and eventually stabilizes at the carrying capacity. In the presence of predators, Y and Z, the population of prey X will decrease due to the attack of predators. The first and second predators attack the prey with an interval-valued rate $\widehat{\beta}_1\,(>0)$ and $\widehat{\beta}_2\,(>0)$ respectively and the Holling type I functional response manner. Holling type function, also known as the functional response curve, is a mathematical model that describes the rate at which a predator consumes prey as a function of prey density. It is named after Canadian ecologist C.S. Holling, who first proposed the idea in the 1950s.

The Holling type function is typically represented by one of three functional forms:

Type I: $f(x) = ax$, where a is a constant that represents the attack rate of the predator.

Type II: $f(x) = \frac{ax}{1+\frac{ax}{h}}$, where h is a constant that represents the handling time, or the time it takes for the predator to consume a single prey item.

Type III: $f(x) = \frac{ax^2}{1+bx+\frac{x^2}{k}}$, where b and k are constants that determine the shape of the curve.

In general, the Holling type function predicts that the rate of predation increases with prey density up to a certain point, after which the rate of predation begins to level off as the predator becomes saturated with prey. The precise shape of the curve depends on the specific functional form of the model. Then the mathematical form of the above assumption is as follows.

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{k}\right) - \widehat{\beta}_1 XY - \widehat{\beta}_2 XZ. \tag{1}$$

Assumption 2. Prey X are food for predators (Y and Z), so in the presence of food, the population density of predators (Y and Z) will increase in the Holling type I functional response manner. We introduced the natural death of predators. Here we also consider the competition between the predators, and for this reason, the population density of both predators will decrease. We consider the competition parameters $\widehat{\delta}_1$ and $\widehat{\delta}_2$ to be imprecise. Hence, from the above assumption, we have

$$\frac{dY}{dt} = \widehat{\beta}_1 XY - d_1 Y - \widehat{\delta}_1 YZ, \tag{2}$$

$$\frac{dZ}{dt} = \widehat{\beta}_2 XZ - d_2 Z - \widehat{\delta}_2 YZ. \tag{3}$$

Therefore, our final mathematical model with four interval-valued parameters is as follows.

$$\begin{aligned} \frac{dX}{dt} &= rX\left(1 - \frac{X}{k}\right) - \widehat{\beta}_1 XY - \widehat{\beta}_2 XZ, \\ \frac{dY}{dt} &= \widehat{\beta}_1 XY - d_1 Y - \widehat{\delta}_1 YZ, \\ \frac{dZ}{dt} &= \widehat{\beta}_2 XZ - d_2 Z - \widehat{\delta}_2 YZ, \end{aligned} \tag{4}$$

where $\widehat{\beta}_1 \in [\beta_{1l}, \beta_{1u}]$, $\widehat{\beta}_2 \in [\beta_{2l}, \beta_{2u}]$, $\widehat{\delta}_1 \in [\delta_{1l}, \delta_{1u}]$, $\widehat{\delta}_2 \in [\delta_{2l}, \delta_{2u}]$, for $\beta_{1l} > 0$, $\beta_{2l} > 0$, $\delta_{1l} > 0$ and $\delta_{2l} > 0$, with initial conditions

$$X(0) > 0, Y(0) > 0, \text{ and } Z(0) > 0. \tag{5}$$

Using the parametric form of interval-valued parameters, the equations (4) can be written in the parametric prey-predator model [16, 17] for $p \in [0,1]$ is as follows:

$$\begin{aligned} \frac{dX}{dt} &= rX\left(1 - \frac{X}{k}\right) - \beta_{1l}^{1-p}\beta_{1u}^{p} XY - \beta_{2l}^{1-p}\beta_{2u}^{p} XZ, \\ \frac{dY}{dt} &= \beta_{1l}^{1-p}\beta_{1u}^{p} XY - d_1 Y - \delta_{1l}^{1-p}\delta_{1u}^{p} YZ, \\ \frac{dZ}{dt} &= \beta_{2l}^{1-p}\beta_{2u}^{p} XZ - d_2 Z - \delta_{2l}^{1-p}\delta_{2u}^{p} YZ, \end{aligned} \tag{6}$$

subject to the initial conditions

$$X(0) > 0, Y(0) > 0 \text{ and } Z(0) > 0. \tag{7}$$

The biological descriptions of each parameter have been discussed in Table 1.

Table 1. Biological meaning of the model parameters

Parameter	Biological meaning
X	Prey species
Y	1^{st} predator species
Z	2^{nd} predator species
r	Intrinsic growth rate
k	Carrying capacity
β_1	Consumption rate of 1^{st} predator
β_2	Consumption rate of 2^{nd} predator
δ_1	Competition rate between the 1^{st} predator (Y) to 2^{nd} predator (Z)
δ_2	Competition rate between the 2^{nd} predator (Z) to 1^{st} predator (Y)
d_1	Natural death rate of 1^{st} predator
d_2	Natural death rate of 2^{nd} predator

3 Dynamical behavior

In this section, we describe the rigorous dynamical behavior of the proposed model system. To do so, we first check the positivity of the solutions of the interval model and the uniform boundedness of the solution of the same model.

Positivity

Theorem 1 *Every solution of system (6) with initial conditions (7) exists in the interval* $[0, \infty)$ *and* $X(0) > 0, Y(0) > 0$ *and* $Z(0) > 0$ *for all* $t \geq 0$.

Proof Since the right-hand side of the system (6) is completely continuous and locally Lipschitzian on C, the solution $(X(t), Y(t), Z(t))$ of (6) with initial conditions (7) exists and is unique on $[0, \xi)$, where $0 < \xi < \infty$.
From system (6) with initial conditions (7), we have

$$X(t) = X(0)exp\left[\left|\int_0^t \{r(1-(X(\theta))/k) - \beta_{1l}^{1-p}\beta_{1u}^{p}Y(\theta) - \beta_{2l}^{1-p}\beta_{2u}^{p}Z(\theta)\}d\theta\right|\right] > 0,$$

$$Y(t) = Y(0)exp\left[\left|\int_0^t \{\beta_{1l}^{1-p}\beta_{1u}^{p}X(\theta) - d_1 - \delta_{1l}^{1-p}\delta_{1u}^{p}Z(\theta)\}d\theta\right|\right] > 0,$$

$$Z(t) = Z(0)exp\left[\left|\int_0^t \{\beta_{2l}^{1-p}\beta_{2u}^{p}X(\theta) - d_2 - \delta_{2l}^{1-p}\delta_{2u}^{p}Y(\theta)\}d\theta\right|\right] > 0,$$

which completes the proof. ∎

Uniform boundedness

Theorem 2 *The solutions of the model system (6) are completely bounded.*

Proof We construct a function such as $\Lambda(t) = X(t) + Y(t) + Z(t)$.
Differentiating both sides with respect to t, we have

$$\frac{d\Lambda(t)}{dt} = \frac{dX(t)}{dt} + \frac{dY(t)}{dt} + \frac{dZ(t)}{dt}.$$

Therefore,

$$\frac{d\Lambda}{dt} = rX(1-\frac{X}{k}) - \beta_{1l}^{1-p}\beta_{1u}^{p}XY - \beta_{2l}^{1-p}\beta_{2u}^{p}XZ$$
$$+\beta_{1l}^{1-p}\beta_{1u}^{p}XY - d_1Y - \delta_{1l}^{1-p}\delta_{1u}^{p}YZ + \beta_{2l}^{1-p}\beta_{2u}^{p}XZ - d_2Z - \delta_{2l}^{1-p}\delta_{2u}^{p}YZ$$
$$= rX(1-\frac{X}{k}) - \left(\delta_{1l}^{1-p}\delta_{1u}^{p} + \delta_{2l}^{1-p}\delta_{2u}^{p}\right)YZ - d_1Y - d_2Z.$$

Now,

$$\frac{d\Lambda}{dt} + \gamma\Lambda == rX(1-\frac{X}{k}) - \left(\delta_{1l}^{1-p}\delta_{1u}^{p} + \delta_{2l}^{1-p}\delta_{2u}^{p}\right)YZ - d_1Y - d_2Z + \gamma\left(X+Y+Z\right).$$
$$\frac{d\Lambda}{dt} + \gamma\Lambda = \left(rX - \frac{rX^2}{k} + \gamma X\right) - \left(d_1-\gamma\right)X - \left(d_2-\gamma\right)Y - \left(\delta_{1l}^{1-p}\delta_{1u}^{p} + \delta_{2l}^{1-p}\delta_{2u}^{p}\right)YZ.$$

Since $\left(\delta_{1l}^{1-p}\delta_{1u}^{p} + \delta_{2l}^{1-p}\delta_{2u}^{p}\right)YZ > 0$ and assuming $\gamma < \min\left(d_1, d_2\right),$ then from the above equation, we have

$$\frac{d\Lambda}{dt} + \gamma\Lambda \leq \left(rX - \frac{rX^2}{k} + \gamma X\right) \leq k\frac{\left(r+\gamma\right)^2}{4r} = A\left(say\right).$$

Applying the result of differential inequality, we obtain,
$0 \leq \Lambda\left(X(t), Y(t), Z(t)\right) \leq \frac{A}{\gamma}\left(1-e^{-\gamma t}\right) + \Lambda\left(X(0), Y(0), Z(0)\right)e^{-\gamma t},$
which implies that $0 \leq \Lambda \leq \frac{A}{\gamma}$ as $t \to \infty$.
Hence all the solutions of (6) is uniformly bounded. ■

4 Equilibrium points and their existence and stability

In this section, we study the existence and stability behavior of the system (6) at equilibrium points of the model system (6) are:
(I) Trivial equilibrium $E_0\left(0,0,0\right)$, (II) Axial equilibrium $E_1\left(k,0,0\right)$, (III) Planar equilibrium (a) $E_2\left(X_2, Y_2, 0\right)$ where $X_2 = \frac{d_1}{\beta_{1l}^{1-p}\beta_{1u}^{p}}$ and $Y_2 = \frac{r}{\beta_{1l}^{1-p}\beta_{1u}^{p}}\left(1-\frac{d_1}{\beta_{1l}^{1-p}\beta_{1u}^{p}k}\right)$, (b) $E_3\left(X_3, 0, Z_3\right)$, where $X_3 = \frac{d_2}{\beta_{2l}^{1-p}\beta_{2u}^{p}}$ and $Z_3 = \frac{r}{\beta_{2l}^{1-p}\beta_{2u}^{p}}\left(1-\frac{d_2}{\beta_{2l}^{1-p}\beta_{2u}^{p}k}\right)$. (IV) Interior equilibrium $E^*\left(X^*, Y^*, Z^*\right)$, where $X^* = \frac{k\left(\beta_{1l}^{1-p}\beta_{1u}^{p}\delta_{1l}^{1-p}\delta_{1u}^{p}d_2 + rd_1\delta_{2l}^{1-p}\delta_{2u}^{p} + \beta_{2l}^{1-p}\beta_{2u}^{p}d_1\delta_{2l}^{1-p}\delta_{2u}^{p}\right)}{r\delta_{1l}^{1-p}\delta_{1u}^{p}\delta_{2l}^{1-p}\delta_{2u}^{p} + k\beta_{1l}^{1-p}\beta_{1u}^{p}\beta_{2l}^{1-p}\beta_{2u}^{p}\left(\delta_{1l}^{1-p}\delta_{1u}^{p} + \delta_{2l}^{1-p}\delta_{2u}^{p}\right)} > 0$, $Y^* = \frac{\beta_{2l}^{1-p}\beta_{2u}^{p}X^* - d_2}{\delta_{2l}^{1-p}\delta_{2u}^{p}}$, and $Z^* = \frac{\beta_{1l}^{1-p}\beta_{1u}^{p}X^* - d_1}{\delta_{1l}^{1-p}\delta_{1u}^{p}}$.
Now $Y^* > 0$ if $\beta_{2l}^{1-p}\beta_{2u}^{p}X^* > d_2$, and $Z^* > 0$ if $\beta_{1l}^{1-p}\beta_{1u}^{p}X^* > d_1$.

Local stability analysis

In this section, we study the local stability of the system (6) at various equilibrium points.

Theorem 3 *The equilibrium point E_0 is always unstable.*

Proof Variational matrix of system (6) at $E_0(0,0,0)$ is given by

$$V(E_0)=\begin{pmatrix} r & 0 & 0\\ 0 & -d_1 & 0\\ 0 & 0 & -d_2\end{pmatrix}.$$

Therefore, eigenvalues of the characteristic equation of $V(E_0)$ are $\lambda_1=r>0$, $\lambda_2=-d_1<0$, $\lambda_3=-d_2<0$. Here, one of the eigenvalues is positive and the other two are negative, so E_0 is always unstable. ∎

Theorem 4 *The equilibrium point E_1 is stable if $\beta_{1l}^{1-p}\beta_{1u}^{p}k<d_1$ and $\beta_{2l}^{1-p}\beta_{2u}^{p}k<d_2$.*

Proof Variational matrix of system (6) at $E_1(k,0,0)$ is given by

$$V(E_1)=\begin{pmatrix} -r & -\beta_{1l}^{1-p}\beta_{1u}^{p}k & -\beta_{2l}^{1-p}\beta_{2u}^{p}k\\ 0 & \beta_{1l}^{1-p}\beta_{1u}^{p}k-d_1 & 0\\ 0 & 0 & \beta_{2l}^{1-p}\beta_{2u}^{p}k-d_2\end{pmatrix}.$$

The eigenvalues of the characteristic equation of $V(E_1)$ are $\lambda_1=-r<0$, $\lambda_2=\beta_{1l}^{1-p}\beta_{1u}^{p}k-d_1$, $\lambda_3=\beta_{2l}^{1-p}\beta_{2u}^{p}k-d_2$. Therefore, E_1 is stable if $\beta_{1l}^{1-p}\beta_{1u}^{p}k<d_1$ and $\beta_{2l}^{1-p}\beta_{2u}^{p}k<d_2$. ∎

Theorem 5 *The equilibrium point E_2 is locally asymptotically stable if $d_2>\beta_{2l}^{1-p}\beta_{2u}^{p}X_2-\delta_{2l}^{1-p}\delta_{2u}^{p}Y_2$, $A_1>0$ and $A_2>0$.*

Proof The variational matrix of system (6) at $E_2\,(X_2,Y_2,0)$ is given by

$$V(E_2)=\begin{pmatrix} m_{11} & m_{12} & m_{13}\\ m_{21} & m_{22} & m_{23}\\ 0 & 0 & m_{33}\end{pmatrix},$$

where $m_{11}=r-\frac{2rX_2}{k}-\beta_{1l}^{1-p}\beta_{1u}^{p}Y_2$, $m_{12}=-\beta_{1l}^{1-p}\beta_{1u}^{p}X_2$, $m_{13}=-\beta_{2l}^{1-p}\beta_{2u}^{p}X_2$, $m_{21}=\beta_{1l}^{1-p}\beta_{1u}^{p}Y_2$, $m_{22}=\beta_{1l}^{1-p}\beta_{1u}^{p}X_2-d_1$, $m_{23}=-\delta_{1l}^{1-p}\delta_{1u}^{p}Y_2$, $m_{33}=\beta_{2l}^{1-p}\beta_{2u}^{p}X_2-d_2-\delta_{2l}^{1-p}\delta_{2u}^{p}Y_2$.
Now the characteristic equation for $V(E_2)$ is $(m_{33}-\lambda)\left\{\lambda^2+A_1\lambda+A_2\right\}=0$, where $A_1=-(m_{11}+m_{22})$ and $A_2=m_{11}m_{22}-m_{12}m_{21}$.
Therefore, one eigenvalue of the characteristic equation above is m_{33}, which is negative as $d_2>\beta_{2l}^{1-p}\beta_{2u}^{p}X_2-\delta_{2l}^{1-p}\delta_{2u}^{p}Y_2$ and the other two eigenvalues are negative if $A_1>0$ and $A_2>0$.
Therefore, the second predator-free equilibrium point $E_2\,(X_2,Y_2,0)$ is locally asymptotically stable if $d_2>\beta_{2l}^{1-p}\beta_{2u}^{p}X_2-\delta_{2l}^{1-p}\delta_{2u}^{p}Y_2$, $A_1>0$, and $A_2>0$, otherwise the system (6) will be unstable. ∎

Theorem 6 *The equilibrium point E_3 is locally asymptotically stable if $d_1>\beta_{1l}^{1-p}\beta_{1u}^{p}X_3-\delta_{1l}^{1-p}\delta_{1u}^{p}Z_3$, $B_1>0$ and $B_2>0$.*

Proof The variational matrix of system (6) at $E_3\,(X_3,0,Z_3)$ is given by

$$V(E_3)=\begin{pmatrix} p_{11} & p_{12} & p_{13}\\ 0 & p_{22} & 0\\ p_{31} & p_{32} & p_{33}\end{pmatrix},$$

where $p_{11} = r - \frac{2rX_3}{k} - \beta_{2l}^{1-p}\beta_{2u}^{p}Z_3$, $p_{12} = -\beta_{1l}^{1-p}\beta_{1u}^{p}X_3$, $p_{13} = -\beta_{2l}^{1-p}\beta_{2u}^{p}X_3$, $p_{22} = \beta_{1l}^{1-p}\beta_{1u}^{p}X_3 - d_1 - \delta_{1l}^{1-p}\delta_{1u}^{p}Z_3$, $p_{31} = \beta_{2l}^{1-p}\beta_{2u}^{p}Z_3$, $p_{32} = -\delta_{2l}^{1-p}\delta_{2u}^{p}Z_3$, $p_{33} = \beta_{2l}^{1-p}\beta_{2u}^{p}X_3 - d_2$.
The characteristic equation for $V(E_3)$ is $(p_{22} - \lambda)\left(\lambda^2 + B_1\lambda + B_2\right) = 0$, where $B_1 = -(p_{11} + p_{33})$ and $B_2 = p_{11}p_{33} - p_{13}p_{31}$.
Therefore, the eigenvalue of the characteristic equation above is p_{22}, which is negative as $d_1 > \beta_{1l}^{1-p}\beta_{1u}^{p}X_3 - \delta_{1l}^{1-p}\delta_{1u}^{p}Z_3$, and the other two eigenvalues are negative if $B_1 > 0$ and $B_2 > 0$.
The first predator-free equilibrium point $E_3(X_3, 0, Z_3)$ is locally asymptotically stable if $d_1 > \beta_{1l}^{1-p}\beta_{1u}^{p}X_3 - \delta_{1l}^{1-p}\delta_{1u}^{p}Z_3$, $B_1 > 0$ and $B_2 > 0$, otherwise the system (6) will be unstable. ■

Theorem 7 *The equilibrium point E^* is locally asymptotically stable if the inequalities $A > 0$, $C > 0$, $AB - C > 0$ are satisfied.*

Proof Variational matrix of system (6) at $E^*(X^*, Y^*, Z^*)$ is given by,

$$V(E^*) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where $a_{11} = r - \frac{2rX^*}{k} - \beta_{1l}^{1-p}\beta_{1u}^{p}Y^* - \beta_{2l}^{1-p}\beta_{2u}^{p}Z^*$, $a_{12} = -\beta_{1l}^{1-p}\beta_{1u}^{p}X^*$, $a_{13} = -\beta_{2l}^{1-p}\beta_{2u}^{p}X^*$, $a_{21} = \beta_{1l}^{1-p}\beta_{1u}^{p}Y^*$, $a_{22} = \beta_{1l}^{1-p}\beta_{1u}^{p}X^* - \delta_{1l}^{1-p}\delta_{1u}^{p}Z^* - d_1$, $a_{23} = -\delta_{1l}^{1-p}\delta_{1u}^{p}Y^*$, $a_{31} = \beta_{2l}^{1-p}\beta_{2u}^{p}Z^*$, $a_{32} = -\delta_{2l}^{1-p}\delta_{2u}^{p}Z^*$, $a_{33} = \beta_{2l}^{1-p}\beta_{2u}^{p}X^* - \delta_{2l}^{1-p}\delta_{2u}^{p}Y^* - d_2$.

Therefore, the characteristic equation of $V(E^*)$ is

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0, \tag{8}$$

where, $A = -(a_{11} + a_{22} + a_{33})$, $B = -(a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32} - a_{11}a_{22} - a_{11}a_{33} - a_{22}a_{33})$, $C = -(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32})$.

According to the Routh-Hurwitz criterion, all eigenvalues of the characteristic equation (8) have negative real parts, which means that the system (6) shows local asymptotic stability at E^* if and only if $A > 0$, $C > 0$, $AB - C > 0$. ■

Remark 1 *When analyzing the stability of a model with exact biological parameters, the results are typically the same as those for the model with the corresponding imprecise biological parameters. The key difference between the two lies in the nature of the parameters used. While exact parameter models use precise numerical values, imprecise biological parameter models employ uncertain parameters that are often in the form of probability distributions or ranges of values. Despite this difference, the stability analysis techniques used for both types of models are essentially the same.*

Global stability analysis

In this section, we discuss the global stability behavior of the system (6) at interior equilibrium point $E^*(X^*, Y^*, Z^*)$. Studying the global stability of the equilibrium points using Lyapunov's direct method has gained popularity in recent years, but constructing suitable Lyapunov functions can be challenging. In general, there are no systematic methods for constructing Lyapunov functions for prey-predator models. However, the most commonly used types of Lyapunov functions are quadratic and Volterra-type functions. In this study, the global stability of the equilibrium states was demonstrated using a Volterra-type Lyapunov function [23–28]. This function was carefully chosen because of its effectiveness in analyzing the stability of dynamical systems with more complex behavior.

Theorem 8 *If E^* is locally asymptotically stable, then E^* is globally asymptotically stable in*

$$G = \{(X, Y, Z) : X > X^*, Y > Y^* \text{ and } Z > Z^* \text{or } X < X^*, Y < Y^* \text{ and } Z < Z^*\}. \tag{9}$$

Proof Let

$$L\left(X, Y, Z\right) = \alpha_1 \left(X - X^* - X^* \ln \frac{X}{X^*}\right) + \alpha_2 \left(Y - Y^* - Y^* \ln \frac{Y}{Y^*}\right) + \alpha_3 \left(Z - Z^* - Z^* \ln \frac{Z}{Z^*}\right),$$

where α_1, α_2 and α_3 are positive constants that will be chosen later.
Define $L_1\left(X\right) = \left(X - X^* - X^* \ln\left(\frac{X}{X^*}\right)\right)$, $L_2\left(Y\right) = \left(Y - Y^* - Y^* \ln\left(\frac{Y}{Y^*}\right)\right)$,
and $L_3\left(Z\right) = \left(Z - Z^* - Z^* \ln\left(\frac{Z}{Z^*}\right)\right)$, therefore, $L\left(X, Y, Z\right) = \alpha_1 L_1\left(X\right) + \alpha_2 L_2\left(Y\right) + \alpha_3 L_3\left(Z\right)$.
Differentiating $L(X, Y, Z)$ along the solution of the system (6) with respect to t, we get

$$\frac{dL}{dt} = \alpha_1 \left(1 - \frac{X^*}{X}\right) \frac{dX}{dt} + \alpha_2 \left(1 - \frac{Y^*}{Y}\right) \frac{dY}{dt} + \alpha_3 \left(1 - \frac{Z^*}{Z}\right) \frac{dZ}{dt}. \tag{10}$$

Linear approximations $X - X^* \cong X$, $Y - Y^* \cong Y$ and $Z - Z^* \cong Z$ are used to compute $\frac{dL_1(X(t))}{dt}$, $\frac{dL_2(Y(t))}{dt}$ and $\frac{dL_3(Z(t))}{dt}$ as follows:

$$\begin{aligned}\frac{dL_1}{dt} &= \left(1 - \frac{X^*}{X}\right) \left[r\left(1 - \frac{X}{k}\right) - \beta_{1l}^{1-p} \beta_{1u}^{p} Y - \beta_{2l}^{1-p} \beta_{2u}^{p} Z\right] X \\ &= -\frac{r}{k}\left(X - X^*\right)^2 - \beta_{1l}^{1-p} \beta_{1u}^{p}\left(X - X^*\right)\left(Y - Y^*\right) - \beta_{2l}^{1-p} \beta_{2u}^{p}\left(Z - Z^*\right)\left(X - X^*\right),\end{aligned}$$

$$\begin{aligned}\frac{dL_2}{dt} &= \left(1 - \frac{Y^*}{Y}\right) \left[\beta_{1l}^{1-p} \beta_{1u}^{p} X - d_1 - \delta_{1l}^{1-p} \delta_{1u}^{p} Z\right] Y \\ &= \beta_{1l}^{1-p} \beta_{1u}^{p}\left(X - X^*\right)\left(Y - Y^*\right) - \delta_{1l}^{1-p} \delta_{1u}^{p}\left(Z - Z^*\right)\left(X - X^*\right),\end{aligned}$$

and

$$\begin{aligned}\frac{dL_3}{dt} &= \left(1 - \frac{Z^*}{Z}\right) \left[\beta_{2l}^{1-p} \beta_{2u}^{p} X - d_2 - \delta_2 Y\right] Z \\ &= \beta_{2l}^{1-p} \beta_{2u}^{p}\left(Z - Z^*\right)\left(X - X^*\right) - \delta_{2l}^{1-p} \delta_{2u}^{p}\left(Y - Y^*\right)\left(Z - Z^*\right).\end{aligned}$$

Now,

$$\frac{dL}{dt} = -\alpha_1 \frac{r}{k} \left(X - X^*\right)^2 - \left[\alpha_1 \beta_{2l}^{1-p} \beta_{2u}^{p} + \alpha_2 \delta_{1l}^{1-p} \delta_{1u}^{p} - \alpha_3 \beta_{2l}^{1-p} \beta_{2u}^{p}\right] \left(Z - Z^*\right) \left(X - X^*\right)$$
$$+ \left(\alpha_2 - \alpha_1\right) \beta_{1l}^{1-p} \beta_{1u}^{p} \left(X - X^*\right) \left(Y - Y^*\right) - \alpha_3 \delta_{2l}^{1-p} \delta_{2u}^{p} \left(Y - Y^*\right) \left(Z - Z^*\right).$$

Let $\alpha_2 = 1$ and $\alpha_3 = 1$, then $\alpha_1 = \alpha_2 = 1$. Hence,

$$\frac{dL}{dt} = -\frac{r}{k} \left(X - X^*\right)^2 - \delta_{1l}^{1-p} \delta_{1u}^{p} \left(Z - Z^*\right) \left(X - X^*\right) - \delta_{2l}^{1-p} \delta_{2u}^{p} \left(Y - Y^*\right) \left(Z - Z^*\right).$$

Now, we see that $\frac{dL}{dt}$ is negative definite in the region:

$$G = \{(X, Y, Z) : X > X^*, Y > Y^* \text{ and } Z > Z^* \text{or } X < X^*, Y < Y^* \text{ and } Z < Z^*\}.$$

Therefore, the theorem follows. ■

5 Numerical simulation

To validate our analytical studies, we performed numerical simulations using hypothetical parameter data. Obtaining real data for this purpose can be complex, and therefore, we chose to use hypothetical parameters for our simulations. This approach allows us to assess the precision of our analytical studies and provides us with a reliable means of testing the effectiveness of our models. In this study, we meticulously examine the influence of four significant parameters on the

Table 2. Value of the parameters for various simulations.

Parameter	Simulation 1	Simulation 2	Simulation 3	Simulation 4
r	0.8	0.8	0.8	0.8
k	5.0	5.0	5.0	5.0
$\widehat{\beta}_1$	$[0.3, 0.5]$	$[0.4, 0.6]$	$[0.3, 0.5]$	$[0.3, 0.5]$
$\widehat{\beta}_2$	$[0.3, 0.5]$	$[0.3, 0.5]$	$[0.3, 0.5]$	$[0.3, 0.5]$
$\widehat{\delta}_1$	$[0.04, 0.06]$	$[0.04, 0.06]$	$[0.05, 0.07]$	$[0.04, 0.06]$
$\widehat{\delta}_2$	$[0.04, 0.06]$	$[0.04, 0.06]$	$[0.04, 0.06]$	$[0.04, 0.06]$
d_1	0.1	0.1	0.1	0.1
d_2	0.1	0.1	0.1	0.1
p	0.5	0.5	0.5	0.0, 0.2, 0.4, 0.6, 0.8, 1.0

model system by using the four-parameter state approach, while simultaneously exploring the potential benefits of incorporating interval numbers into our analysis. In doing so, our objective is to expand our understanding of the model system and to provide valuable insight into its behavior under varying conditions.

For the parameter set of **simulation 1**, we find that equilibrium points of the model are $E_0\left(0, 0, 0\right)$, $E_1\left(5, 0, 0\right)$, $E_2\left(0.2582, 1.9589, 0\right)$, $E_3\left(0.2582, 0, 1.9589\right)$ and $E^*\left(0.3789, 0.9545, 0.9545\right)$.

And corresponding eigenvalues are $-0.1000, -0.1000, 0.8000; -0.8000, 1.8365, 1.8365; -0.0207 \pm 0.2747i$,
$-0.0960; -0.0207 \pm 0.2747i, -0.0960$ and $-0.0297 \pm 0.3313i, -0.0012$. Among these points, E_2, E_3 and E^* are stable. Fig. 1 supports our results.

To study the effect of the transmission coefficient ($\widehat{\beta}_1$ and $\widehat{\beta}_2$) on the model, we change the value of the parameter $\widehat{\beta}_1$ in **simulation 2** compared to **simulation 1**. For the parameter set of **simulation 2**, $E_0\left(0, 0, 0\right)$, $E_1\left(5, 0, 0\right)$, $E_2\left(0.2041, 1.5663, 0\right)$, $E_3\left(0.2582, 0, 1.9589\right)$ and $E^*\left(0.3277, 0.5492, 1.2355\right)$

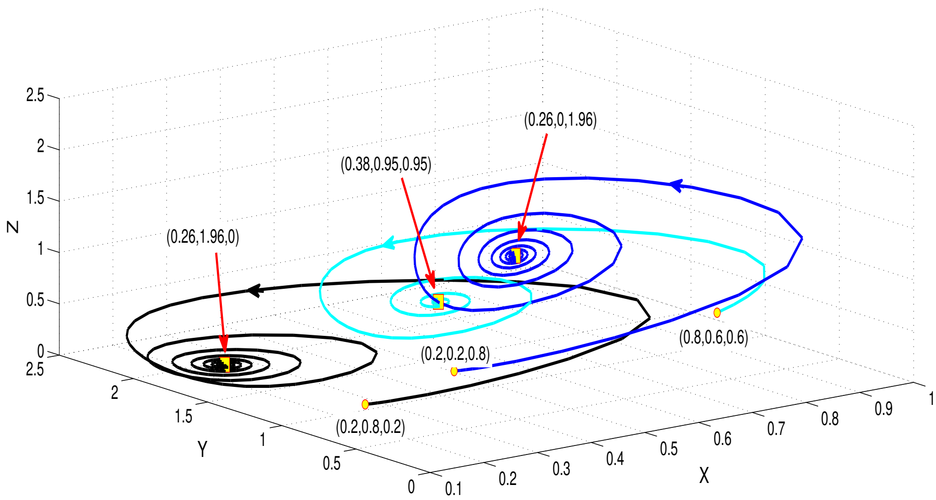

Figure 1. Dynamical behaviour for the equilibrium points.

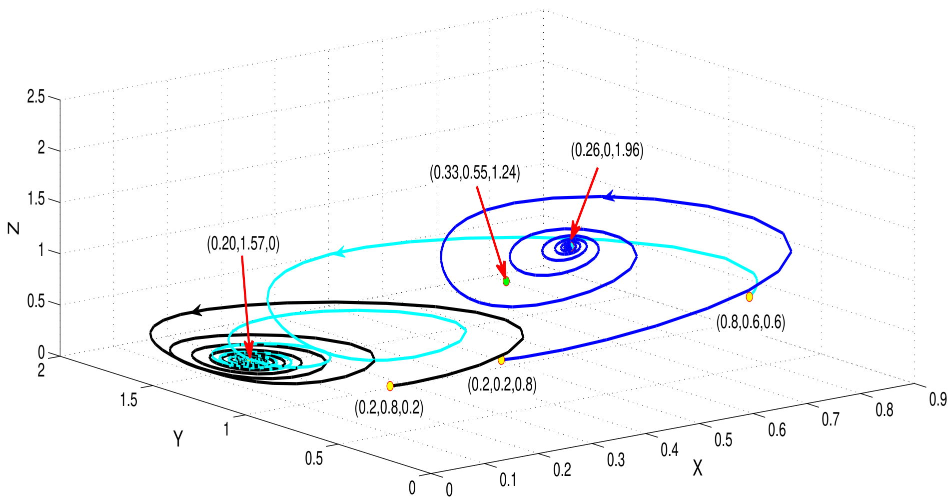

Figure 2. Effect of consumption parameter on the system.

are equilibrium points of our model system. The corresponding eigenvalues are $-0.1000, -0.1000, 0.8000$; $-0.8000, 2.3495, 1.8365$; $-0.0163 \pm 0.2765i, -0.0977$; $-0.0205 \pm 0.2746i, -0.0694$ and $-0.0258 \pm 0.3238i, -0.0008$. Out of these points E_2, E_3 and E^* are stable. Fig. 2 and Fig. 3 are the graphical representation of our analysis based on the set of parameters of **simulation 2**. We found that the initial value and the transmission coefficient ($\widehat{\beta}_1$, and $\widehat{\beta}_2$) are sensitive issues in this system. Due to the change in $\widehat{\beta}_1$, Fig. 2 shows a change compared to Fig. 1 for the same initial condition. We notice another change in Fig. 3 compared to Fig. 2 for different initial conditions and the same parameter values.

To study the effect of the competition coefficient ($\widehat{\delta}_1$ and $\widehat{\delta}_2$) on the model, we change the value of the parameter $\widehat{\delta}_1$ in **simulation 3** compared to **simulation 1**. For the parameter set of **simulation 3** $E_0\ (0, 0, 0)$, $E_1\ (5, 0, 0)$, $E_2\ (0.2582, 1.9589, 0)$, $E_3\ (0.2582, 0, 1.9589)$ and $E^*\ (0.3899, 1.0418, 0.8627)$ are the equilibrium points of our model.
And the corresponding eigenvalues are $-0.1000, -0.1000, 0.8000$; $-0.8000, 1.8365, 1.8365$; $-0.0207 \pm$

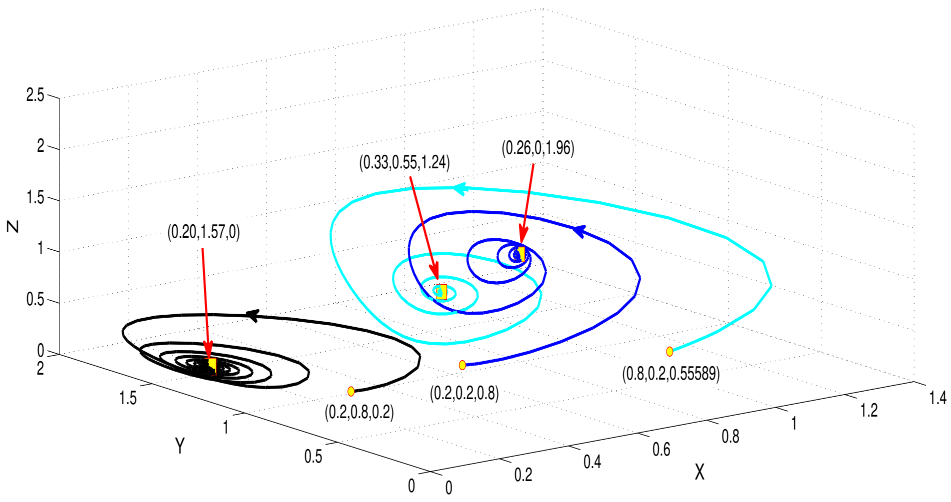

Figure 3. Dynamical behaviour for the equilibrium points.

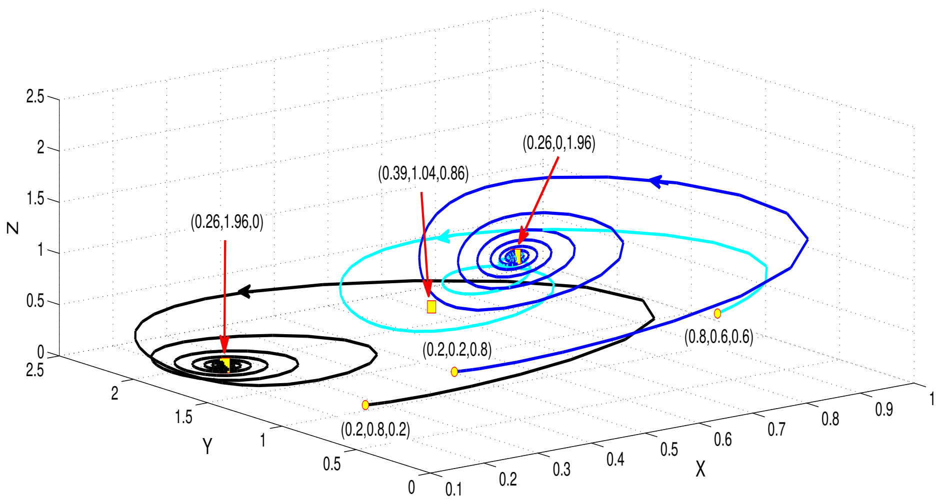

Figure 4. Effect of the competition parameter on the model.

$0.2747i, -0.0960; -0.0207 \pm 0.2747i, -0.1159$ and $-0.0281 \pm 0.3360i, -0.0061$.

Among these points, we identify three equilibrium points, namely E_2, E_3, and E^*, with stable dynamics. The graphical representation of our findings based on the set of parameters used in **simulation 3** is presented in Figs. 4 and 5. Our analysis reveals that the initial values of the competition coefficients ($\widehat{\delta}1$ and $\widehat{\delta}2$) are critical determinants of the behavior of the system. Specifically, even slight changes in $\widehat{\delta}_1$ can significantly alter the system's dynamics, as evident from the comparison between Figs. 1 and 4 for the same initial conditions. Furthermore, we observe another significant change in Fig. 5 compared to Fig. 4 when there is a change in the initial state. These findings highlight the importance of carefully selecting and monitoring initial conditions and competition coefficients in ecological systems to ensure their long-term sustainability. Here, we explore the impact of varying the values of the parameter p on the equilibrium points of the model system. The interior equilibrium points, the corresponding eigenvalues, and the equilibrium characteristics for different values of p are presented in Table 3, based on the parameters used in

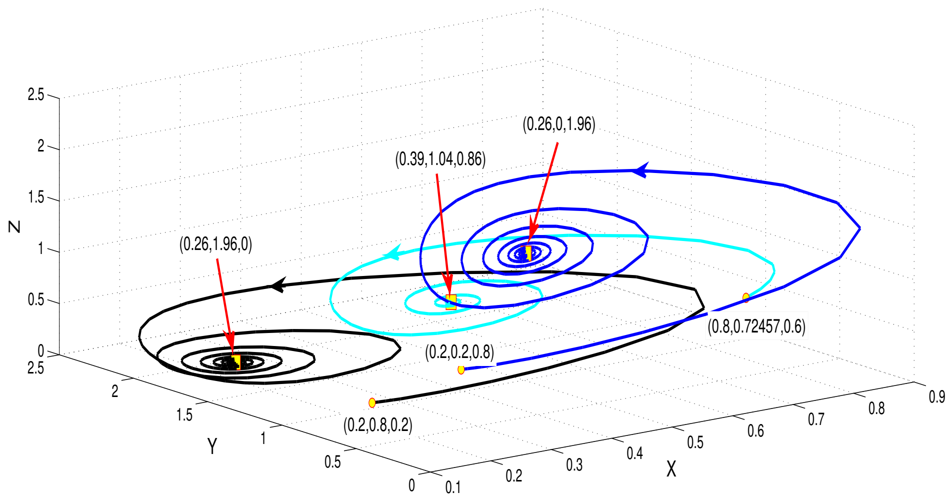

Figure 5. Dynamical behaviour for the equilibrium points.

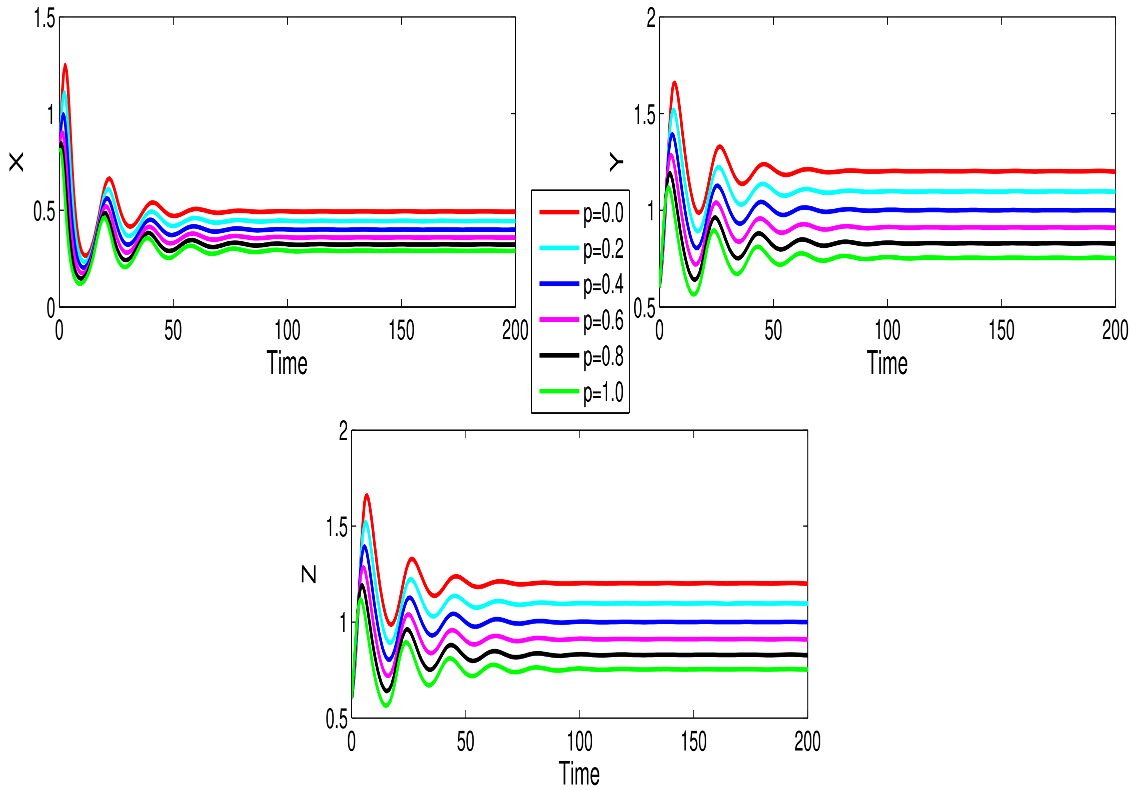

Figure 6. Time history for different value of p.

simulation 4. The results reveal that as p increases, the equilibrium population levels of both the prey and predator species exhibit a gradual decline. This finding suggests that changes in the parameter p have a significant impact on the stability and behavior of the model system.

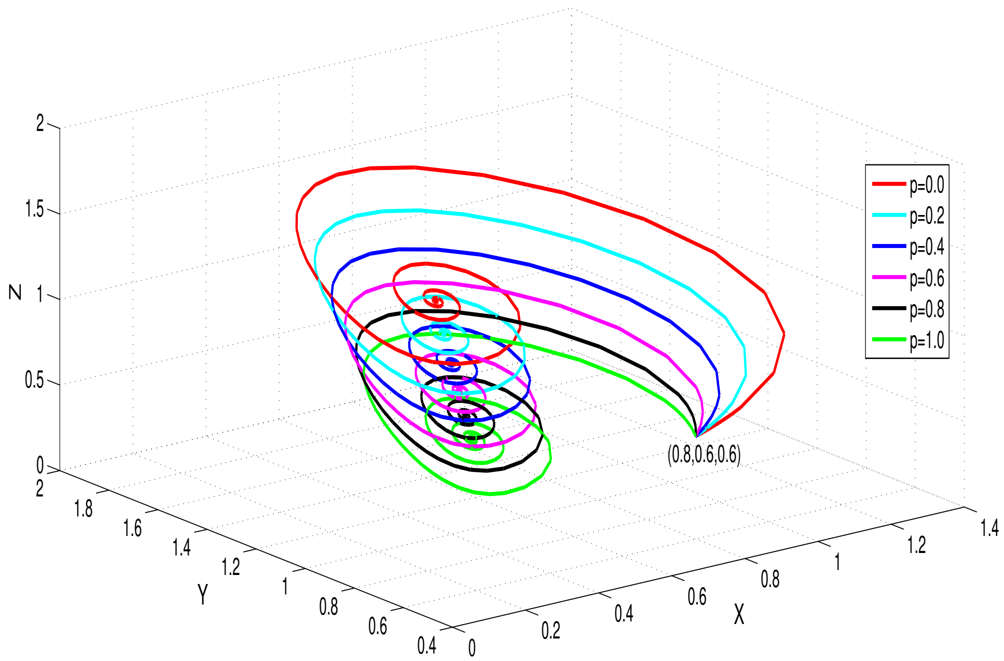

Figure 7. Phase portrait for different value of p.

Table 3. Equilibrium points, eigenvalues, and their nature for different p values.

p	Equilibrium	Eigenvalue	Nature
0	$(0.49356, 1.2017, 1.20172)$	$-0.0386 \pm 0.3278i, -0.0017$	Stable
0.2	$(0.44416, 1.09690, 1.09690)$	$-0.0348 \pm 0.3294i, -0.0015$	Stable
0.4	$(0.39957, 1.00006, 1.00006)$	$-0.0313 \pm 0.3307i, -0.0013$	Stable
0.6	$(0.35934, 0.91083, 0.91083)$	$-0.0282 \pm 0.3317i, -0.0011$	Stable
0.8	$(0.32309, 0.82880, 0.82880)$	$-0.0254 \pm 0.3325i, -0.0010$	Stable
1	$(0.29042, 0.75353, 0.75353)$	$-0.0228 \pm 0.3330i, -0.0009$	Stable

In this study, we analyze the population dynamics of prey and predator species over time, starting with initial population values of $X = 0.8$ (prey), $Y = 0.6$ (first predator), and $Z = 0.6$ (second predator), for various values of $p \in [0, 1]$. The results are presented in Fig. 6, while the corresponding phase portrait is depicted in Fig. 7. From these figures, it is evident that an increase in the value of p is associated with a gradual decrease in population density. These findings provide valuable information on the sensitivity of the model system to changes in parameter values and highlight the importance of understanding the underlying mechanisms that govern predator-prey interactions.

6 Conclusion

In this article, we have presented a three-species prey-predator model that incorporates imprecise biological parameters using the concept of interval numbers. Through the analysis of the model, we have demonstrated that the interval number method is a simple and effective tool for examining the impact of imprecise parameters on the behavior of the system.

Our analysis included checking the positivity and boundedness of the model, as well as performing a stability analysis of the five equilibrium points. The results of our analysis provide valuable insights into the dynamics of the prey-predator system and the effects of imprecision in the parameters.

In conclusion, our work highlights the importance of considering imprecision in biological parameters when modeling ecological systems. The interval number method provides a powerful approach to this challenge, enabling researchers to better capture the complexity of ecological systems and make more accurate predictions about their behavior. We believe that our findings will be of significant value to researchers working in the field of ecological modeling and contribute to the development of more accurate and reliable models of complex ecological systems.

Declarations

Consent for publication

Not applicable.

Conflicts of interest

The authors declare that they have no conflict of interest.

Funding

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Author's contributions

D.G.: Conceptualization, Methodology, Investigation, Writing-Original draft. P.K.S.: Conceptualization, Investigation, Software, Writing-Original draft, Supervision. G.S.M.: Visualization, Writing-Reviewing and Editing, Supervision. All authors discussed the results and contributed to the final manuscript.

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