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HEDONIC COALITION FORMATION GAMES: NASH STABILITY UNDER DIFFERENT MEMBERSHIP RIGHTS

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ABSTRACT
Purpose - We study hedonic coalition formation games in which each agent has preferences over the coalitions she is a member of. Hedonic coalition formation games are used to model economic, social, and political instances in which people form coalitions. The outcome of a hedonic coalition formation game is a partition. We consider stability concepts of a partition that are based on a single-agent deviation under different membership rights, that is, we study Nash stability under different membership rights. We revisit the conditions that guarantee the existence of Nash stable partitions and provide examples of hedonic coalition formation games satisfying these conditions.
Methodology – While analyzing a stability notion for hedonic coalition formation games, two crucial points are considered: i) who can deviate from the given partition, ii) what are the allowed movements for the deviator(s), i.e., what deviators are entitled to do. For the first point, the deviation of a single agent is considered for Nash stabilities. For the second point, the allowed movements for deviators are determined by specifying membership rights, that is, membership rights describe whose approval is needed for a particular deviation. So, we reconsider stability concepts by using membership rights based on individual deviations, i.e., we consider Nash stability under different membership rights for hedonic coalition formation games.
Findings- A classification of stability concepts based on a single-agent deviation for hedonic coalition formation games are provided by employing membership rights. The conditions in the literature guaranteeing the existence of Nash stable partitions for all membership rights are revisited. For each condition, an example of a hedonic coalition formation game satisfying the condition is given. Hence, a complete analysis of sufficient conditions for all Nash stability concepts are provided.
Conclusion- To choose the correct stability notion one first should understand the membership rights in the environment that she studies. Then, for hedonic coalition formation problems, the appropriate Nash stability notion consistent with the ongoing membership rights should be chosen when single-agent deviation is considered.

Keywords: Coalition formation, Hedonic games, Nash stability, membership rights, separable preferences
JEL Codes: C71, C78, D71

1. INTRODUCTION

In many economic, social, and political instances people prefer to form groups (coalitions) rather than staying alone. For example, individuals form hobby groups, students form assignment groups in a course, researchers form research teams, and political parties form coalitions. Some of these instances of forming groups can be modeled as a hedonic coalition formation game.

A hedonic coalition formation game (shortly, a hedonic game) consists of a finite set of agents and a list of agents' preferences. A non-empty subset of the agents is called a coalition, and each agent's preferences depend only on the coalitions of which she is a member. That is, each agent only cares about which other agents are in her coalition and does not care how other agents who are not in her coalition behave. This is called the *hedonic aspect of preferences* by Drèze and Greenberg (1980). The formal model of hedonic coalition formation games was introduced by Banerjee et al. (2001) and Bogomolnaia and Jackson (2002). Marriage problems and roommate problems (Gale and Shapley, 1962; Roth and Sotomayor, 1990) are special hedonic games in which the size of every coalition can be a maximum of two.

An outcome of a hedonic game is a partition (a coalition structure) that is a collection of pairwise disjoint coalitions such that their union is equal to the set of agents. The stability properties of partitions have been studied in the literature.[1] Stability notions concern individual as well as coalitional deviations from a given partition. One of the most studied stability concepts for hedonic coalition formation games is Nash stability.[2] A partition for a given hedonic game is Nash stable if there is no agent such that she gets strictly better off by leaving her current coalition and joining an existing coalition of the partition or she forms a singleton coalition by herself (in this case we say that she joins to the empty set). The agent who moves to another coalition of the partition does not consider whether the agents who are in the coalition she leaves, and joins are affected negatively, and hence she does not ask their permission for her movement.

As mentioned in Karakaya (2011), when a stability notion is analyzed, we consider agent(s) who can deviate and what are the allowed movements for the deviator(s), that is, what deviators are entitled to do. The allowed movements for deviators are determined by specifying membership rights, that is, membership rights describe whose approval is needed for a particular deviation. Sertel (1992) introduced four membership rights: Free Exit and Free Entry (FX-FE) membership rights, Free Exit and Approved Entry (FX-AE) membership rights, Approved Exit and Free Entry (AX-FE) membership rights, and Approved Exit and Approved Entry (AX-AE) membership rights.

Free Exit and Free Entry (FX-FE) membership rights describe situations in which an agent does not need the consent of anyone when leaving one coalition and joining another, that is, she asks permission from neither the remaining agents in the coalition she leaves nor the agents in the coalition she joins. For example, when a family moves from one neighborhood to another while changing houses, it does not need the permission of anyone (neither the families in the neighborhood they leave nor the families in the new neighborhood they move to).

Free Exit and Approved Entry (FX-AE) membership rights describe situations in which an agent does not need the consent of remaining agents in the coalition she leaves but she needs the consent of agents of the coalition she joins. For example, joining a new friendship group or a new hobby group, an agent must have the consent of everyone in the new group but does not need anyone’s permission when leaving her old group.

Approved Exit and Free Entry (AX-FE) membership rights describe situations in which an agent does not need the consent of the agents of the coalition she joins but needs the consent of the remaining agents in the coalition she leaves. As an example, we can consider a person who would like to join a voluntary-based organization. Joining such an organization is usually easy and does not require permission. However, a person working in such an organization cannot leave her job without the consent of the people in the organization.

Approved Exit and Approved Entry (AX-AE) membership rights describe situations in which an agent needs the consent of both the remaining agents in the coalition she leaves and the agents in the coalition she joins. As an example, to quit a job and start another one, an agent must first terminate the contract with her current workplace and then start a contract with the new workplace, that is, the agent needs the permission of both workplaces.

In this study, we reconsider the stability properties of partitions regarding single-agent deviations under different membership rights together with conditions that guarantee the existence of such partitions. We revisit these sufficient conditions given in the literature by providing hedonic games satisfying them.

The Nash stability of a partition that we mentioned above is defined under FX-FE membership rights, and hence we call it FX-FE Nash stability. That is, a partition is **FX-FE Nash stable** if there does not exist an agent and an existing coalition of the partition (or, the empty set) such that she gets strictly better off by moving to this coalition (that is, she leaves her current coalition and joins the coalition of the partition) without asking the permission of anyone else.

[1]We refer the reader to Hajduková (2006) and Aziz and Savani (2016) for the survey of the hedonic coalition formation literature and Sung and Dimitrov (2007) for the taxonomy of stability concepts.

[2]The other one is core stability. A partition for a given hedonic game is core stable if there is no coalition such that each agent in the coalition prefers it to her coalition under the partition. That is, each agent in this coalition leaves her current coalition and then they form a new coalition among themselves. Note that when an agent in this coalition leaves her current coalition, she does not require any permission from members of her current coalition of the partition.

Nash stability under FX-AE membership rights is called *individual stability* in the literature (Bogomolnaia and Jackson, 2002). We call it FX-AE Nash stability. A partition is **FX-AE Nash stable** if there does not exist an agent and a coalition of the partition such that the agent gets strictly better off by moving to this coalition and each member of the coalition gets weakly better off (that is, all members of the coalition to which she moves approve her joining to their coalition).

Nash stability under AX-FE membership rights is called *contractual Nash stability* in the literature (Sung and Dimitrov, 2007). We call it AX-FE Nash stability. A partition is **AX-FE Nash stable** if there does not exist an agent and a coalition of the partition such that the agent gets strictly better off by moving to this coalition and all members of her coalition under the given partition she leaves get weakly better off (that is, all members of her coalition under the partition that she leaves approve her leaving).

Nash stability under AX-AE membership rights is called *contractual individual stability* in the literature (Bogomolnaia and Jackson, 2002). We call it AX-AE Nash stability. A partition is **AX-AE Nash stable** if there does not exist an agent and a coalition of the partition such that the agent gets strictly better off by moving to this coalition and all members of the coalition she leaves and joins gets weakly better off (that is, all members of coalitions that she leaves and joins approve her movement).

If a partition is Nash stable under some membership rights and when the membership rights are restricted, then it is also Nash stable under the restricted membership rights. That is, if a partition is FX-FE Nash stable, then it is also FX-AE Nash stable, AX-FE Nash stable, and AX-AE Nash stable. If a partition is FX-AE Nash stable, then it is also AX-AE Nash stable. In the same manner, if a partition is AX-FE Nash stable, then it is also AX-AE Nash stable. However, FX-AE Nash stability and AX-FE Nash stability are independent of each other.

The concepts of FX-FE Nash stability (or, Nash stability), FX-AE Nash stability (individual stability), and AX-AE Nash stability (contractual individual stability) were first introduced and studied by Bogomolnaia and Jackson (2002) without referring to membership rights. The concept of AX-FE Nash stability (contractual Nash stability) was first introduced by Sung and Dimitrov (2007) without referring to membership rights, but they introduced the taxonomy for stability concepts. Karakaya (2011) considered coalitional extension of Nash stability by employing membership rights in the context of hedonic games and introduced and analyzed the notion of *strong Nash stability* under different membership rights.[3]

Bogomolnaia and Jackson (2002) proved that if a hedonic game is *additively separable* and *symmetric*, then there exists an FX-FE Nash stable partition. They proved that the symmetry property is crucial and cannot be replaced by *mutuality*, otherwise, there would be no FX-FE Nash stable partition. Dimitrov and Sung (2004) and Dimitrov et al. (2006) introduced the properties of *appreciation of friends* and *aversion to enemies*. Dimitrov and Sung (2004) proved that if a hedonic game satisfies the properties of appreciation of friends and mutuality or the properties of aversion to enemies and mutuality, then there exists an FX-FE Nash stable partition. Dimitrov and Sung (2006) proved that if a hedonic game satisfies the *top responsiveness property* (Alcalde and Revilla, 2004) and *mutuality* (with respect to top responsiveness), then there exists an FX-FE Nash stable partition. Suksompong (2015) introduced the properties called *subset neutrality* and *neutral anonymity* and showed that if a hedonic game satisfies the subset neutrality property or the neutral anonymity property, then there exists an FX-FE Nash stable partition.

Burani and Zwicker (2003) introduced *descending separable preferences* and showed that they guarantee the existence of a partition that is both FX-FE Nash stable and core stable. Karakaya (2011) showed that descending separable preferences are indeed sufficient for the existence of an FX-FE strongly Nash stable partition. He also introduced a sufficient condition called *weak top-choice property* (by using the weak top-coalition notion of Banerjee et al. (2001)) and proved that if a hedonic game satisfies the weak top-choice property, then there exists an FX-FE strongly Nash stable partition. Aziz and Brandl (2012) proved that if a hedonic game satisfies the *top responsiveness property* and *mutuality* (with respect to top responsiveness) or *bottom responsiveness property* (Suzuki and Sung, 2010) and *mutuality* (with respect to bottom responsiveness), then there exists an FX-FE stronly Nash stable partition.[4]

[3]A partition is FX-FE strongly Nash stable if there exists no subset of agents who reach a new partition via movements among the coalitions of the given partition (e.g., these movements include but not restricted to forming a new coalition, joining existing coalitions individually or as a group, exchanging their current coalitions, or shuffling their coalitions, etc.) such that these agents strictly prefer the new partition to the initial one. This definition uses the reachability approach that we also adopt to define Nash stabilities under different membership rights. We also note that FX-FE strong Nash stability is stronger than core stability and FX-FE Nash stability.

[4]In this study, we do not provide the definitions of *descending separable preferences*, *weak top-choice property*, *top responsiveness property*, and *bottom responsiveness property* since these conditions guarantee the existence of an FX-FE strongly Nash stable partition. For these properties, we refer readers to Burani and Zwicker (2003), Karakaya (2011), Dimitrov and Sung (2006), Suzuki and Sung (2010), and Aziz and Brandl (2012).

Bogomolnaia and Jackson (2002) also studied FX-AE Nash stability and showed that if a hedonic game satisfies the *ordered characteristics property,* then there exists an FX-AE Nash stable partition. Suksompong (2015) also proved that if a hedonic game satisfies the *common ranking property* (Farrell and Scotchmer, 1988), then there exists an FX-AE Nash stable partition.

Sung and Dimitrov (2007) showed that if a hedonic game satisfies the *separability* and *weak mutuality*, then there exists an AX-FE Nash stable partition.

Bogomolnaia and Jackson (2002) proved that every hedonic game has an AX-AE Nash stable partition and introduced an algorithm that brings an AX-AE Nash stable partition which is also Pareto optimal and individually rational when preferences of each agent are strict. Ballester (2004) showed that every hedonic game has an AX-AE Nash stable partition.

In the field of electrical and electronics engineering, the problem of allocating different and complex tasks to a swarm of autonomous robots is a well-known problem. That problem is a hedonic game and the Nash stability of partitions of the given swarm is analyzed via experimental and computational methods. For such studies, we refer the reader to Czarnecki and Dutta (2021), Jang et al. (2018), and Xiong and Xie (2023). In the field of computer science, Nash stability under different membership rights is also studied. In these studies, the computational complexity analysis of the problem for finding a Nash stable partition is investigated for different domains of hedonic games, and algorithms that yield Nash stable partitions are studied. For such studies, we refer the reader to Aziz et al. (2011), Ballester (2004), Bilò et al. (2018), Kerkmann and Rothe (2019), Olsen (2009), and Sung and Dimitrov (2010).

The paper is organized as follows. In Chapter 2, we introduce the model of hedonic coalition formation games. In Chapter 3, we study FX-FE Nash stability and conditions (symmetric and additively separable preferences, appreciation of friends and aversion to enemies, subset neutrality and neutral anonymity) that guarantee its existence by providing examples for each of them. In Chapter 4, we study FX-AE Nash stability with common ranking and ordered characteristics properties each of which suffices for the existence of an FX-AE Nash stable partition. We also provide hedonic games that satisfy these properties. In Chapter 5, we study AX-FE Nash stability with separability and weak mutuality. We provide an example that is separable and weakly mutual and hence has an AX-FE Nash stable partition. In Chapter 6, we study AX-AE Nash stability and provide the proof ideas for the existence of such partitions for every hedonic game by Bogomolnaia and Jackson (2002) and Ballester (2004). In the concluding chapter (Chapter 7), the entire study is summarized and further comments on Nash stability are included.

2. HEDONIC COALITION FORMATION

Let $N = \{1,2, \dots, n\}$ be a finite set of agents with $n \geq 2$. A nonempty subset S of N is called a *coalition of* N. For each agent $i \in N$, let $\mathcal{C}_i^N = \{S \subseteq N \mid i \in S\}$ denote the set of all coalitions of N containing agent i.

Each agent $i \in N$ has complete and transitive preferences (*weak preferences*) $\succcurlyeq_i$ over $\mathcal{C}_i^N$.[5] For each $i \in N$ and each $S, T \in \mathcal{C}_i^N$ with $S \neq T$, $S \succ_i T$ if and only if $S \succcurlyeq_i T$ but not $T \succcurlyeq_i S$, that is, agent i *strictly prefers* S *to* T; and $S \sim_i T$ if and only if both $S \succcurlyeq_i T$ and $T \succcurlyeq_i S$ hold, that is, agent i *is indifferent between* S *and* T. For instance, let $i \in N$ and $S, T, U, V \in \mathcal{C}_i^N$, then $[\succcurlyeq_i : S \succ_i T \sim_i U \succ_i V \sim_i \{i\} \succ_i \dots]$ means that the best coalition for agent i is S (that is, S is strictly preferred to any other coalition that contains i), agent i is indifferent between coalitions T and U, and these are strictly preferred to coalition V, and agent i is indifferent between coalition V and being single ($\{i\}$), etc.

For each $i \in N$, let $\mathcal{R}_i$ denote the set of all preferences of agent i over $\mathcal{C}_i^N$, and let $\mathcal{R}^N = \prod_{i \in N} \mathcal{R}_i$ denote the set of all preference profiles of agents in N.

A ***hedonic coalition formation game***, or simply a ***hedonic game***, consists of a finite set of agents N and their preferences $\succcurlyeq = (\succcurlyeq_1, \succcurlyeq_2, \dots, \succcurlyeq_n) \in \mathcal{R}^N$ and is denoted by $(N, \succcurlyeq)$.

A ***partition*** for a hedonic game $(N, \succcurlyeq)$ is a set $\pi = \{S_1, S_2, \dots, S_K\}$ ($K \leq |N|$ is a positive integer) such that (i) for any $k \in \{1, \dots, K\}$, $S_k \neq \emptyset$, (ii) $\bigcup_{k=1}^K S_k = N$, and (iii) for any $k, l \in \{1, \dots, K\}$ with $k \neq l$, $S_k \cap S_l = \emptyset$.

[5]A preference relation $\hat{\succcurlyeq}$ over $\mathcal{C}_i^N$ satisfies *completeness* if for all $S, T \in \mathcal{C}_i^N$, $S \mathrel{\hat{\succcurlyeq}} T$ or $T \mathrel{\hat{\succcurlyeq}} S$, and it satisfies *transitivity* if for all $S, T, U \in \mathcal{C}_i^N$, if $S \mathrel{\hat{\succcurlyeq}} T$ and $T \mathrel{\hat{\succcurlyeq}} U$, then $S \mathrel{\hat{\succcurlyeq}} U$.

Given any partition π and any $i \in N$, we let $\pi(i)$ denote the unique coalition in π that contains agent i. We denote the set of all partitions for hedonic game $(N, \succcurlyeq)$ by $\Pi(N, \succcurlyeq)$. Since agents only care about their own coalitions, preferences over coalitions are extended to over partitions as follows: for each $i \in N$ and partitions $\pi, \pi' \in \Pi(N, \succcurlyeq)$, $\pi \succcurlyeq_i \pi'$ if and only if $\pi(i) \succcurlyeq_i \pi'(i)$.

Next, we introduce the classic voluntary participation concept, individual rationality, and the most common efficiency concept, Pareto optimality.

A partition π is ***individually rational*** for hedonic game $(N, \succcurlyeq)$ if for all $i \in N$, $\pi(i) \succcurlyeq_i \{i\}$.

A partition π is ***Pareto optimal*** for hedonic game $(N, \succcurlyeq)$ if there does not exist another partition $\pi' \in (\Pi(N, \succcurlyeq) \backslash \{\pi\})$ such that for all $i \in N$, $\pi'(i) \succcurlyeq_i \pi(i)$ and for some $j \in N$, $\pi'(j) \succ_j \pi(j)$.

Given a partition $\pi \in \Pi(N, \succcurlyeq)$ and an agent $i \in N$, when agent i deviates from π, she leaves her current coalition $\pi(i)$ and moves to another coalition $S \in (\pi \cup \{\emptyset\})$ of the partition π (or to the empty set). With this deviation of agent i from π, another partition $\pi' \in (\Pi(N, \succcurlyeq) \backslash \{\pi\})$ is obtained. Following Karakaya (2011), we call this case as π' ***is reachable from*** π ***via agent*** i (denoted by $\pi \xrightarrow{i} \pi'$), that is,

- $\pi'(i) = S \cup \{i\}$,
- for each $j \in S$, $\pi'(j) = S \cup \{i\}$,
- for each $k \in \pi(i)$, $\pi'(k) = \pi(i) \backslash \{i\}$, and
- for each $h \in N$ such that $h \notin \pi(i)$ and $h \notin S$, $\pi'(h) = \pi(h)$.

In the following sections, we introduce Nash stability under different membership rights and revisit the sufficient conditions that guarantee the existence of such stable partitions.

3. FREE EXIT - FREE ENTRY (FX-FE) NASH STABILITY

In this section, we introduce the definition of Nash stability under Free Exit and Free Entry membership rights, Free Exit - Free Entry (FX-FE) Nash stability and consider the sufficient conditions that guarantee the existence of such partitions.

Definition 1. FX-FE Nash Stability

*Let $(N, \succcurlyeq)$ be a hedonic game. A partition $\pi \in \Pi(N, \succcurlyeq)$ is **FX-FE Nash stable** if there does not exist a pair (i, S), where $i \in N$ and $S \in (\pi \cup \{\emptyset\})$, such that $S \cup \{i\} \succ_i \pi(i)$. If such a pair (i, S) exists, we say that agent i **FX-FE Nash blocks** π (by joining coalition S).*

We can redefine FX-FE Nash stability by using the reachability notion as follows:

A partition $\pi \in \Pi(N, \succcurlyeq)$ is ***FX-FE Nash stable*** if there does not exist a pair (i, π'), consisting of an agent $i \in N$ and another partition $\pi' \in (\Pi(N, \succcurlyeq) \backslash \{\pi\})$, such that $\pi \xrightarrow{i} \pi'$ (π' is reachable from π via agent i) and $\pi'(i) \succ_i \pi(i)$. If such a pair (i, π') exists, we say that agent i **FX-FE Nash blocks** π (by inducing π').

3.1. Symmetric and Additively Separable Preferences

The notion of *additive separability* was introduced by Banerjee et al. (2001) and Bogomolnaia and Jackson (2002). Bogomolnaia and Jackson (2002) showed that if a hedonic game is additively separable and symmetric, then there exists an FX-FE Nash stable partition.

A hedonic game is additively separable if each agent's preferences are representable by an additively separable utility function.

Definition 2. Additive Separability

*A hedonic game $(N, \succcurlyeq)$ is **additively separable** if for any $i \in N$, there exists a function $v_i : N \to \mathbb{R}$ such that for any $S, T \in \mathcal{C}_i^N$,*

$$S \succcurlyeq_i T \Leftrightarrow \sum_{j \in S} v_i(j) \geq \sum_{j \in T} v_i(j), \text{ where } v_i(j) = 0 \text{ for } i = j.$$

*An additively separable hedonic game $(N, \succcurlyeq)$ satisfies **symmetry** if for any $i, j \in N$, we have $v_i(j) = v_j(i)$, and satisfies **mutuality** if for any $i, j \in N$, we have $v_i(j) \geq 0 \Leftrightarrow v_j(i) \geq 0$.*

For each agent i, v_i denotes her utility function that assigns a cardinal utility for every agent in N, where she assigns zero value to herself. For any coalition $S \in \mathcal{C}_i^N$, the total payoff that agent i obtains from being a member of this coalition is the sum of the utilities that she assigns to each agent in S, that is, $\sum_{j\in S} v_i\,(j)$. Then, any two coalitions containing agent i are compared according to the total payoffs that she obtains from these coalitions, that is, for any $S, T \in \mathcal{C}_i^N$, agent i prefers S to T if and only if the total payoff that i obtains from S is as big as the total payoff that she obtains from T.

Additively separable preferences are symmetric if every two agents assign the same utilities to each other, that is, for every agent i and j, $v_i(j) = v_j(i)$. Additively separable preferences are mutual if it holds for every two agents i and j that i assigns a positive (negative or zero, respectively) value to j if and only if j assigns a positive (negative or zero, respectively) value to i. Note that symmetry implies mutuality.

The hedonic game given in the following example is additively separable and symmetric and hence it has an FX-FE Nash stable partition.

Example 1. Symmetric and Additively Separable Hedonic Game

Let $(N, \succcurlyeq)$ be a hedonic game with $N = \{1,2,3\}$ and additively separable and symmetric preferences are represented by following functions: $v_1(2) = v_2(1) = 2$, $v_1(3) = v_3(1) = -1$, $v_2(3) = v_3(2) = 1$. That is, we have following preferences:

$\succcurlyeq_1: \{1,2\} \succ_1 \{1,2,3\} \succ_1 \{1\} \succ_1 \{1,3\}$,

$\succcurlyeq_2: \{1,2,3\} \succ_2 \{1,2\} \succ_2 \{2,3\} \succ_2 \{2\}$,

$\succcurlyeq_3: \{2,3\} \succ_3 \{1,2,3\} \sim_3 \{3\} \succ_3 \{1,3\}$.

The partitions $\pi^1 = \{\{1,2\},\{3\}\}$ and $\pi^2 = \{\{1,2,3\}\}$ are FX-FE Nash stable. □

Bogomolnaia and Jackson (2002) mentioned that a hedonic game with mutual and additively separable preferences may not have an FX-FE Nash stable partition, so symmetry is a critical property, and it cannot be weakened to mutuality for the existence of FX-FE Nash stable partitions.

3.2. Appreciation of Friends and Aversion to Enemies

Dimitrov and Sung (2004) and Dimitrov et al. (2006) introduced the *appreciation of friends* and the *aversion to enemies* properties. These properties are based on the cardinality of friends and the cardinality of enemies in each coalition.

Let $(N, \succcurlyeq)$ be a hedonic game. For each $i \in N$, we say that $F_i = \{j \in N \mid \{i,j\} \succcurlyeq_i \{i\}\}$ is the ***set of friends of agent*** i, and $E_i = \{j \in N \mid \{i\} \succ_i \{i,j\}\} = N \backslash F_i$ is the ***set of enemies of agent*** i. Note that for each $i \in N$, $i \in F_i$.

Definition 3. Appreciation of Friends

Let $(N, \succcurlyeq)$ be a hedonic game. We say that $(N, \succcurlyeq)$ satisfies the ***appreciation of friends property*** if, for each $i \in N$ and each $S, T \in \mathcal{C}_i^N$, $S \succcurlyeq_i T$ if and only if

- $|S \cap F_i| > |T \cap F_i|$ or,
- $|S \cap F_i| = |T \cap F_i|$ and $|S \cap E_i| \leq |T \cap E_i|$.

A hedonic game $(N, \succcurlyeq)$ having the *appreciation of friends property* also satisfies ***mutuality*** (with respect to appreciation of friends) if for each $i, j \in N$ with $i \neq j$, $i \in F_j$ if and only if $j \in F_i$.[6]

The appreciation of friends property means that: For an agent, when comparing two coalitions containing her, she first looks at the number of friends in these coalitions and that she prefers the one with more friends. If the two coalitions have the same number of friends, then she looks at the number of enemies in these coalitions and prefers the one with fewer enemies. If the numbers of friends and enemies are equal in these two coalitions then she is indifferent between these two coalitions.

[6] This is equivalent to that $i \in E_j$ if and only if $j \in E_i$, since if $i \notin F_j$ we then have $i \in E_j$.

Definition 4. Aversion to Enemies

Let $(N, \succcurlyeq)$ be a hedonic game. We say that $(N, \succcurlyeq)$ satisfies the ***aversion to enemies property*** if, for each $i \in N$ and each $S, T \in \mathcal{C}_i^N$, $S \succcurlyeq_i T$ if and only if

- $|S \cap E_i| < |T \cap E_i|$ or,
- $|S \cap E_i| = |T \cap E_i|$ and $|S \cap F_i| \geq |T \cap F_i|$.

A hedonic game $(N, \succcurlyeq)$ having the *aversion to enemies' property* also satisfies ***mutuality*** (with respect to aversion to enemies property) if for each $i, j \in N$ with $i \neq j$, $i \in E_j$ if and only if $j \in E_i$.[7]

The aversion to enemies' property means that: For an agent, when comparing two coalitions containing her, she first looks at the number of enemies in these coalitions and that she prefers the one with fewer enemies. If the two coalitions have the same number of enemies, then she looks at the number of friends in these coalitions and prefers the one with more friends. If the numbers of enemies and friends are equal in these two coalitions then she is indifferent between these two coalitions.

When appreciation of friends or aversion to enemies' property is satisfied for a hedonic game $(N, \succcurlyeq)$, then for any agent $i \in N$, the best coalition is F_i and the worst coalition is $E_i \cup \{i\}$ according to her preferences. Moreover, the domain of preferences at which appreciation of friends or aversion to enemies property is satisfied is a proper sub-domain of additively separable preference profiles.[8] If a hedonic game $(N, \succcurlyeq)$ satisfies the appreciation of friends property then it is additively separable where for each $i \in N$ the function $v_i: N \to \mathbb{R}$ is defined as follows: for each $j \in N \backslash \{i\}$, $v_i(j) = n$ if $j \in F_i$, $v_i(j) = -1$ if $j \in E_i$, and $v_i(i) = 0$. If a hedonic game $(N, \succcurlyeq)$ satisfies the aversion to enemies property then it is additively separable where for each $i \in N$ the function $v_i: N \to \mathbb{R}$ is defined as follows: for each $j \in N \backslash \{i\}$, $v_i(j) = 1$ if $j \in F_i$, $v_i(j) = -n$ if $j \in E_i$, and $v_i(i) = 0$.

Dimitrov and Sung (2004) showed that if a hedonic game satisfies the appreciation of friends' property and mutuality (with respect to appreciation of friends) then it has an FX-FE Nash stable partition, and similarly, if a hedonic game satisfies the aversion to enemies' property and mutuality (with respect to aversion to enemies) then it has an FX-FE Nash stable partition.[9]

We now provide a hedonic game that satisfies the appreciation of friends' property and mutuality (with respect to appreciation of friends), and hence it has an FX-FE Nash stable partition.

Example 2. A Hedonic Game Satisfying Appreciation of Friends and Mutuality

Let $(N, \succcurlyeq)$ be a hedonic game with $N = \{1,2,3,4\}$. Let $F_1 = \{1,2,3\}$, $F_2 = \{1,2\}$, $F_3 = \{1,3\}$, and $F_4 = \{4\}$. Then, agents' preferences satisfying the appreciation of friends' property are as follows:

$\succcurlyeq_1: \{1,2,3\} \succ_1 \{1,2,3,4\} \succ_1 \{1,2\} \sim_1 \{1,3\} \succ_1 \{1,2,4\} \sim_1 \{1,3,4\} \succ_1 \{1\} \succ_1 \{1,4\}$.

$\succcurlyeq_2: \{1,2\} \succ_2 \{1,2,3\} \sim_2 \{1,2,4\} \succ_2 \{1,2,3,4\} \succ_2 \{2\} \succ_2 \{2,3\} \sim_2 \{2,4\} \succ_2 \{2,3,4\}$,

$\succcurlyeq_3: \{1,3\} \succ_3 \{1,2,3\} \sim_3 \{1,3,4\} \succ_3 \{1,2,3,4\} \succ_3 \{3\} \succ_3 \{2,3\} \sim_3 \{3,4\} \succ_3 \{2,3,4\}$,

$\succcurlyeq_4: \{4\} \succ_4 \{1,4\} \sim_4 \{2,4\} \sim_4 \{3,4\} \succ_4 \{1,2,4\} \sim_4 \{1,3,4\} \sim_4 \{2,3,4\} \succ_4 \{1,2,3,4\}$.

We note that this hedonic game also satisfies mutuality (with respect to appreciation of friends): $2 \in F_1$ and $1 \in F_2$, $3 \in F_1$ and $1 \in F_3$, $4 \notin F_1$ and $1 \notin F_4$, $3 \notin F_2$ and $2 \notin F_3$, $4 \notin F_2$ and $2 \notin F_4$, and $4 \notin F_3$ and $3 \notin F_4$.

The partition $\pi = \{\{1,2,3\}, \{4\}\}$ is FX-FE Nash stable. □

[7] This is equivalent to that $i \in F_j$ if and only if $j \in F_i$, since if $i \notin E_j$ we then have $i \in F_j$.

[8] See Definition 2 for additively separable preferences.

[9] Suzuki and Sung (2010) showed that hedonic games that satisfy the appreciation of friends' property also satisfy the top responsiveness property, and hedonic games that satisfy the aversion to enemies' property also satisfy the bottom responsiveness property. Therefore, the results in Aziz and Brandl (2012) also hold for the appreciation of friends and aversion to enemies' properties. If a hedonic game satisfies the appreciation of friends and mutuality (with respect to appreciation of friends), then there exists an FX-FE strongly Nash stable partition and likewise if a hedonic game satisfies the aversion to enemies and mutuality (with respect to aversion to enemies), then there exists an FX-FE strongly Nash stable partition.

We now provide a hedonic game that satisfies the aversion to enemies' property and mutuality (with respect to aversion to enemies), and hence it has an FX-FE Nash stable partition.

Example 3. A Hedonic Game Satisfying Aversion to Enemies and Mutuality

Let $(N, \succcurlyeq)$ be a hedonic game with $N = \{1,2,3,4\}$. Let $F_1 = \{1,2,3\}$, $F_2 = \{1,2\}$, $F_3 = \{1,3\}$, and $F_4 = \{4\}$. Then, agents' preferences satisfying the aversion to enemies' property are as follows:

$\succcurlyeq_1: \{1,2,3\} \succ_1 \{1,2\} \sim_1 \{1,3\} \succ_1 \{1\} \succ_1 \{1,2,3,4\} \succ_1 \{1,2,4\} \sim_1 \{1,3,4\} \succ_1 \{1,4\}$.

$\succcurlyeq_2: \{1,2\} \succ_2 \{2\} \succ_2 \{1,2,3\} \sim_2 \{1,2,4\} \succ_2 \{2,3\} \sim_2 \{2,4\} \succ_2 \{1,2,3,4\} \succ_2 \{2,3,4\}$,

$\succcurlyeq_3: \{1,3\} \succ_3 \{3\} \succ_3 \{1,2,3\} \sim_3 \{1,3,4\} \succ_3 \{2,3\} \sim_3 \{3,4\} \succ_3 \{1,2,3,4\} \succ_3 \{2,3,4\}$,

$\succcurlyeq_4: \{4\} \succ_4 \{1,4\} \sim_4 \{2,4\} \sim_4 \{3,4\} \succ_4 \{1,2,4\} \sim_4 \{1,3,4\} \sim_4 \{2,3,4\} \succ_4 \{1,2,3,4\}$.

This hedonic game also satisfies mutuality (with respect to aversion to enemies): $2 \notin E_1$ and $1 \notin E_2$, $3 \notin E_1$ and $1 \notin E_3$, $4 \in E_1$ and $1 \in E_4$, $3 \in E_2$ and $2 \in E_3$, $4 \in E_2$ and $2 \in E_4$, and $4 \in E_3$ and $3 \in E_4$.

The partitions $\pi^1 = \{\{1,2\},\{3\},\{4\}\}$ and $\pi^2 = \{\{1,3\},\{2\},\{4\}\}$ are both FX-FE Nash stable. □

3.3. Subset Neutrality and Neutral Anonymity

Suksompong (2015) introduced conditions called *subset neutrality* and *neutral anonymity*. He showed that if a hedonic game satisfies the subset neutrality property or the neutral anonymity property, then there always exists an FX-FE Nash stable partition.

Definition 5. Subset Neutrality

A hedonic game $(N, \succcurlyeq)$ is ***subset neutral*** if there exists a function $\omega: 2^N \setminus \{\emptyset\} \to \mathbb{R}$ such that for each $i \in N$ and each $S, T \in \mathcal{C}_i^N$,

$$S \succcurlyeq_i T \Leftrightarrow \sum_{i \in \bar{S} \subseteq S} \omega(\bar{S}) \geq \sum_{i \in \bar{T} \subseteq T} \omega(\bar{T})$$

The function ω is defined on the set of all coalitions, and it assigns a numerical value to each coalition. For an agent, when comparing two coalitions containing her, she compares the sums of the numerical values of all sub-coalitions that contain her assigned by ω for the two coalitions and prefers the coalition with the larger sum.

We note that a hedonic game that is additively separable and symmetric satisfies subset neutrality, when we define the ω as follows: for each coalition $| S | > 2$ we let $\omega(S) = 0$. However, a hedonic game satisfying subset neutrality might not be additively separable.

A hedonic game $(N, \succcurlyeq)$ satisfies ***anonymity*** if for each $i \in N$ and each $S, T \in \mathcal{C}_i^N$ with $| H | = | T |$ we have $H \sim_i T$.

Definition 6. Neutral Anonymity

A hedonic game $(N, \succcurlyeq)$ satisfies ***neutral anonymity*** if there exists a function such that for each $i \in N$ and each $S, T \in \mathcal{C}_i^N$,

$$S \underset{i}{\succcurlyeq} T \text{ if and only if } f(|S|) \geq f(|T|).$$

Neutral anonymity means that: There exists a function that assigns a numerical value for each coalition size, and when an agent compares two coalitions containing her, she prefers the coalition whose size is assigned a larger numerical value than the other one. We note that a neutrally anonymous hedonic game is also subset neutral.

We now continue with a hedonic game that satisfies the subset neutrality and neutral anonymity properties, and hence it has an FX-FE Nash stable partition.

Example 4. A Hedonic Game Satisfying Subset Neutrality and Neutral Anonymity

Let $(N, \succcurlyeq)$ be a hedonic game with $N = \{1,2,3\}$ and a function $\omega: 2^N \setminus \{\emptyset\} \to \mathbb{R}$ is as follows: for each $i \in N$, $\omega((i)) = 0$, for each $i, j \in N$ with $i \neq j$, $\omega(\{i, j\}) = 2$, and $\omega(\{1,2,3\}) = 4$. Then, players' preferences derived from the function ω satisfying the subset neutrality property are as follows:

$\succcurlyeq_1: \{1,2,3\} \succ_1 \{1,2\} \sim_1 \{1,3\} \succ_1 \{1\}$,

$\succcurlyeq_2: \{1,2,3\} \succ_2 \{1,2\} \sim_2 \{2,3\} \succ_2 \{2\}$,

$\succcurlyeq_3: \{1,2,3\} \succ_3 \{1,3\} \sim_3 \{2,3\} \succ_3 \{3\}$.

This hedonic game also satisfies the neutral anonymity property. We define a function $f: \{1,2,3\} \to \mathbb{R}$ for each coalition size $t \in \{1,2,3\}$ as $f(t) = 2t - 2$, that is, $f(1) = 0$, $f(2) = 2$, and $f(3) = 4$. Now, for each $i \in N$ and each $S, T \in \mathcal{C}_i^N$ we have that $S \succcurlyeq_i T$ if and only if $f(|S|) \geq f(|T|)$, e.g., for agent 1, $\{1,2,3\} \succ_1 \{1,2\}$ since $f|\{1,2,3\}| = f(3) = 4$, $f|\{1,2\}| = f(2) = 2$ and $4 > 2$.

The partition $\pi^1 = \{\{1,2,3\}\}$ is FX-FE Nash stable. □

4. FREE EXIT - APPROVED ENTRY (FX-AE) NASH STABILITY (INDIVIDUAL STABILITY)

In this section, we introduce Nash stability under Free Exit and Approved Entry membership rights, Free Exit - Approved Entry (FX-AE) Nash stability and consider the sufficient conditions that guarantee the existence of FX-AE Nash stable partitions.

Definition 7. FX-AE Nash Stability

*Let $(N, \succcurlyeq)$ be a hedonic game. A partition $\pi \in \Pi(N, \succcurlyeq)$ is **FX-AE Nash stable** if there does not exist a pair (i, S), where $i \in N$ and $S \in (\pi \cup \{\emptyset\})$, such that*

(i) $S \cup \{i\} \succ_i \pi(i)$ *and*

(ii) *for all* $j \in S$, $S \cup \{i\} \succcurlyeq_j S$.

*If such a pair (i, S) exists, we say that agent i **FX-AE Nash blocks** π (by joining coalition S).*

We can redefine FX-AE Nash Stability by using the reachability notion as follows:

A partition $\pi \in \Pi(N, \succcurlyeq)$ is ***FX-AE Nash stable*** if there does not exist a pair (i, π'), consisting of an agent $i \in N$ and another partition $\pi' \in (\Pi(N, \succcurlyeq) \setminus \{\pi\})$, such that $\pi \overset{i}{\to} \pi'$ (π' is reachable from π via agent i), $\pi'(i) \succ_i \pi(i)$, and for all $j \in (\pi'(i) \setminus \{i\})$, $\pi'(j) \succcurlyeq_j \pi(j)$. If such a pair (i, π') exists, we say that agent i **FX-AE Nash blocks** π (by inducing π').

4.1. Common Ranking Property

The *common ranking property* was introduced by Farrell and Scotchmer (1988). The common ranking property requires that there is a linear order on the set of all coalitions which coincides with any agent's preference ordering over her coalitions. Suksompong (2015) proved that when a hedonic game satisfies the common ranking property, there exists an FX-AE Nash stable partition.

Definition 8. Common Ranking Property

A hedonic game $(N, \succcurlyeq)$ satisfies the ***common ranking property*** if there exists an ordering $\succcurlyeq$ over $2^N \setminus \{\emptyset\}$ such that for each $i \in N$ and each $S, T \in \mathcal{C}_i^N$, we have

$$S \underset{i}{\succcurlyeq} T \text{ if and only if } S \succcurlyeq T.$$

We now give an example of a hedonic game that satisfies the common ranking property and has an FX-AE Nash stable partition.

Example 5. A Hedonic Game Satisfying the Common Ranking Property

Let $(N, \succcurlyeq)$ where $N = \{1,2,3\}$ and the preferences of agents are as follows:

$\succcurlyeq_1: \{1,2\} \succ_1 \{1\} \sim_1 \{1,2,3\} \succ_1 \{1,3\}$,

$\succcurlyeq_2: \{1,2\} \succ_2 \{2\} \sim_2 \{1,2,3\} \succ_2 \{2,3\}$,

$\succcurlyeq_3: \{1,2,3\} \succ_3 \{1,3\} \sim_3 \{2,3\} \succ_3 \{3\}$.

This hedonic game satisfies the common ranking property with respect to the ordering $\succcurlyeq$, where [$\succcurlyeq: \{1,2\} \succ \{1\} \sim \{2\} \sim \{1,2,3\} \succ \{1,3\} \sim \{2,3\} \succ \{3\}$].

The partition $\pi = \{\{1,2\}, \{3\}\}$ is FX-AE Nash stable. □

4.2. Ordered Characteristics

Bogomolnaia and Jackson (2002) showed that if a hedonic game satisfies the *ordered characteristics* property, then there exists an FX-AE Nash stable partition. We will follow Bogomolnaia and Jackson (2002) to define the ordered characteristics property.

Let each coalition $S \subseteq N$ be described by a *characteristic* $c(S)$ that lies in $\{0,1,\dots,|S|\}$. Let each agent $i \in N$ has single-peaked preferences on $\{0,1,\dots,n\}$ with peaks denoted by p_i such that $p_i \geq 1$. Agents' preferences over coalitions correspond to the preference ranking of $c(S)$, that is, for all $i \in N$, all $S,T \in \mathcal{C}_i^N$, $S \succcurlyeq_i T$ if and only if $c(S) \succcurlyeq_i c(T)$.

Definition 9. Ordered Characteristics

A hedonic game $(N, \succcurlyeq)$ has ***ordered characteristics*** if agents' preferences over coalitions depend on single-peaked preferences over characteristics $c(S)$ where:

(i) If $c(S) < |S|$ then $c(S) = p_j$ for some $j \in S$, and

(ii) If $i \notin S$, $j \notin S$, and $p_i \geq p_j$, then $c(S \cup \{i\}) \geq c(S \cup \{j\})$. Moreover, if $c(S \cup \{i\}) > p_i$, then $c(S \cup \{i\}) = c(S \cup \{j\})$.

The first condition states that if a characteristic of a coalition is smaller than the size of that coalition, then the characteristic is the peak of some agent in that coalition. The first part of the second condition states that when comparing any two coalitions which differ by only one agent, the characteristics of these coalitions are ordered by the peaks of the agents who differ. The second part states that if the peak of the agent who has a higher peak than other agent is smaller than the characteristic of the coalition that contains her, then the characteristics of these two coalitions that differ by one agent are equal.

Bogomolnaia and Jackson (2002) noted that if in a hedonic game, agents' preferences are anonymous and single-peaked on the sizes of the coalitions to which they belong, then this hedonic game satisfies the ordered characteristics property.

The following hedonic game is taken from Bogomolnaia and Jackson (2002). It satisfies the ordered characteristics property and hence has an FX-AE Nash stable partition.

Example 6. A Hedonic Game Satisfying Ordered Characteristics Property

Let $(N, \succcurlyeq)$ be a hedonic game with $N = \{1,2,3,4\}$. Agents' preferences are anonymous and single-peaked over the sizes of coalition that they belong. For each coalition $S \subseteq N$ we have $c(S) = |S|$ and the peaks of the agents are as follows: $p_1 = 4$, $p_2 = 3$, and $p_3 = p_4 = 2$. Moreover, agents' preferences over sizes of coalitions are $[4 \succ_1 3 \succ_1 2 \succ_1 1]$, $[3 \succ_2 2 \succ_2 1 \succ_2 4]$, $[2 \succ_3 3 \succ_3 1 \succ_3 4]$, and $[2 \succ_4 3 \succ_4 1 \succ_4 4]$.

This hedonic game satisfies the ordered characteristics property, and the preferences of agents are as follows:

$\succcurlyeq_1: \{1,2,3,4\} \succ_1 \{1,2,3\} \sim_1 \{1,2,4\} \sim_1 \{1,3,4\} \succ_1 \{1,2\} \sim_1 \{1,3\} \sim_1 \{1,4\} \succ_1 \{1\}$,

$\succcurlyeq_2: \{1,2,3\} \sim_2 \{1,2,4\} \sim_2 \{2,3,4\} \succ_2 \{1,2\} \sim_2 \{2,3\} \sim_2 \{2,4\} \succ_2 \{2\} \succ_2 \{1,2,3,4\}$,

$\succcurlyeq_3: \{1,3\} \sim_3 \{2,3\} \sim_3 \{3,4\} \succ_3 \{1,2,3\} \sim_3 \{1,3,4\} \sim_3 \{2,3,4\} \succ_3 \{3\} \succ_3 \{1,2,3,4\}$,

$\succcurlyeq_4: \{1,4\} \sim_4 \{2,4\} \sim_4 \{3,4\} \succ_4 \{1,2,4\} \sim_4 \{1,3,4\} \sim_4 \{2,3,4\} \succ_4 \{4\} \succ_4 \{1,2,3,4\}$.

The partition $\pi = \{\{1,2\}), \{3,4\}\}$ is FX-AE Nash stable. □

5. APPROVED EXIT - FREE ENTRY (AX-FE) NASH STABILITY (CONTRACTUAL NASH STABILITY)

We now introduce Nash stability under Approved Exit and Free Entry membership rights, Approved Exit - Free Entry (AX-FE) Nash stability and consider the sufficient condition that guarantee the existence of AX-FE Nash stable partitions.

Definition 10. AX-FE Nash Stability

*Let $(N, \succcurlyeq)$ be a hedonic game. A partition $\pi \in \Pi(N, \succcurlyeq)$ is **AX-FE Nash stable** if there does not exist a pair (i, S), where $i \in N$ and $S \in (\pi \cup \{\emptyset\})$, such that*

(i) $S \cup \{i\} \succ_i \pi(i)$ and

(ii) for all $k \in (\pi(i) \backslash \{i\})$, $\pi(i) \backslash \{i\} \succcurlyeq_k \pi(i)$.

*If such a pair (i, S) exists, we say that agent i **AX-FE Nash blocks** π (by joining coalition S).*

We can redefine AX-FE Nash Stability by using the reachability notion as follows:

A partition $\pi \in \Pi(N, \succcurlyeq)$ is ***AX-FE Nash stable*** if there does not exist a pair (i, π'), consisting of an agent $i \in N$ and another partition $\pi' \in (\Pi(N, \succcurlyeq)\backslash\{\pi\})$, such that $\pi \overset{i}{\to} \pi'$ (π' is reachable from π via agent i), $\pi'(i) \succ_i \pi(i)$, and for all $k \in (\pi(i)\backslash\{i\})$, $\pi'(k) \succcurlyeq_k \pi(k)$. If such a pair (i, π') exists, we say that agent i **AX-FE Nash blocks** π (by inducing π').

5.1. Separable Preferences

The notion of *separable preferences* was introduced by Banerjee et al. (2001) and Bogomolnaia and Jackson (2002) in the context of hedonic games. Sung and Dimitrov (2007) proved that when a hedonic game satisfies separability and weak mutuality, then there exists an AX-FE Nash stable partition.

Definition 11. Separability

A hedonic game $(N, \succcurlyeq)$ is ***separable*** if for each $i \in N$, each $S \in \mathcal{C}_i^N$ and each $j \notin S$, we have $[S \cup \{j\} \succcurlyeq_i S \Leftrightarrow \{i,j\} \succcurlyeq_i \{i\}]$ *and* $[S \succcurlyeq_i S \cup \{j\} \Leftrightarrow \{i\} \succcurlyeq_i \{i,j\}]$.

A separable hedonic game $(N, \succcurlyeq)$ satisfies ***mutuality*** if for each $i, j \in N$, we have $[\{i,j\} \succcurlyeq_i \{i\} \Leftrightarrow \{i,j\} \succcurlyeq_j \{j\}]$ *and* $[\{i\} \succcurlyeq_i \{i,j\} \Leftrightarrow \{j\} \preccurlyeq_j \{i,j\}]$.

A separable hedonic game $(N, \succcurlyeq)$ satisfies ***weak mutuality*** if for each $i \in N$, if there exists $j \in N\backslash\{i\}$ such that $\{i,j\} \succcurlyeq_i \{i\}$, then there exists $k \in N\backslash\{i\}$ such that $\{i,k\} \succcurlyeq_k \{k\}$.

Separability means that for any agent $i \in N$, any coalition S containing agent i, and any other agent j not in S, i prefers to cooperate with j rather than staying alone if and only if i prefers $S \cup \{j\}$ to S. In the same way, i prefers to stay alone rather than cooperating with j if and only if i prefers S to $S \cup \{j\}$. A separable hedonic game satisfies mutuality if, for each pair of agents, one agent prefers to cooperate with the other to stay alone if and only if the other behaves in the same way, and one agent prefers to be alone to cooperate with the other if and only if the other behaves in the same way. A separable hedonic game satisfies weak mutuality if, for each agent, there exists an agent with whom she prefers to stay together rather than to be alone, then there exists another agent who prefers staying with her rather than being alone.

The following hedonic game satisfies separability and weak mutuality, and it has an AX-FE Nash stable partition.

Example 7. A Hedonic Game Satisfying the Separability and Weak Mutuality

Let $(N, \succcurlyeq)$ where $N = \{1,2,3\}$ and the preferences of agents are as follows:

$\succcurlyeq_1: \{1,2\} \succ_1 \{1,2,3\} \succ_1 \{1\} \succ_1 \{1,3\}$,

$\succcurlyeq_2: \{2,3\} \succ_2 \{1,2,3\} \succ_2 \{2\} \succ_2 \{1,2\}$,

$\succcurlyeq_3: \{1,3\} \succ_3 \{1,2,3\} \succ_3 \{3\} \succ_3 \{2,3\}$.

For agent 1, we have that $[\{1,2\} \succ_1 \{1\}$ and $\{1,2,3\} \succ_1 \{1,3\}]$, $[\{1\} \succ_1 \{1,3\}$ and $\{1,2\} \succ_1 \{1,2,3\}]$. For agent 2, we have that $[\{2,3\} \succ_2 \{2\}$ and $\{1,2,3\} \succ_2 \{1,2\}]$, $[\{2\} \succ_2 \{1,2\}$ and $\{2,3\} \succ_2 \{1,2,3\}]$. For agent 3, we have $[\{1,3\} \succ_3 \{3\}$ and $\{1,2,3\} \succ_3 \{2,3\}]$, $[\{3\} \succ_3 \{2,3\}$ and $\{1,3\} \succ_3 \{1,2,3\}]$. So, this hedonic game satisfies separability.

This hedonic game also satisfies weak mutuality since we have that $\{1,2\} \succ_1 \{1\}$, $\{1,3\} \succ_3 \{3\}$, and $\{2,3\} \succ_2 \{2\}$.

The partition $\pi = \{\{1,2,3\}\}$ is AX-FE Nash stable. □

6. APPROVED EXIT - APPROVED ENTRY (AX-AE) NASH STABILITY (CONTRACTUAL INDIVIDUAL STABILITY)

In this section, we introduce Nash stability under Approved Exit and Approved Entry membership rights, Approved Exit - Approved Entry (AX-AE) Nash stability. We describe the proof techniques offered by Bogomolnaia and Jackson (2002) and Ballester (2004) to show the existence of an AX-AE Nash stable partition for every hedonic game.

Definition 12. AX-AE Nash Stability

*Let $(N, \succcurlyeq)$ be a hedonic game. A partition $\pi \in \Pi(N, \succcurlyeq)$ is **AX-AE Nash stable** if there does not exist a pair (i, S), where $i \in N$ and $S \in (\pi \cup \{\emptyset\})$, such that*

(i) $S \cup \{i\} \succ_i \pi(i)$ *and*

(ii) $for\ all\ j\ \in S,\ S\ \cup\{i\}\ \succcurlyeq_j\ S$, and

(iii) *for all* $k\ \in (\pi(i)\backslash\{i\}),\ \pi(i)\backslash\{i\}\ \succcurlyeq_k\ \pi(i)$.

*If such a pair (i, S) exists, we say that agent i **AX-AE Nash blocks** π (by joining coalition S).*

We can redefine AX-AE Nash Stability by using the reachability notion as follows:

A partition $\pi \in \Pi(N, \succcurlyeq)$ is ***AX-AE Nash stable*** if there does not exist a pair (i, π'), consisting of an agent $i \in N$ and another partition $\pi' \in (\Pi(N, \succcurlyeq)\backslash\{\pi\})$, such that $\pi \overset{i}{\rightarrow} \pi'$ (π' is reachable from π via agent i), $\pi'(i) \succ_i \pi(i)$, and for all $l \in (N\backslash\{i\})$, $\pi'(l) \succcurlyeq_l \pi(l)$. If such a pair (i, π') exists, we say that agent i **AX-AE Nash blocks** π (by inducing π').

Bogomolnaia and Jackson (2002) proved that every hedonic game has an AX-AE Nash stable partition by showing that any Pareto optimal partition is AX-AE Nash stable. They also constructed an algorithm and showed that for every hedonic game with agents having strict preferences, the result of the algorithm is an AX-AE Nash stable partition which is also Pareto optimal and individually rational.

The algorithm works as follows: Let $(N, \succ)$ be a hedonic game such that $N = \{1,2, \ldots, n\}$ and each agent has strict preferences. Consider the first agent, that is, agent 1, and call her i_1. We choose the coalition that agent i_1 prefers the most out of all the individually rational coalitions of N, that is, the best coalition for agent i_1 in the set $\{S \subseteq N \mid i_1 \in S$ and for each $j \in S, S \succcurlyeq_j \{j\}\}$ is chosen, and we call it S_1. If $N\backslash S_1 \neq \emptyset$, we continue with $N_2 = N\backslash S_1$. Consider the first agent in N_2 and call her i_2. We choose the coalition that agent i_2 prefers the most out of all the individually rational coalitions of N_2, that is, the best coalition for agent i_2 among the coalitions of the set $\{S \subseteq N_2 \mid i_2 \in S$ and for each $j \in S, S \succcurlyeq_j \{j\}\}$ is chosen, and we call it S_2. If $N\backslash(S_1 \cup S_2) \neq \emptyset$, we continue with $N_3 = N\backslash(S_1 \cup S_2)$. The algorithm continues like this and since we have a finite set of agents, there exists a positive integer K such that $N\backslash(S_1 \cup \ldots \cup S_K) = \emptyset$. That is, the algorithm terminates, and the resulting partition is $\pi^* = \{S_1, S_2, \ldots, S_K\}$ that consists of all coalitions that are chosen in the algorithm. It is clear that π^* is individually rational, and Bogomolnaia and Jackson (2002) showed that π^* is Pareto optimal and AX-AE Nash stable.

Ballester (2004) also proved that every hedonic game has an AX-AE Nash stable partition by introducing a different approach. Ballester's approach is as follows: Choose a partition randomly. If it is AX-AE Nash stable, then we are done. If not, then there exists an agent who AX-AE Nash blocks the partition by inducing another partition. If the induced partition is AX-AE Nash stable, then we are done. If not, then there exists an agent who AX-AE Nash blocks the partition by inducing a new partition, and so on. Since the preferences are transitive, this procedure will never be cyclic, that is, an already induced partition will not be reached again. When passing from one partition to another by an AX-AE Nash blocking of an agent, at least one agent is made strictly better off without hurting other agents. Since the set of agents for a hedonic game is finite, we have a finite number of coalitions and of partitions. So, an agent can be made finite times strictly better off. Hence, this process stops, and an AX-AE Nash stable partition is obtained.

7. CONCLUSION AND FURTHER COMMENTS

In this study, we focused on hedonic coalition formation games. Hedonic coalition formation games are used to model and analyze economic, social, and political instances where agents form coalitions. A hedonic coalition formation game consists of a finite set of agents and a preference list of agents such that each agent has preferences over all coalitions that contain her. An outcome of a hedonic coalition formation game is a collection of coalitions that are pairwise disjoint, and their union is equal to the set of agents, and it is called a partition. We considered Nash stable partitions under different membership rights. We revisited the (sufficient) conditions that guarantee the existence of a Nash stable partition for each membership rights and provided examples of hedonic coalition formation games satisfying these sufficient conditions.

We note that there are sufficient conditions in the literature for the existence of a Nash stable partition under different membership rights. The existence of a necessary and sufficient condition for the existence of a Nash stable partition under different membership rights is not yet known and it is still an open research question.

Conditions that we reconsidered in this survey are imposed either on the preferences of the agents or on the preference profiles. Pápai (2007) studied hedonic coalition formation models with preferences over permissible coalitions and presented sufficient conditions that guarantee the existence of an FX-FE Nash stable and an FX-AE Nash stable partitions in these models. The existence of an AX-FE Nash stable partition or the existence of an AX-AE Nash stable partition in hedonic coalition formation models with preferences over permissible coalitions has yet to be investigated and this is still an open research question.

In the hedonic coalition formation model that we considered in this study, the agents are myopic, they are not farsighted, that is they are unable to look many steps ahead and consider credible outcomes. We refer readers to Diamantoudi and Xue (2003) about how stability notions are analyzed in the hedonic coalition formation models where agents are endowed with foresight.

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