Mathematical Modelling and Numerical Simulation with Applications, 2021, 1(1), 44–55

https://www.mmnsa.org
ISSN Online: 2791-8564 / Open Access
https://doi.org/10.53391/mmnsa.2021.01.005

RESEARCH PAPER

Stability analysis of an incommensurate fractional-order SIR model

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Abstract

In this paper, a fractional-order generalization of the susceptible-infected-recovered (SIR) epidemic model for predicting the spread of an infectious disease is presented. Also, an incommensurate fractional-order differential equations system involving the Caputo meaning fractional derivative is used. The equilibria are calculated and their stability conditions are investigated. Finally, numerical simulations are presented to illustrate the obtained theoretical results.

Key words: *SIR* mathematical model; incommensurate order differential equation; fractional-derivative; stability analysis
AMS 2020 Classification: 34A08; 34D20; 34K60; 92C50; 92D30

1 Introduction

The topic of fractional calculus (FC) has gained considerable popularity and importance in the last three decades, mainly because of its wide variety of applications in science and engineering. Also, it has been found that many systems can be described with fractional differential equations in many interdisciplinary fields [1]. Fractional-order differential equation (FODE) models have advantages over classical ordinary differential equation (ODE) and/or delayed differential equation models because integer derivatives are used to obtain information about only local properties of a state, while fractional derivatives describe the entire space. In other words, in FODE models, the next precise location for a physical phenomenon depends not only on the current situation, but also on all historical situations. Thus, these models not only give more realistic biological models involving memory but also expand the stability region of states [2]. Fractional-order systems (FOSs) are can be considered in two parts, as commensurate FOS (CFOS) and incommensurate FOS (IFOS) according to the derivative orders in the system. CFOS can be considered as a special case of derivative orders in IFOS [3]. Given the fact that the stability theorem of fractional differential equations favors stability analysis and controller synthesis, this motivates us to adopt stability criteria for the field of incommensurate fractional-order nonlinear systems and give sufficient conditions for determining stability [4]. Therefore, modeling of biological dynamics with IFOS is more comprehensive in terms of predicting the behavior of the system [5]. Furthermore, theorems of existence, uniqueness and dependence upon initial conditions according to some special conditions of IFOS are given in [6, 7].
There are many recent studies in the literature on the stability of IFOS [8, 9, 10, 11]. In addition, modeling and stability analysis of biological systems by IFOS has been frequently discussed in the literature recently [12, 13, 5, 7, 14] and CFOS [15, 16, 17, 18, 19, 20, 21, 22, 23, 24] .
In the field of epidemiology, many schemes have been developed to mathematically model various infectious epidemics. Compartment models such as SIR modeling, which divide communities into certain main classes, are the most widely used models. The interactions between these classes are mainly determined by certain pre-mathematical formulas. The classical SIR epidemiological model was first introduced by Kermack and McKendrick in 1927. This ordinary differential equation system (ODES) models the

➤ **Received:** 17.09.2021 ➤ **Revised:** 28.09.2021 ➤ **Accepted:** 29.09.2021 ➤ **Published:** 30.09.2021

spread of an epidemic in a population. More recently, there has been increased interest in extending SIR models through the inclusion of fractional derivatives [27]. Modified SIR mathematical modeling through CFOS are in recent years analyzed in [28, 29, 30]. In here, the time-dependent changes in sizes of susceptible, infected and recovered individuals in a population in case of an infectious disease were investigated by mathematically modeling with IFOS. An innovation has been presented to the literature in terms of the use of IFOS in the model. In addition, the results were supported by numerical studies.
The remainder of the article is arranged as follows:

- In Section 2, the existence of equilibrium points of the proposed model and their stabilities are analyzed.
- In Section 3, the mathematical formulation of the proposed SIR model is presented. Furthermore, the threshold parameter is presented.
- Section 4 proposes the stability conditions of the mentioned biological system.
- Section 5 backs up the qualitative analysis results of the proposed IFOS. In this respect, numerical simulations are performed.
- The article ends in Section 6 with some concluding remarks.

2 Preliminaries and definitions

In here, it is given some basic definitions and notations with respect to follows: FODE with Caputo derivatives and locally asymptotically stability (LAS) of the equilibrium point of an n-dimensional FOS, respectively.

Definition 1 *According to the definition of Caputo sense, the fractional derivative of the function $f(t)$ is defined as*

$$ {}^{C}D_{t}^{\alpha}\left(f(t)\right)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}(t-x)^{n-\alpha-1}\left(\frac{d}{dx}\right)^{n}f(x)\,dx,\ n-1<\alpha\leq n, \tag{1} $$

where $\Gamma(.)$ is the Gamma function, which is described by $\Gamma(x)=\int_{0}^{\infty}t^{x-1}e^{-t}dt$, $f\colon(0,+\infty)\to\mathbb{R}$ and $\alpha>0$ [31].

The Caputo fractional order sense is used in this study.

Remark 1 *The nonlinear FOS can be defined as following*

$$ \frac{d^{\overline{\alpha}}X(t)}{dt^{\overline{\alpha}}}=F(t,X(t)), \tag{2} $$

where it is considered initial conditions by $X(0)=X_0$, the state vectors by $X(t)=[x_1(t),x_2(t),\ldots,x_n(t)]^{T}\in\mathbb{R}^{n}$, the functions by $F=[f_1,f_2,\ldots,f_n]^{T}\in\mathbb{R}^{n}$, $f_i\colon[0,+\infty)\,x\mathbb{R}^{n}\to\mathbb{R}$, $(i=1,2,\ldots,n)$ and the derivative-orders by $\overline{\alpha}=[\alpha_1,\alpha_2,\ldots,\alpha_n]^{T}$ such that $\frac{d^{\overline{\alpha}}X(t)}{dt^{\overline{\alpha}}}=\left[\frac{d^{\alpha_1}x_1(t)}{dt^{\alpha_1}},\frac{d^{\alpha_2}x_2(t)}{dt^{\alpha_2}},\ldots,\frac{d^{\alpha_n}x_n(t)}{dt^{\alpha_n}}\right]^{T}$ [32].

For the rest of the article, α_i is in $(0,1]$.

Definition 2 *For system (2), autonomous IFOS can be presented as*

$$ \frac{d^{\overline{\alpha}}X(t)}{dt^{\overline{\alpha}}}=F(X(t)),\ X(0)=X_0. \tag{3} $$

Also, the equilibrium point of system (3) is found from $F\left(\overline{X}\right)=0$ for $\overline{X}=(\overline{x_1},\overline{x_2},\ldots,\overline{x_n})$ [6].

Lemma 1 *Eigenvalues λ_i for $i=1,2,\ldots,m(\alpha_1+\alpha_2+\cdots+\alpha_n)$ of system (3) are obtained from*

$$ det\left(diag\left(\lambda^{m\alpha_1},\lambda^{m\alpha_2},\ldots,\lambda^{m\alpha_n}\right)-J\left(\overline{X}\right)\right)=0, \tag{4} $$

where m is the smallest of the common multiples of the denominators of rational numbers $\alpha_1,\alpha_2,\ldots,\alpha_n$ and $J\left(\overline{X}\right)=\frac{\partial F}{\partial X}\Big|_{X=\overline{X}}$. If all eigenvalues λ_i obtained from equation (4) satisfy

$$ |arg(\lambda_i)|>\frac{\pi}{2m}, \tag{5} $$

then $\overline{X}$ is LAS for system (3) [33, 34].

As a result, Figure 1 shows the stability conditions of the incommensurate order SIR model given in (3), where $\alpha_1\neq\alpha_2\neq\cdots\neq\alpha_n<1$ and λ_i for $i=1,2,\ldots,m(\alpha_1+\alpha_2+\cdots+\alpha_n)$.

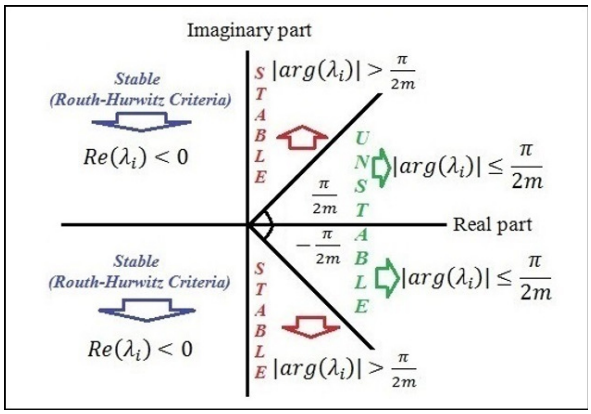

Figure 1. Regions of stability and instability of the equilibrium point in terms of the roots of the characteristic equation of the system (3) [5].

3 The SIR model through IFOS

We consider a *SIR* epidemic disease model. Define the following dependent-time t:

Table 1. State variables and their meanings

State Variable	Meaning
$S(t)$	The susceptible individuals at the t-time
$I(t)$	The infected symptomatic individuals at the t-time
$R(t)$	The recovered individuals at the t-time

Therefore, the dynamics is governed by a system of three FODE as follows:

$$\begin{aligned} \frac{d^{\alpha_1} S(t)}{dt^{\alpha_1}} &= \Lambda + \nu R - \eta IS - (\mu + b) S, \\ \frac{d^{\alpha_2} I(t)}{dt^{\alpha_2}} &= \eta IS - (\gamma + d + b) I, \\ \frac{d^{\alpha_3} R(t)}{dt^{\alpha_3}} &= \mu S + \gamma I - (\nu + b) R, \end{aligned} \tag{6}$$

where $t \geq 0$, $\alpha_i \in (0,1]$ for $i = 1,2,3$. Also, the initial conditions are $S(t_0) = S_0 > 0$, $I(t_0) = I_0 > 0$ and $R(t_0) = R_0 > 0$ for $t > t_0$. Restrictions are imposed on the parameters to ensure that solutions are nonnegative. Therefore, the following conditions hold

$$\Lambda,\ \nu,\ \eta,\ \mu,\ b,\ \gamma,\ d > 0. \tag{7}$$

In Table 2, it is illustrated parameters with their meaning.

Table 2. Parameters and their meanings in the proposed model

Parameter	Meaning
Λ	The constant birth number in the overall population
ν	The immunity loss rate of recovered individuals
η	The contact number, the average number of successful contacts resulting in infection and made by one infected individual
μ	Rate of the vaccinated susceptible individuals
b	The death rate due to the different conditions other than the disease for the overall population.
γ	Recovery rate of the infected individual
d	Average fatality rate of the infected individual due to infectious disease

Therefore, Figure 2 is obtained from system (6).

Definition 3 *The baseline reproduction number, often denoted as $\mathcal{R}_0$, describes the average number of secondary infections caused by an infected individual in a fully susceptible population. This number indicates whether the infection will spread to the population or not [35].*

For the proposed model, it is described this parameter as

$$\mathcal{R}_0 = \frac{\eta}{(\gamma + d + b)} \frac{\Lambda}{b} \frac{(b + \nu)}{(b + \nu + \mu)}. \tag{8}$$

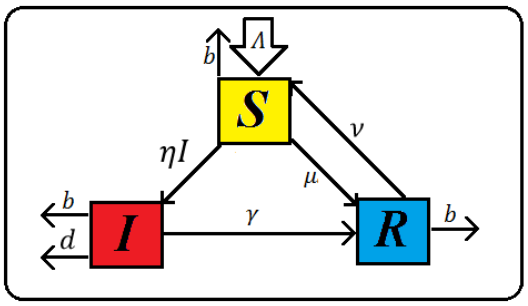

Figure 2. The movement of the individuals between compartments in the proposed model

It is clear that

$$\mathcal{R}_0 > 0, \tag{9}$$

due to the inequalities in (7).

4 Stability analysis

Proposition 1 *Let us consider the equations,* $\frac{d^{\alpha_1}S(t)}{dt^{\alpha_1}} = 0, \frac{d^{\alpha_2}I(t)}{dt^{\alpha_2}} = 0, \frac{d^{\alpha_3}R(t)}{dt^{\alpha_3}} = 0,$ *for equilibrium points. The proposed model has two types of the equilibrium points. These are disease free equilibrium point* $E_0\left(\frac{\Lambda}{b}\frac{(b+\nu)}{(b+\nu+\mu)}, 0, \frac{\Lambda}{b}\frac{\mu}{(b+\nu+\mu)}\right)$ *and the endemic equilibrium* $E_1(S^*, I^*, R^*)$ *when*

$$\mathcal{R}_0 > 1. \tag{10}$$

In here, it is

$$\begin{cases} S^* = \frac{(\gamma+d+b)}{\eta}, \\ I^* = \frac{S^*(\mathcal{R}_0-1)(b+\nu+\mu)}{\left(\gamma+d+b+\nu\frac{d}{b}+\nu\right)}, \\ R^* = \left(\frac{S^*(\mathcal{R}_0-1)(b+\nu+\mu)}{\left(\gamma+d+b+\nu\frac{d}{b}+\nu\right)} + \frac{S^*}{\gamma}\mu\right)\frac{\gamma}{(b+\nu)}. \end{cases} \tag{11}$$

Proposition 2 *Considering the proposed model in* (6)*, there are follows.*

i. *Let* $\alpha_1 = \alpha_2 = \alpha_3 \leq 1$*. For CFOS, it is satisfied the followings:*

a) *If* $\mathcal{R}_0 < 1$*, the equilibrium point* E_0*, namely trivial disease-free equilibrium, is LAS.*
b) *If*

$$((\eta I^* + (\mu + b)) + (\nu + b))(\eta I^*((\nu + b) + \eta S^*) + b(\mu + \nu + b)) - \eta I^*((\nu + b)\eta S^* + \gamma\nu) > 0, \tag{12}$$

then the equilibrium point E_1*, existing biologically meaning when* $\mathcal{R}_0 > 1$*, is LAS.*

ii. *For IFOS in system* (6)*, where* $\alpha_1 \neq \alpha_2 \neq \alpha_3 < 1$*, it is satisfied the followings:*

a) *If* $\mathcal{R}_0 < 1$ *and all roots* λ_i *for* $i = 1, 2, \ldots, m(\alpha_1 + \alpha_3)$ *founded from the equation*

$$\lambda^{m(\alpha_1+\alpha_3)} + \lambda^{m\alpha_1}(\nu + b) + \lambda^{m\alpha_3}(\mu + b) + b(\mu + \nu + b) = 0$$

satisfy Routh-Hurwitz stability criteria [36] or the condition $|arg(\lambda_i)| > \frac{1}{m}\frac{\pi}{2}$ *[37] as seen inequalities* (5)*, then the equilibrium point* E_0 *is LAS.*
b) *Let* $\mathcal{R}_0 > 1$*. If all roots* λ_i *for* $i = 1, 2, \ldots, m(\alpha_1 + \alpha_2 + \alpha_3)$ *founded from the equation*
$\lambda^{m(\alpha_1+\alpha_2+\alpha_3)} + \lambda^{m(\alpha_1+\alpha_2)}(\nu + b) + \lambda^{m(\alpha_2+\alpha_3)}(\eta I^* + (\mu + b)) + \lambda^{m\alpha_2}(\eta I^*(\nu + b) + b(\mu + \nu + b))$
$+ \lambda^{m\alpha_3}(\gamma + d + b)\eta I^* + \eta I^*((\gamma + d + b)(\nu + b) + \gamma\nu) = 0$ *satisfy Routh-Hurwitz stability criteria or the condition* $|arg(\lambda_i)| > \frac{1}{m}\frac{\pi}{2}$*, then the equilibrium point* E_1 *is LAS.*

Proof By the equations in (6), the Jacobian matrix evaluated at the equilibrium point $E_i\left(\bar{S},\ \bar{I}, \bar{R}\right)$ for $i = 0, 1$ is

$$J\left(E_i\right) = \begin{pmatrix} -\left(\eta\bar{I} + (\mu + b)\right) & -\eta\bar{S} & \nu \\ \eta\bar{I} & \left(\eta\bar{S} - (\gamma + d + b)\right) & 0 \\ \mu & \gamma & -(\nu + b) \end{pmatrix}. \tag{13}$$

i. The system in (6) translates to CFOS, when $0 < \alpha_1 = \alpha_2 = \alpha_3 \leq 1$.

a) For E_0, the eigenvalues are obtained by considering the equation $Det\left(J_{(S,I,\ R)=E_0\left(\frac{\Lambda}{b}\frac{(b+\nu)}{(b+\nu+\mu)}, 0, \frac{\Lambda}{b}\frac{\mu}{(b+\nu+\mu)}\right)} - \lambda I_{3x3}\right) = 0$. Accordingly, it is

$$(\lambda - (\gamma + d + b)(\mathcal{R}_0 - 1))\left(\lambda^2 + \lambda((\nu + b) + (\mu + b)) + (\mu + \nu + b)\right) = 0. \tag{14}$$

Therefore, the eigenvalues obtained from equation in (14) are determined as followings:

$$\lambda_1 = (\gamma + d + b)(\mathcal{R}_0 - 1), \tag{15}$$

and λ_2 and λ_3 are found by solving the equation

$$\lambda^2 + \lambda((\nu + b) + (\mu + b)) + (\mu + \nu + b). \tag{16}$$

It can be observed that $((\nu + b) + (\mu + b)) > 0$ and $(\mu + \nu + b) > 0$, due to inequalities in (7). The LAS conditions for E_0 are provided for the eigenvalues λ_2 and λ_3. Thus, it is sufficient to examine the sign of λ_1. If

$$\mathcal{R}_0 < 1, \tag{17}$$

then λ_1 is a negative real number due to inequalities (7). Routh-Hurwitz stability conditions are satisfied. In this case, E_0 is LAS.

b) Let $\mathcal{R}_0 > 1$. There is positive equilibrium point. Charasteristic equation obtained from $Det\left(J_{(S,I,\ R)=E_1(S^*,I^*,R^*)} - \lambda I_{3x3}\right) = 0$ for the equilibrium point E_1 is founded as

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{18}$$

where

$$a_1 = \left(\eta I^* + (\mu + b) + (\nu + b)\right), a_2 = \left(\eta I^*\left((\nu + b) + \eta S^*\right) + b(\mu + \nu + b)\right), a_3 = \eta I^*\left((\nu + b)\eta S^* + \gamma_I\nu\right). \tag{19}$$

Let us consider that Routh-Hurwitz stability criteria. It is already clear that $a_1, a_3 > 0$ due to inequalities in (7) and (9). In addition, we have

$$a_1a_2 - a_3 = \left(\left(\eta I^* + (\mu + b)\right) + (\nu + b)\right)\left(\eta I^*\left((\nu + b) + \eta S^*\right) + b(\mu + \nu + b)\right) - \eta I^*\left((\nu + b)\eta S^* + \gamma\nu\right).$$

If

$$\left(\left(\eta I^* + (\mu + b)\right) + (\nu + b)\right)\left(\eta I^*\left((\nu + b) + \eta S^*\right) + b(\mu + \nu + b)\right) - \eta I^*\left((\nu + b)\eta S^* + \gamma\nu\right) > 0, \tag{20}$$

then $a_1a_2 - a_3 > 0$. Hence, E_1 is LAS when inequality in (20) is satisfied.

ii. In case of $0 < \alpha_1 \neq \alpha_2 \neq \alpha_3 < 1$, we have IFOS of (6). In this sense, the determinant found by the equation

$$det\left(diag\left(\lambda^{m\alpha_1},\ \lambda^{m\alpha_2}, \lambda^{m\alpha_3}\right) - J_{(x_1,\ x_2,\ldots,\ x_n)=(\overline{x_1},\overline{x_2},\ldots,\overline{x_n})}\right) = 0 \tag{21}$$

is

$$\begin{vmatrix} \lambda^{m\alpha_1} + \left(\eta\bar{I} + (\mu + b)\right) & -\eta\bar{S} & \nu \\ \eta\bar{I} & \lambda^{m\alpha_2} - \left(\eta\bar{S} - (\gamma + d + b)\right) & 0 \\ \mu & \gamma_I & \lambda^{m\alpha_3} + (\nu + b) \end{vmatrix} = 0. \tag{22}$$

a) Firstly, if the determinant in (22) evaluates in the point $E_0\left(\frac{\Lambda}{b}\frac{(b+\nu)}{(b+\nu+\mu)}, 0, \frac{\mu}{(\nu+b)}\frac{\Lambda}{b}\frac{(b+\nu)}{(b+\nu+\mu)}\right)$ or $E_0\left(\frac{(\gamma+d+b)}{\eta}\mathcal{R}_0, 0, \frac{\mu}{(\nu+b)}\frac{(\gamma+d+b)}{\eta}\mathcal{R}_0\right)$ with respect to (8), then it is achieved the equations:

$$\lambda^{m\alpha_2} - (\gamma + d + b)(\mathcal{R}_0 - 1) = 0, \tag{23}$$

and

$$\left(\lambda^{m\alpha_1} + (\mu + b)\right)\left(\lambda^{m\alpha_3} + (\nu + b)\right) - \mu\nu = 0. \tag{24}$$

These equations are examined as the followings: Taking into consideration the equation in (23), it is found that $\lambda^{m\alpha_2} = (\gamma + d + b)(\mathcal{R}_0 - 1)$. If

$$\mathcal{R}_0 < 1, \tag{25}$$

then $\lambda^{m\alpha_2}$ is a negative real number due to inequalities in (7). Otherwise, at least one root of (23) would be a positive real number, in which case the equilibrium point E_0 would be unstable. By De-Moivre formulas, we have $\lambda^{m\alpha_2} = \overbrace{(\gamma + d + b)(1 - \mathcal{R}_0)}^{>0 \text{ due to } (7),(25)} cis\pi$, and so, $\lambda_k = \left[(\gamma + d + b)(1 - \mathcal{R}_0)\right]^{\frac{1}{m\alpha_2}} cis\left(\frac{\pi + 2k\pi}{m\alpha_2}\right)$ for $k = 0, 1, 2, \ldots, (m\alpha_2 - 1)$, such that $cis\pi = \cos\pi + i\sin\pi$, $i = \sqrt{-1}$. Also, we have

$$\begin{cases} |arg(\lambda_0)| = \dfrac{\pi}{m\alpha_2}, \\ |arg(\lambda_1)| = \dfrac{3\pi}{m\alpha_2}, \\ \vdots \\ \left|arg\left(\lambda_{(m\alpha_2-1)}\right)\right| = \dfrac{(2m\alpha_2 - 1)\pi}{m\alpha_2}. \end{cases} \tag{26}$$

Considering the conditions $|arg(\lambda)| > \frac{\pi}{2m}$ for the stability of the equilibrium point, the stability condition for E_0 is given as $\frac{\pi}{m\alpha_2}, \frac{3\pi}{m\alpha_2}, \ldots, \frac{(2m\alpha_2-1)\pi}{m\alpha_2} > \frac{\pi}{2m}$, and so,

$$\begin{cases} \alpha_2 < 2, \\ \alpha_2 < 6, \\ \vdots \\ \alpha_2 < 2(2m\alpha_2 - 1). \end{cases} \tag{27}$$

Inequalities in (27) have been always provided since the derivative-orders $0 < \alpha_1, \alpha_2, \alpha_3 \leq 1$ in (6) are already satisfied. On the other hand, we have considered the equation (24)). If this equation is arranged, then

$$\lambda^{m(\alpha_1+\alpha_3)} + \lambda^{m\alpha_1}(\nu + b) + \lambda^{m\alpha_3}(\mu + b) + b(\mu + \nu + b) = 0 \tag{28}$$

is obtained. If the eigenvalues, which are the roots of equation (28), satisfy Routh-Hurwitz stability condition or the conditions $|arg(\lambda_i)| > \frac{\pi}{2m}$ for $i = 1, 2, \ldots, m(\alpha_1 + \alpha_3)$, then E_0 is LAS.

b) Let $\mathcal{R}_0 > 1$. In this case, the equilibrium point E_1 emerges as positive definite. By calculating the determinant (22) at this equilibrium point, it is obtained the following characteristic equation

$$\begin{aligned} &\lambda^{m(\alpha_1+\alpha_2+\alpha_3)} + \lambda^{m(\alpha_1+\alpha_2)}(\nu + b) + \lambda^{m(\alpha_2+\alpha_3)}(\eta I^* + (\mu + b)) + \\ &\lambda^{m\alpha_2}(\eta I^*(\nu + b) + b(\mu + \nu + b)) + \\ &\lambda^{m\alpha_3}(\gamma + d + b)\eta I^* + \eta I^*((\gamma + d + b)(\nu + b) + \gamma\nu) = 0. \end{aligned} \tag{29}$$

When the signs of the terms of the last equation are examined according to Descartes' sign rule [38], it is clear that the equation does not have a positive real root. This does not disturb the stability of the equilibrium point. Therefore, if the eigenvalues λ_i for $i = 1, 2, \ldots, m(\alpha_1 + \alpha_2 + \alpha_3)$, which are the roots of the equation (29), satisfy Routh-Hurwitz stability condition or the conditions $|arg(\lambda_i)| > \frac{1}{m}\frac{\pi}{2}$, the equilibrium point E_1 is LAS.

Therefore, the proof is completed. ■
As a result, it can be reached to Table 3.

Corollary 1 *Let us consider Table 3. If $\mathcal{R}_0 < 1$ and some additional conditions are satisfied, then the equilibrium point E_0, always existing, is LAS. However, the equilibrium point E_1 biologically exists when $\mathcal{R}_0 > 1$. In this context, it can be said the followings:*

i. In case the unexistence of E_1, E_0 can be a stable equilibrium point,
ii. In case the unstability of E_0, where $\mathcal{R}_0 > 1$, E_1 exists.

Therefore, these two points cannot be stable under the same conditions.

Table 3. The existence conditions for the equilibrium points of system (6) and the stability conditions of these points according to different states of its derivative orders

Equilibrium Point	The existence condition	Derivative-orders	Stability conditions
$E_0\left(\frac{\Lambda}{b}\frac{(b+\nu)}{(b+\nu+\mu)}, 0, \frac{\Lambda}{b}\frac{\mu}{(b+\nu+\mu)}\right)$	Always	$\alpha_1=\alpha_2=\alpha_3\leq 1$	If $\mathcal{R}_0 < 1$
$E_0\left(\frac{\Lambda}{b}\frac{(b+\nu)}{(b+\nu+\mu)}, 0, \frac{\Lambda}{b}\frac{\mu}{(b+\nu+\mu)}\right)$	Always	$\alpha_1\neq\alpha_2\neq\alpha_3$, $\alpha_1, \alpha_2, \alpha_3 \in (0,1)$	If $\mathcal{R}_0 < 1$ and all roots λ_i for $i = 1, 2, \ldots, m(\alpha_1+\alpha_3)$ founded from the equation $\lambda^{m(\alpha_1+\alpha_3)} + \lambda^{m\alpha_1}(\nu+b) + \lambda^{m\alpha_3}(\mu+b) + b(\mu+\nu+b) = 0$ satisfy Routh-Hurwitz stability criteria or the condition $\lvert arg(\lambda_i)\rvert > \frac{1}{m}\frac{\pi}{2}$.
$E_1(S^*, I^*, R^*)$	$\mathcal{R}_0 > 1$	$\alpha_1=\alpha_2=\alpha_3\leq 1$	$((\eta I^* + (\mu+b)) + (\nu+b))(\eta I^*((\nu+b) + \eta S^*) + b(\mu+\nu+b)) - \eta I^*((\nu+b)\eta S^* + \gamma\nu) > 0$,
$E_1(S^*, I^*, R^*)$	$\mathcal{R}_0 > 1$	$\alpha_1\neq\alpha_2\neq\alpha_3$, $\alpha_1, \alpha_2, \alpha_3 \in (0,1)$	If all roots λ_i for $i = 1, 2, \ldots, m(\alpha_1+\alpha_2+\alpha_3)$ founded from the equation $\lambda^{m(\alpha_1+\alpha_2+\alpha_3)} + \lambda^{m(\alpha_1+\alpha_2)}(\nu+b) + \lambda^{m(\alpha_2+\alpha_3)}(\eta I^* + (\mu+b)) + \lambda^{m\alpha_2}(\eta I^*(\nu+b) + b(\mu+\nu+b)) + \lambda^{m\alpha_3}(\gamma+d+b)\eta I^* + \eta I^*((\gamma+d+b)(\nu+b) + \gamma\nu) = 0$ satisfy Routh-Hurwitz stability criteria or the condition $\lvert arg(\lambda_i)\rvert > \frac{1}{m}\frac{\pi}{2}$,
where $\mathcal{R}_0$ is in (8) and the components S^*, I^* and R^* of E_1 are in (11).			

5 Numerical results

To highlight the stability analysis results of this work using the proposed model for both CFOS and IFOS, two numerical examples are investigated. To do this, it is examined the behavior of the solutions of the model by valuing the parameters. It has been used Matlab R2012b. The parameter values are given in Table 4.

Table 4. The considered values of the parameters

Parameter	Value[1]	Value[2]	Unit
Λ	100	1000	individuals
ν	0.001	0.01	day $^{-1}$
η	0.0001	0.0001	day $^{-1}$
μ	0.045	0.05	day $^{-1}$
b	0.0032	0.15	day $^{-1}$
γ	0.25	0.25	day $^{-1}$
d	0.022	0.022	day $^{-1}$
α_1	0.9	0.8	Rational number
α_2	0.9	0.6	Rational number
α_3	0.9	0.4	Rational number
Value[1] is used in numerical study 1. **Value[2]** is used in numerical study 2.			

Numerical study 1

Consider Value[1] in Table 4. It is found as $\mathcal{R}_0 \approx 0.969$. This only means the existence of the equilibrium point $E_0(2668, 0, 28582)$. In addition, since $\mathcal{R}_0 < 1$, it is seen that the equilibrium point E_0 for CFOS ($\alpha_1 = \alpha_2 = \alpha_3 = 0.9$) is stable according to Table 3 . This situation with initial conditions $[S_0\ I_0\ R_0] = [1000\ 1000\ 10000]$ can be seen in Figures 3 and 4.

Numerical study 2

When the values in Table 4 are used, the threshold parameter is found as $\mathcal{R}_0 \approx 1.2$. Also, the trivial equilibrium point is $E_0(5079, 0, 1587)$. Since $\mathcal{R}_0 > 1$, the positive equilibrium point $E_1(4220, 409, 1958)$ exists and E_0 is an unstable point according to Table 3. Only the stability of E_1 can be examined.

Derivatives-orders are given as $[\alpha_1\ \alpha_2\ \alpha_3] = [0.8\ 0.6\ 0.4]$. Since m is the least common multiple of the denominators of derivative-orders, it is 5. Equation (29) translates to

$$\lambda^9 + 0.16\lambda^7 + 0.2409\lambda^5 + 0.038044\lambda^3 + 0.0172598\lambda^2 + 0.002863818 = 0. \tag{30}$$

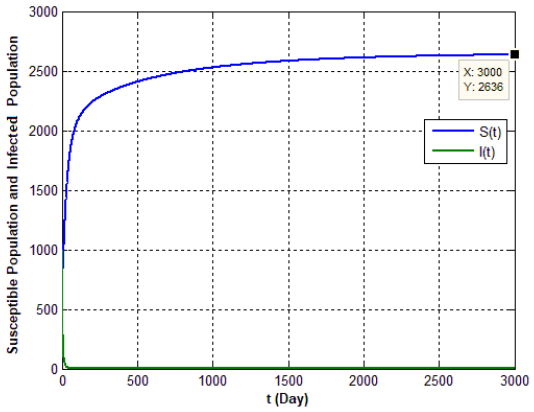

Figure 3. Time-dependent variation of susceptible and infectious populations for CFOS in (6)

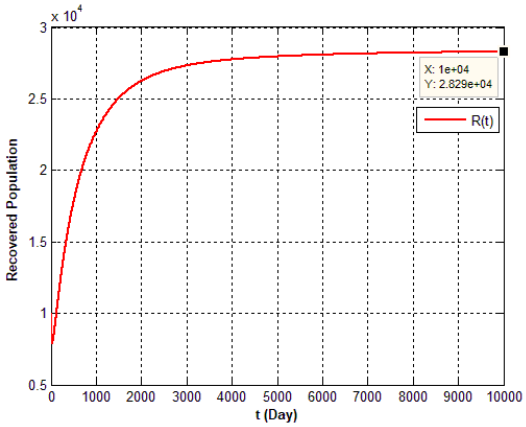

Figure 4. Time-dependent variation of the recovered population for CFOS in (6)

Roots of (30) are

$$\lambda_1 = -0.4990 + 0.5267i,$$

$$\lambda_2 = -0.4990 - 0.5267i,$$

$$\lambda_3 = 0.5039 + 0.4542i,$$

$$\lambda_4 = 0.5039 - 0.4542i,$$

$$\lambda_5 = -0.4042,$$

$$\lambda_6 = 0.2018 + 0.3738i,$$

$$\lambda_7 = 0.2018 - 0.3738i,$$

$$\lambda_8 = -0.0047 + 0.4025i,$$

$$\lambda_9 = -0.0047 - 0.4025i.$$

Also, we have

$$arg\lambda_1 = 133.4530°,$$

$$arg\lambda_2 = 226.5470°,$$

$$arg\lambda_3 = 42.0305°,$$

$$arg\lambda_4 = 317.9695°,$$

$$arg\lambda_5 = 180°,$$

$$arg\lambda_6 = 61.6371°,$$

$$arg\lambda_7 = 298.3629°,$$

$$arg\lambda_8 = 90.6690°,$$

$$\lambda_9 = 269.3310°.$$

Eigenvalues λ_i for $i = 1, 2, \ldots, 9$ are greater than $\frac{\pi}{2m} = 18°$. Therefore E_1 is LAS.

Let the initial conditions by $[S_0\ I_0\ R_0] = [10000\ 100\ 100]$. In this case, the numerical simulation is obtained along the following Figures 5, 6 and 7.

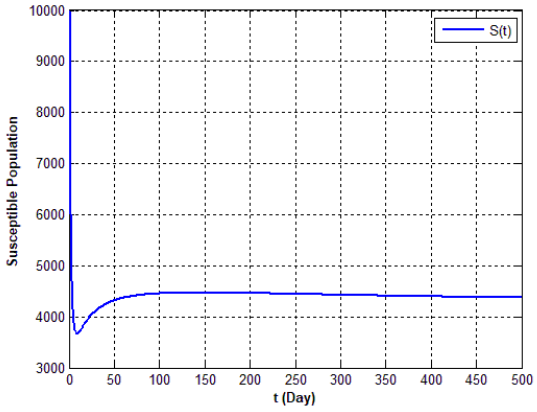

Figure 5. Time-dependent variation of susceptible population for IFOS in (6)

6 Conclusions

In this study, it is suggested the newly IFOS SIR model including the three time-dependent variables: susceptible, infected and recovered individuals in a population. This model proposed in system (6) is the form of nonlinear IFOS with the Caputo fractional derivative, accepted as rational numbers in the interval $(0, 1]$. In this context, the general situation regarding the stability of proposed model was investigated. Considering the derivative-orders, a new perspective was presented to the literature.

The model has an infection-free equilibrium point $E_0\left(\frac{\Lambda}{b}\frac{(b+\nu)}{(b+\nu+\mu)}, 0, \frac{\Lambda}{b}\frac{\mu}{(b+\nu+\mu)}\right)$ and a positive equilibrium point $E_1\left(S^* = \frac{(\gamma+d+b)}{\eta}, I^* = \frac{S^*(\mathcal{R}_0-1)(b+\nu+\mu)}{\left(\gamma+d+b+\nu\frac{d}{b}+\nu\right)}, R^* = \left(\frac{S^*(\mathcal{R}_0-1)(b+\nu+\mu)}{\left(\gamma+d+b+\nu\frac{d}{b}+\nu\right)} + \frac{S^*}{\gamma}\mu\right)\frac{\gamma}{(b+\nu)}\right)$. For these equilibrium points, their existence were analyzed according to the threshold parameter $\mathcal{R}_0$ and their stability were examined according to both $\mathcal{R}_0$ and eigenvalues obtained from characteristic equation roots. These results about the stability analysis are summarized in Table 3. In general, the *SIR* models in literature trying to explain the infection progress in a population with respect to the only parameter R_0. According to qualitative analysis of our model, it was found followings:

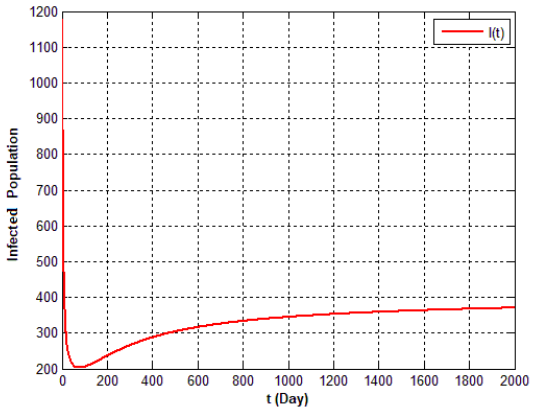

Figure 6. Time-dependent variation of infected population for IFOS in (6)

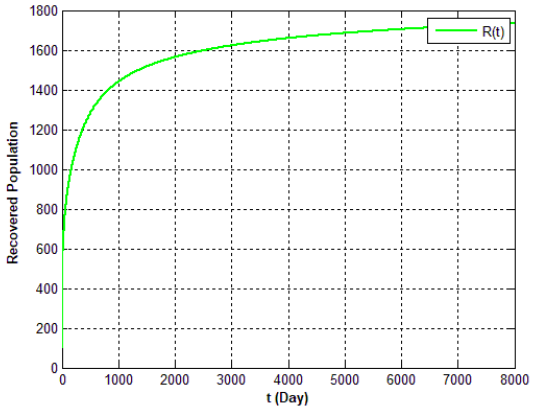

Figure 7. Time-dependent variation of recovered population for IFOS in (6)

i. Disease-free equilibrium point always exists and is LAS,

$$\begin{cases} \text{If } R_0 < 1 & \text{in case of } \alpha_1=\alpha_2=\alpha_3\leq 1. \\ \text{If } R_0 < 1 \text{ and (28) meet conditions } |arg(\lambda_i)| > \frac{\pi}{2m} & \text{in other cases.} \end{cases}$$

ii. Positive equilibrium point exists when $R_0>1$. This point is LAS,

$$\begin{cases} \text{If } R_0>1 \text{ (also the existence condition)} & \text{in case of } \alpha_1=\alpha_2=\alpha_3\leq 1. \\ \text{If (29) meet conditions } |arg(\lambda_i)| > \frac{\pi}{2m} & \text{in other cases.} \end{cases}$$

In numerical studies, the results of the qualitative analysis given in Table 3 are supported by graphics for the proposed SIR model. For this, the stability of E_0 for CFOS is shown in the first numerical study, while the stability of E_1 for IFOS is shown in the second numerical study.

Declarations

Consent for publication

Not applicable.

Conflicts of interest

The author declares that there is no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding

The author declares that there is no funding source for the reported research.

Author's contributions

The research was carried out by the author and he accepts that the contributions and responsibilities belong to the author.

Acknowledgements

Not applicable.

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